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Wavy spirals and their fractal connection with chirps

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Abstract. We study the fractal oscillatory of a class of smooth real functions near infinity by connecting their *oscillatory* and *phase dimensions*, defined as the box dimension of their graphs and of the corresponding phase spirals, respectively. In particular, we introduce *wavy spirals*, which exhibit *non-monotone* radial convergence to the origin.

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1. Introduction

fractal analysis of differential equations since emerged in the last decades as an important tool in better understanding the behavior of their oscillatory solutions. The main focus of fractal analysis in dynamics is on fractal dimension theory. Its goal is to determine complexity of invariant sets and measures using fractal dimensions. The fractal dimension has been successfully used in studying, for instance, the logistic map, the Smale horseshoe, Lorenz and Hénon attractors, Julia and Mandelbrot sets, spiral trajectories, infinite-dimensional dynamical systems and even in the probability theory; see [26].

In this paper we are focused on studying the connection between the fractal dimension of graphs of oscillatory solutions and the fractal dimension of the associated phase portraits. In particular, we use the box dimension, which we exploit instead of the Hausdorff dimension. Due to the countable stability of the Hausdorff dimension, its value is trivial on all smooth nonrectifiable curves, while the box dimension is nontrivial, that is, larger than 1. From the point of view of fractal analysis of trajectories and graphs of solutions of differential equations, most interesting are solutions having phase plots and graphs of an infinite length. The Hausdorff dimension, unlike the box dimension, is not suitable to classify these solutions.

Our work was initially inspired by Tricot [20], where the box dimension of graphs of a simple spiral ($r = \varphi^{-\alpha}$, $\alpha \in (0,1)$, in polar coordinates) and of an (α,β)-*chirp* ($f(t) = t^\alpha \cos t^{-\beta}$, $\alpha > 0$, $\beta > 0$) has been computed near the origin. Since then, these results have been generalized to some more general spiral trajectories of dynamical systems and to chirp-like functions. Fractal properties of spiral trajectories of

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dynamical systems in the phase plane have been studied by Žubrinić and Županović; see [23, 24, 25]. An interesting behavior of the box dimension of spiral trajectories has been discovered and related to the bifurcation of a system, in particular to the Hopf bifurcation. On the other hand, the chirp-like behavior of solutions of different types of second-order linear differential equations is also of interest. The Euler type, half linear and Bessel equations have been studied by Pašić, Tanaka and Wong; see [13, 14, 21]. More specifically, this work has been motivated by Pašić, Žubrinić and Županović [15], containing the first results connecting fractal properties of chirps and spirals, with applications to Liénard and Bessel equations.

All of this encouraged us to study and analyze the connection between chirp-like functions and the corresponding spiral trajectories in the phase plane and vice versa. There are two possible ways of looking at solutions: using the graph of a solution, or using the phase plot of the solution, and the latter was first theoretically developed by Poincaré. Our main results are obtained in Theorems 4 and 7. An application to the Bessel equation can be found in [8].

A specific type of a spiral associated to the oscillatory solutions of Bessel equations emerged in our study of phase portraits, converging to the origin in a *non-monotone* way as a function of φ. We call it the *wavy spiral*; see Definition 10. It also appears in the study of the curves obtained via the parametrization of the oscillatory integrals studied in Arnol′d, Guseĭn-Zade and Varchenko, [1, Part II]. These curves can exhibit even more complex behavior, having self-intersections. The oscillatory integrals from [1] are naturally related to generalized Fresnel integrals, and fractal properties of the associated spirals studied in [7].

Techniques of fractal analysis have also been successfully applied to the study of bifurcations (see, e.g., Horvat Dmitrović [5], Li and Wu [22], Mardešić, Resman and Županović [9], Resman [17]), as well as to the case of the Hopf bifurcation at infinity (see Radunović, Žubrinić and Županović [16]), and to the infinite-dimensional dynamical systems related to a class of Schrödinger equations (see Milišić, Žubrinić and Županović [10]).

2. Definitions and notation

Given a bounded subset A of $\mathbb{R}^N$, we define the ε-neighborhood of A by $A_\varepsilon := \{y \in \mathbb{R}^N : d(y,A) < \varepsilon\}$, where $d(y,A)$ denotes the Euclidean distance from y to A. The *lower s-dimensional Minkowski content* of A, where $s \geq 0$, is defined by $\mathcal{M}_*^s(A) := \liminf_{\varepsilon\to 0} \frac{|A_\varepsilon|}{\varepsilon^{N-s}}$, and analogously the *upper s-dimensional Minkowski content* $\mathcal{M}^{*s}(A)$. If both of these quantities coincide, the common value is called the *s-dimensional Minkowski content of* A, and denoted by $\mathcal{M}^s(A)$. Now we introduce the *lower and upper box dimensions* of A by $\underline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}_*^s(A) = 0\}$, and $\overline{\dim}_B A := \inf\{s \geq 0 : \mathcal{M}^{*s}(A) = 0\}$, respectively. If these two values coincide, we call it simply the box dimension of A, and denote it by $\dim_B A$.

Definition 1 (The Minkowski nondegeneracy). *If $0 < \mathcal{M}_*^d(A) \leq \mathcal{M}^{*d}(A) < \infty$ for some d, then we say that A is* Minkowski nondegenerate. *In this case obviously $d = \dim_B A$.*

More details on these definitions can be found in Falconer [3] and Tricot [20]. Some generalizations are given in [9].

Definition 2 (The oscillatory function near ∞ and 0)**.** *Let $x : [t_0, \infty) \to \mathbb{R}$, where $t_0 > 0$, be a continuous function. We say that the function x is* oscillatory *near $t = \infty$ if there exists a sequence $t_k \to \infty$, such that $x(t_k) = 0$ and the functions $x|_{(t_k, t_{k+1})}$ alternately change a sign for $k \in \mathbb{N}$.*

Analogously, let $u : (0, t_0] \to \mathbb{R}$, where $t_0 > 0$, be a continuous function. We say that the function u is oscillatory *near the origin if there exists a sequence s_k such that $s_k \searrow 0$ as $k \to \infty$, $u(s_k) = 0$ and the restrictions $u|_{(s_{k+1}, s_k)}$ alternately change a sign for $k \in \mathbb{N}$.*

Definition 3 (The d-dimensional fractal oscillatory function (see Pašić [11]))**.** *Suppose that $v : I \to \mathbb{R}$, where $I = (0, 1]$, is an oscillatory function near the origin and $d \in [1, 2)$. We say that v is the* d-dimensional fractal oscillatory *function near the origin if $\dim_B G(v) = d$ and $0 < \mathcal{M}_*^d(G(v)) \leq \mathcal{M}^{*d}(G(v)) < \infty$, where $G(v)$ denotes the graph of v.*

Assume that the function $x : [t_0, \infty) \to \mathbb{R}$ is oscillatory near $t = \infty$. Let us define $X : (0, 1/t_0] \to \mathbb{R}$, by $X(\tau) = x(1/\tau)$. It is clear that the function $X = X(\tau)$ is oscillatory near the origin. We measure the rate of oscillatority of $x = x(t)$ near $t = \infty$ by the rate of oscillatority of $X(\tau)$ near $\tau = 0$.

Definition 4 (The oscillatory dimension)**.** *The* oscillatory dimension $\dim_{osc}(x)$ *(near $t = \infty$) is defined as the box dimension of the graph of the function $X = X(\tau)$ near $\tau = 0$, $\dim_{osc}(x) = \dim_B G(X)$, provided the box dimension exists.*

Definition 5 (The spiral)**.** *By a (positively oriented)* spiral *we mean the graph of a function $r = f(\varphi)$, for $\varphi \geq \varphi_1 > 0$, in polar coordinates, where:*

$$f : [\varphi_1, \infty) \to (0, \infty), \;\; f(\varphi) \to 0 \;\text{ as } \varphi \to \infty,$$

and f is radially decreasing *(i.e., for any fixed $\varphi \geq \varphi_1$ the function $\mathbb{N} \ni k \mapsto f(\varphi + 2k\pi)$ is decreasing).*

This definition appears in [23]. By a negatively oriented spiral we mean the graph of a function $r = g(\varphi)$, for $\varphi \leq \varphi_1' < 0$, in polar coordinates, such that the curve defined as the graph of $r = g(-\varphi)$, $\varphi \geq |\varphi_1'| > 0$, given in polar coordinates, satisfies the conditions of Definition 5. It is easy to see that the spiral defined by a function $g(\varphi)$ is a mirror image of the spiral defined by $g(-\varphi)$, with respect to the x-axis. Both types of spirals will be called the *spiral*, in short. We also say that the graph of a function $r = f(\varphi)$, for $\varphi \geq \varphi_1 > 0$, defined in polar coordinates, is a *spiral near the origin* if there exists $\varphi_2 \geq \varphi_1$, such that the graph of the function $r = f(\varphi)$, for $\varphi \geq \varphi_2$, viewed in polar coordinates, is the spiral.

Assume now that a function x is of class C^1. We say that the function x is *phase oscillatory* if the set $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$ in the plane is a spiral converging to the origin.

Definition 6 (The phase dimension)**.** *The* phase dimension $\dim_{ph}(x)$ *of a function $x : [t_0, \infty) \to \mathbb{R}$ of class C^1 is defined as the box dimension of the corresponding planar curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$.*

The oscillatory and phase dimensions are fractal dimensions, introduced in the study of chirp-like solutions of second order ODEs; see [15].

For any two real functions $f(t)$ and $g(t)$ of a real variable we write $f(t) \simeq g(t)$ as $t \to 0$ (resp., as $t \to \infty$) if there exist two positive constants C and D such that $C\, f(t) \leq g(t) \leq D\, f(t)$ for all t sufficiently close to $t = 0$ (resp., for all t sufficiently large). For a function $F : U \to V$, with $U, V \subset \mathbb{R}^2$, $V = F(U)$, we write $|F(x_1) - F(x_2)| \simeq |x_1 - x_2|$ if F is a bi-Lipschitz mapping, i.e., both F and F^{-1} are Lipschitz functions.

Definition 7 (The k-similarity)**.** *Let k be a fixed positive integer and let f and g be two functions of class C^k. For any nonzero integer $j \leq k$, we say that $f^{(j)}(t) \sim g^{(j)}(t)$ as $t \to 0$ (resp., as $t \to \infty$) if $f^{(j)}(t)/g^{(j)}(t) \to 1$ as $t \to 0$ (resp., as $t \to \infty$). If for all $j = 0, 1, ..., k$ we have that $f^{(j)}(t) \sim g^{(j)}(t)$ as $t \to 0$ (resp., as $t \to \infty$), then we write that $f(t) \sim_k g(t)$ as $t \to 0$ (resp., as $t \to \infty$).*

Analogously, if k is a fixed positive integer, for any two given functions f and g of class C^k we write that $f(t) \simeq_k g(t)$ as $t \to 0$ (resp., as $t \to \infty$), if $f^{(j)}(t) \simeq g^{(j)}(t)$ as $t \to 0$ (resp., as $t \to \infty$) for all $j = 0, 1, ..., k$.

We write $f(t) = O(g(t))$ as $t \to 0$ (as $t \to \infty$) if there exists a positive constant C such that $|f(t)| \leq C|g(t)|$ for all t sufficiently close to $t = 0$. (for all t sufficiently large). Similarly, we write $f(t) = o(g(t))$ as $t \to \infty$ if for every positive constant ε it holds $|f(t)| \leq \varepsilon |g(t)|$ for all t sufficiently large.

Definition 8 (The (α, β)-chirp-like function)**.** *A function of the following form, $y = P(x)\sin(Q(x))$ or $y = P(x)\cos(Q(x))$, where $P(x) \simeq x^\alpha$, $Q(x) \simeq_1 x^{-\beta}$ as $x \to 0$, with $\alpha > 0$ and $\beta > 0$, is called the (α, β)-*chirp-like function *near $x = 0$. A special case is the (α, β)-*chirp*, defined by $P(x) = x^\alpha$ and $Q(x) = x^\beta$.*

3. Spirals generated by chirps

We study spirals generated by chirps in the sense of Theorem 4; see Definitions 5 and 8. To prove Theorem 4 about the box dimension of a spiral generated by a chirp we need a new version of [23, Theorem 5]. Let us first recall [23, Theorem 5], cited here in a more condensed form, suitable for our purposes. The following theorem extends a result about the box dimension of a spiral due to Dupain, Mendès France and Tricot; see [2, 20].

Theorem 1 (Theorem 5 from [23])**.** *Let $f : [\varphi_1, \infty) \to (0, \infty)$ be a decreasing function of class C^2, such that $f(\varphi) \to 0$ as $\varphi \to \infty$. Let $\alpha \in (0, 1)$. Assume that there exist positive constants $\underline{m}$, $\overline{m}$, M_1, M_2 and M_3 such that for all $\varphi \geq \varphi_1 > 0$,*

$$\underline{m}\, \varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\, \varphi^{-\alpha}, \quad M_1 \varphi^{-\alpha-1} \leq |f'(\varphi)| \leq M_2 \varphi^{-\alpha-1}, \quad |f''(\varphi)| \leq M_3 \varphi^{-\alpha}.$$

Let Γ be the graph of $r = f(\varphi)$ in polar coordinates. Then $\dim_B \Gamma = 2/(1 + \alpha)$.

Now we provide an adapted version of Theorem 1.

Theorem 2 (The dimension of a piecewise smooth nonincreasing spiral)**.** *Let $f : [\varphi_1, \infty) \to (0, \infty)$ be a nonincreasing and radially decreasing function, as well as a* *continuous and piecewise continuously differentiable function. We assume that the number of smooth pieces of f in $[\varphi_1, \overline{\varphi}_1]$ is finite, for any $\overline{\varphi}_1 > \varphi_1$. Assume that there exist positive constants α, $\underline{m}$, $\overline{m}$, $\underline{a}$ and M such that for all $\varphi \geq \varphi_1$,*

$$\underline{m}\varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\varphi^{-\alpha}, \quad \underline{a}\varphi^{-\alpha-1} \leq f(\varphi) - f(\varphi + 2\pi),$$

and for all φ where $f(\varphi)$ is differentiable, $|f'(\varphi)| \leq M\varphi^{-\alpha-1}$. Let Γ be the graph of $r = f(\varphi)$ in polar coordinates. If $\alpha \in (0,1)$ then $\dim_B \Gamma = 2/(1+\alpha)$.

Remark 1. *Notice the difference between the assumptions of Theorems 1 and 2. In Theorem 1, the function f is decreasing and of class C^2. By careful examination of the proof of [23, Theorem 5], one can see that f being decreasing is used only in the sense of nonincreasing, that is, not strictly decreasing, hence in Theorem 1 we can assume that f is nonincreasing. The additional smoothness of f and additional conditions regarding constants M_1 and M_3 in Theorem 1 are used only in the calculation of the Minkowski content in [23, Theorem 5] which we exclude from our Theorem 2. Further reduction in smoothness of f from a continuously differentiable to the piecewise continuously differentiable function can be found in Lemma 1.*

For the proof of Theorems 2 and 4 below, we need the following lemma, which is a generalization of [23, Lemma 1] dealing with smooth spirals.

Lemma 1 (The excision property for piecewise smooth curves)**.** *Let Γ be the image of a continuous and piecewise continuously differentiable function $h : [\varphi_1, \infty) \to \mathbb{R}^2$ (piecewise in the sense of Theorem 2). Assume that $\underline{\dim}_B\Gamma > 1$, $\Gamma_1 := h((\overline{\varphi}_1, \infty))$, for some fixed $\overline{\varphi}_1 > \varphi_1$, and $h([\varphi_1, \overline{\varphi}_1]) \bigcap \Gamma_1 = \emptyset$. Then $\underline{\dim}_B\Gamma_1 = \underline{\dim}_B\Gamma$ and $\overline{\dim}_B\Gamma_1 = \overline{\dim}_B\Gamma$.*

Proof. The proof is analogous to the proof of [23, Lemma 1], but with the following difference. Here, the curve $\Gamma_2 := \Gamma \backslash \Gamma_1 = h([\varphi_1, \overline{\varphi}_1])$ is rectifiable due to the piecewise rectifiability of h and due to the finite number of pieces in the segment $(\varphi_1, \overline{\varphi}_1]$. The function h is piecewise rectifiable due to its piecewise smoothness and continuity. Also, by careful examination of the proof of [23, Lemma 1], it follows that we can substitute the injectivity assumption on h with the weaker condition that $h([\varphi_1, \overline{\varphi}_1]) \bigcap \Gamma_1 = \emptyset$. (For more details, see [23, Lemma 1].) ☐

Proof of Theorem 2. The proof is analogous to the proof of [23, Theorem 5], but using the new Lemma 1. ☐

Theorem 3 deals with a spiral Γ' described by $r = f(\varphi)$, where $f(\varphi) \to 0$ as $\varphi \to \infty$ in a nonmonotonous way; see Definitions 9 and 10 below. Such a property of Γ' is called the *spiral waviness* and it is defined below.

Definition 9 (The wavy function). *Let $r : [t_0, \infty) \to (0, \infty)$ be a C^1 function. Assume that $r'(t_0) \leq 0$. We say that $r = r(t)$ is the* wavy function *if the sequence (t_n) defined inductively by*

$$t_{2k+1} = \inf\{t : t > t_{2k}, r'(t) > 0\},\ t_{2k+2} = \inf\{t : t > t_{2k+1}, r(t) = r(t_{2k+1})\},\ k \in \mathbb{N}_0,$$

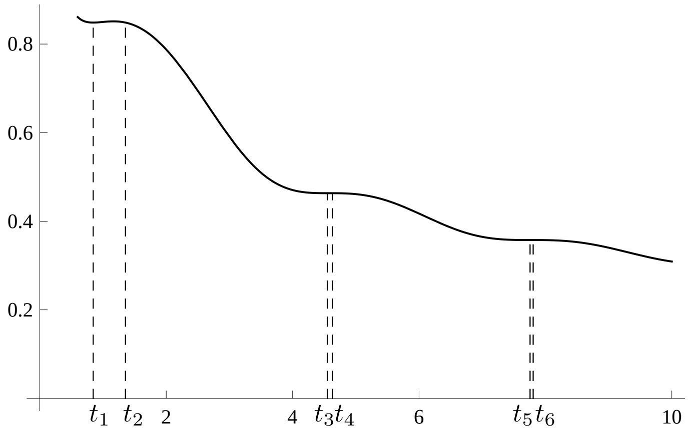

Figure 1: *The function $r(t)$ for $p(t) = t^{-1/2}$, with $t_0 = 0.6$; see Lemma 3. This is a wavy function; see Definition 9, with local minima at t_{2k+1}, $k = 0, 1, \ldots$*

is well-defined, and satisfies the waviness conditions*:*

$$\begin{aligned}
&(i)\ \textit{The sequence } (t_n) \textit{ is increasing and } t_n \to \infty \textit{ as } n \to \infty.\\
&(ii)\ \textit{There exists } \varepsilon > 0, \textit{ such that for all } k \in \mathbb{N}_0 \textit{ we have } t_{2k+1} - t_{2k} \geq \varepsilon.\\
&(iii)\ \textit{For all } k \textit{ sufficiently large it holds } \operatorname*{osc}_{t \in [t_{2k+1}, t_{2k+2}]} r(t) = o\left(t_{2k+1}^{-\alpha-1}\right),\\
&\qquad \alpha \in (0,1),
\end{aligned} \tag{1}$$

where $\operatorname*{osc}_{t \in I} r(t) := \max_{t \in I} r(t) - \min_{t \in I} r(t)$.

Notice that $\min_{t \in [t_{2k+1}, t_{2k+2}]} r(t) = r(t_{2k+1})$. Condition (i) means that the property of waviness of $r = r(t)$ is global on the whole domain. Condition (ii) is connected to an assumption of Lemma 3. Condition (iii) is a condition on a decay rate on the sequence of oscillations of r on $I_k = [t_{2k+1}, t_{2k+2}]$, for k sufficiently large. Also, observe that the condition $r'(t_0) \leq 0$ assures that t_1 is well-defined; see Figure 1.

Definition 10 (The wavy spiral)**.** *Let a spiral Γ' be given in polar coordinates by $r = f(\varphi)$, where f is a given function. If there exists an increasing or decreasing function of class C^1, $\varphi = \varphi(t)$, such that $r(t) = f(\varphi(t))$ is the wavy function, then we say Γ' is the* wavy spiral*.*

For an example of a spiral Γ', see Figure 2. Now, using Theorem 2 and Lemma 1 we prove the following Theorem 3.

Theorem 3 (The box dimension of a wavy spiral)**.** *Let $t_0 > 0$ and assume $r : [t_0, \infty) \to (0, \infty)$ is a wavy function. Assume that $\varphi : [t_0, \infty) \to [\varphi_0, \infty)$ is an increasing function of class C^1 such that $\varphi(t_0) = \varphi_0 > 0$ and there exists $\bar{\varphi}_0 \in \mathbb{R}$ such that*

$$|(\varphi(t) - \bar{\varphi}_0) - (t - t_0)| \to 0 \quad \textit{as} \quad t \to \infty. \tag{2}$$

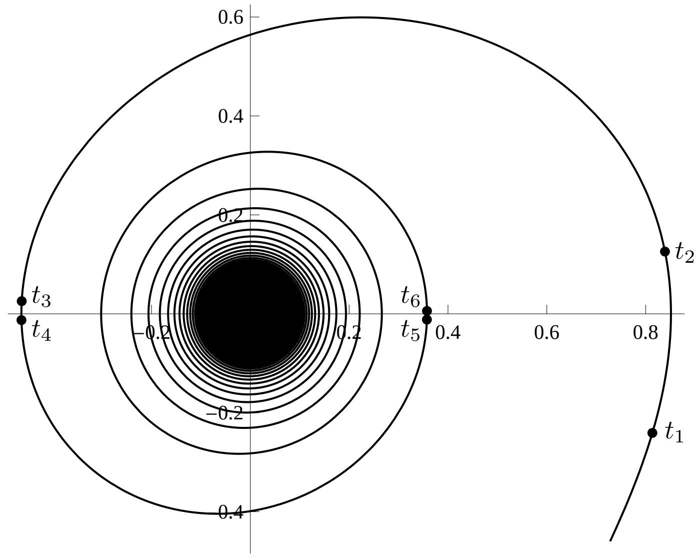

Figure 2: *The spiral* Γ' *for* $p(t) = t^{-1/2}$, *with* $t_0 = 0.6$*; see Lemma 3, is a wavy spiral; see Definition 10. Highlighted points correspond to parameters* t_k, $k = 1, 2, \ldots$

Let $f : [\varphi_0, \infty) \to (0, \infty)$ *be defined by* $f(\varphi(t)) = r(t)$. *Assume that* Γ' *is a spiral defined in polar coordinates by* $r = f(\varphi)$, *satisfying Definition 5. Let* $\alpha \in (0, 1)$ *be the same value as in* $(1)(iii)$ *for the wavy function* r, *and assume* ε' *is such that* $0 < \varepsilon' < \varepsilon$, *where* ε *is defined by* $(1)(ii)$ *for the wavy function* r. *Assume that there exist positive constants* $\underline{m}$, $\overline{m}$, $\underline{a}'$ *and* M *such that for all* $\varphi \ge \varphi_0$,

$$\underline{m}\varphi^{-\alpha} \le f(\varphi) \le \overline{m}\varphi^{-\alpha}, \tag{3}$$

$$|f'(\varphi)| \le M\varphi^{-\alpha-1}, \tag{4}$$

and for all $\triangle\varphi$, *such that* $\theta \le \triangle\varphi \le 2\pi + \theta$, *there holds*

$$\underline{a}'\varphi^{-\alpha-1} \le f(\varphi) - f(\varphi + \triangle\varphi), \tag{5}$$

where $\theta := \min\{\varepsilon', \pi\}$. *Then* Γ' *is the wavy spiral and* $\dim_B \Gamma' = 2/(1+\alpha)$.

The proof of Theorem 3 is given in [8]. Now, Theorem 3 enables us to calculate the box dimension of the spiral generated by a chirp, which is one of the main results of this paper.

Theorem 4 (The chirp–spiral comparison)**.** *Let* $\alpha > 0$. *Assume that* $X : (0, 1/\tau_0] \to \mathbb{R}$, $\tau_0 > 0$, $X(\tau) = P(\tau)\sin 1/\tau$, *where* $P(\tau)$ *is a positive function such that* $P(\tau) \sim_3 \tau^\alpha$ *as* $\tau \to 0$. *Define* $x(t) := X(1/t)$ *and a continuous function* $\varphi(t)$ *by* $\tan\varphi(t) = \frac{\dot{x}(t)}{x(t)}$.

(i) *If* $\alpha \in (0, 1)$, *then the planar curve* $\Gamma := \{(x(t), \dot{x}(t)) \in \mathbb{R} : t \in [\tau_0, \infty)\}$ *generated by* X *is a wavy spiral* $r = f(\varphi)$, $\varphi \in (-\infty, -\phi_0]$ *near the origin. We have* $f(\varphi) \simeq |\varphi|^{-\alpha}$ *as* $\varphi \to -\infty$, *and* $\dim_{ph}(x) := \dim_B \Gamma = 2/(1+\alpha)$.

(ii) *If* $\alpha > 1$*, then the planar curve* $\Gamma := \{(x(t), \dot{x}(t)) \in \mathbb{R} : t \in [\tau_0, \infty)\}$ *is a rectifiable wavy spiral near the origin.*

The proof of Theorem 4 consists of checking the conditions of Theorem 3. The following lemmas make this verification easy.

Lemma 2. *Let* $\alpha > 0$ *and assume that* $P(\tau)$*,* $\tau \in (0, 1/t_0]$*,* $t_0 > 0$*, is such that* $P(\tau) \sim_3 \tau^\alpha$ *as* $\tau \to 0$*. Then* $p(t) := P(\frac{1}{t}) \sim_3 t^{-\alpha}$ *as* $t \to \infty$ *and vice versa. Furthermore, we have:*

$$\lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \ \lim_{t\to\infty} \frac{p''(t)}{p(t)} = 0, \tag{6}$$

$$-\frac{p(t)}{p'(t)} \sim \frac{t}{\alpha}, \quad -\frac{2p'(t)}{p''(t)} \sim \frac{2t}{\alpha+1} \text{ as } t \to \infty, \tag{7}$$

$$\sup_{t\in[t_0,\infty)} \left(-\frac{p(t)}{p'(t)}\right)' < \infty, \quad \sup_{t\in[t_0,\infty)} \left(-\frac{2p'(t)}{p''(t)}\right)' < \infty. \tag{8}$$

The claims of Lemma 2 follow directly from the assumptions.

Lemma 3. *Let* $\alpha \in (0, 1)$ *and*

$$r(t) = p(t)\sqrt{1 + \frac{[p'(t)]^2}{[p(t)]^2}[\sin t]^2 + \frac{p'(t)}{p(t)}\sin 2t}, \quad t \in [t_0, \infty), \ t_0 > 0,$$

where $p(t) \sim_1 t^{-\alpha}$ *as* $t \to \infty$*.*

Let $C \in \mathbb{R}$ *and assume that* $t(\varphi) = \varphi + C + O(\varphi^{-1})$ *as* $\varphi \to \infty$*. Let* $\triangle\varphi > 1$ *be fixed. Then there exists a constant* $k > 0$*, independent of* φ *and* $\triangle\varphi$*, such that for all* φ *sufficiently large it holds* $r(t(\varphi)) - r(t(\varphi + \triangle\varphi)) \geq k\varphi^{-\alpha-1}(1 + O(\varphi^{-1}))$*.*

The proof of Lemma 3 easily follows using Lemma 2, and will be omitted.

Proof of Theorem 4. (i) *Step 1.* (The box dimension is invariant with respect to mirroring of a spiral.) We will prove the equivalent claim, that the planar curve $\Gamma' = \{(x(t), -\dot{x}(t)) : t \in [\tau_0, \infty)\}$ is a wavy spiral defined by $r = f(\varphi)$, $\varphi \in [\phi_0, \infty)$, near the origin, satisfying $f(\varphi) \simeq \varphi^{-\alpha}$, in polar coordinates, near the origin, and $\dim_B \Gamma' = \frac{2}{1+\alpha}$. It is easy to see that the curve Γ is a mirror image of the curve Γ', with respect to the x-axis and hence Γ is the wavy spiral. Reflecting with respect to the x-axis in the plane is an isometric map. As the isometric map is bi-Lipschitz and therefore it preserves the box dimension (see [3, p. 44]), we see that $\dim_B \Gamma = \dim_B \Gamma' = \frac{2}{1+\alpha}$.

Step 2. (Checking condition (3).) From $x(t) = p(t)\sin t$ and $\dot{x}(t) = p'(t)\sin t + p(t)\cos t$, where $p(t) := P(1/t)$, we compute

$$\tan\varphi(t) = -\frac{\dot{x}(t)}{x(t)} = -\frac{p'(t)}{p(t)} - \frac{1}{\tan t}. \tag{9}$$

By differentiating (9) we obtain

$$\frac{d\varphi}{dt}(t) = [\cos\varphi(t)]^2 \left[\frac{[p'(t)]^2 - p(t)p''(t)}{[p(t)]^2} + \frac{1}{[\sin t]^2}\right]. \tag{10}$$

Using (9) again, we have

$$[\cos\varphi(t)]^2 = \frac{1}{1+[\tan\varphi(t)]^2} = \frac{[p(t)\sin t]^2}{[p(t)]^2+[p'(t)\sin t]^2+2p(t)p'(t)\sin t\cos t}. \tag{11}$$

Substituting into (10) and using (6) we get

$$\lim_{t\to\infty}\frac{d\varphi}{dt}(t) = 1. \tag{12}$$

From (12), it follows that $\varphi \simeq t$ as $t\to\infty$ and

$$[r(t)]^2 = [x(t)]^2 + [-\dot{x}(t)]^2 = [p(t)]^2 + [p'(t)\sin t]^2 + p(t)p'(t)\sin 2t \tag{13}$$

implies that

$$f(\varphi(t)) = r(t) \simeq t^{-\alpha} \simeq \varphi^{-\alpha} \text{ as } t\to\infty. \tag{14}$$

Notice that from (13) it follows that the function $r(t)$ is of class C^2 and by substituting (11) into (10), taking (13) into account, we see that the function $\varphi(t)$ is of class C^1.

Step 3. (Checking condition (4).) On the other hand, differentiating (13) we obtain that

$$\frac{dr}{dt}(t) = \left[2p(t)p'(t)[\cos t]^2 + \frac{2[p'(t)]^2+p(t)p''(t)}{2}\sin 2t + p'(t)p''(t)[\sin t]^2\right]\frac{1}{r(t)}. \tag{15}$$

Also, from (15) we have

$$\frac{dr}{dt}(t) = \frac{2p(t)p'(t)}{r(t)}[\cos t]^2 + O(t^{-\alpha-2}) \text{ as } t\to\infty. \tag{16}$$

Since $\frac{dr}{dt}(t) = f'(\varphi)\cdot\frac{d\varphi}{dt}(t)$ and since by (12) we have $\frac{d\varphi}{dt}(t) \simeq 1$ as $t\to\infty$, there exists $C_0 > 0$ and $C_1 > C_0$ such that $|f'(\varphi)| \le C_0 t^{-\alpha-1} \le C_1\varphi^{-\alpha-1}$ as $\varphi\to\infty$.

Step 4. (Checking condition (2).) Using (9) and [8, Lemma 7], we obtain $\tan\varphi(t) = -(\cot t + O(t^{-1})) = -\cot(t+O(t^{-1})) = \tan(t+\frac{\pi}{2}+O(t^{-1}))$ as $t\to\infty$. Since the function $\varphi(t)$ is continuous by the definition and $O(t^{-1}) < \pi$ for t sufficiently large, then there exists $k\in\mathbb{Z}$ such that $\varphi(t) = (t+\frac{\pi}{2}+k\pi)+O(t^{-1})$ as $t\to\infty$. From the definition of $\varphi(t)$ we conclude that we may take without loss of generality $k=0$. Finally, we get

$$\varphi(t) = \left(t+\frac{\pi}{2}\right) + O(t^{-1}) \text{ as } t\to\infty. \tag{17}$$

Step 5. (Checking condition (5).) From (12) it follows that there exists $\tau_1 \ge \tau_0$ such that $\frac{d\varphi}{dt}(t) > 0$ for all $t \ge \tau_1$. Hence, the function $\varphi(t)$ is increasing for all t sufficiently large. As the function $\varphi(t)$ is continuous, we conclude that for all φ sufficiently large there exists the inverse function $t = t(\varphi)$ of the function $\varphi = \varphi(t)$ and $t(\varphi) = \left(\varphi - \frac{\pi}{2}\right) + O(\varphi^{-1})$ as $\varphi\to\infty$. Define the value $\phi_1 := \varphi(\tau_1)$ and notice that we can take τ_1 sufficiently large such that $\phi_1 \ge \phi_0$.

From (13), we obtain $r(t) = p(t)\sqrt{1 + \frac{[p'(t)]^2}{[p(t)]^2}[\sin t]^2 + \frac{p'(t)}{p(t)}\sin 2t}$. By Lemma 3 we conclude that for fixed $\triangle\varphi > 1$ we have

$$f(\varphi) - f(\varphi + \triangle\varphi) = r(t(\varphi)) - r(t(\varphi + \triangle\varphi)) \geq k_1\varphi^{-\alpha-1}, \tag{18}$$

provided that φ is sufficiently large. Moreover, by careful examination of the proof of Lemma 3, we conclude that equation (18) holds uniformly for every $\triangle\varphi$ from a bounded interval whose lower bound is greater than 1, also provided φ is sufficiently large. (We note that we will have to require that θ from Theorem 3 is larger than 1.)

Step 6. (Γ' is a spiral near the origin.) Now we can prove that Γ' is a spiral near the origin, that is, $f(\varphi)$ satisfies Definition 5 near the origin. First, from (14) it follows that $f(\varphi) \to 0$ as $\varphi \to \infty$. Second, from (18) it follows that $f(\varphi)$ is radially decreasing for all φ sufficiently large, that is, there exists $\phi_2 \geq \phi_1$ such that $f|_{[\phi_2,\infty)}$ is radially decreasing.

Step 7. (The box dimension is invariant with respect to taking τ_0 and ϕ_0 sufficiently large.) First, we define τ_2 to be such that $\varphi(\tau_2) = \phi_2$. Notice that τ_2 is well-defined and $\tau_2 \geq \tau_1$. As $p(t) > 0$, from (13) and the definition of $x(t)$ and $\dot{x}(t)$, it follows that $r(t) > 0$, that is, $r(t)$ is a strictly positive function. This means that there exists a constant $m_1 > 0$ such that $r(t) > m_1$ for all $t \in [\tau_0, \tau_2]$. Observe that $\phi_2 \geq \phi_1 \geq \phi_0$. From (14) it follows that $r(t) \to 0$ as $t \to \infty$, so there exists $\tau_3 \geq \tau_2$ such that $r(t) < m_1$ for all $t \in [\tau_3, \infty)$. We define $\phi_3 := \varphi(\tau_3)$. Notice that we could increase τ_3 and ϕ_3 to accommodate all requirements, in different parts of the proof, on t or φ being sufficiently large. Now, using the upper and lower bounds on $r(t)$, we conclude that $\Gamma'|_{[\tau_0,\tau_2]} \bigcap \Gamma'|_{(\tau_3,\infty)} = \emptyset$. As $f|_{[\phi_2,\infty)}$ is radially decreasing and $\varphi'(t) > 0$ for all $t \in [\tau_2, \infty)$, it follows that $\Gamma'|_{(\tau_2,\infty)}$ does not have self intersections, so that $\Gamma'|_{[\tau_2,\tau_3]} \bigcap \Gamma'|_{(\tau_3,\infty)} = \emptyset$.

Finally, we conclude that $\Gamma'|_{[\tau_0,\tau_3]} \bigcap \Gamma'|_{(\tau_3,\infty)} = \emptyset$. Now, we can apply Lemma 1 to the curve Γ'. Using Lemma 1 we see that we can assume without loss of generality that τ_0 and ϕ_0 appearing in the assumptions of the theorem, are sufficiently large. Informally, we can always remove any rectifiable part from the beginning of Γ', without changing the box dimension of Γ'.

Step 8. (Checking waviness conditions (1).) By factoring (15), we get

$$\frac{dr}{dt}(t) = \left(1 + \frac{p'(t)}{p(t)}\tan t\right)\left(1 + \frac{p''(t)}{2p'(t)}\tan t\right)\frac{2p(t)p'(t)}{r(t)}[\cos t]^2, \tag{19}$$

for every $t \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ ($\cos t \neq 0$). By Lemma 6 and Remark 6 (see below) and using (7) and (8), there exists $k_0 \in \mathbb{N}_0$ such that the equations $\tan t = -\frac{p(t)}{p'(t)}$ and $\tan t = -\frac{2p'(t)}{p''(t)}$, have unique solutions $\hat{t}_{2k}$ and t_{2k-1}, respectively, in the intervals $((k+k_0)\pi - \pi, (k+k_0)\pi - \frac{\pi}{2})$, for each $k \in \mathbb{N}_0$, since $-\frac{p(t)}{p'(t)} \sim \frac{t}{\alpha}$ and $-\frac{2p'(t)}{p''(t)} \sim \frac{2t}{\alpha+1}$ as $t \to \infty$. Moreover, by taking k_0 to be sufficiently large, from (7) and using inequalities $1 < 2/(\alpha+1) < 1/\alpha$, we see that $\hat{t}_{2k}$ and t_{2k-1} even lie in the smaller intervals

$$((k+k_0)\pi - \frac{\pi}{2} - \frac{\pi}{3}, (k+k_0)\pi - \frac{\pi}{2}), \tag{20}$$

for each $k \in \mathbb{N}_0$. (The statement is true for interval of any length provided the upper bound is $(k+k_0)\pi - \frac{\pi}{2}$. We choose the value $\pi/3$, because it is convenient later in the proof.)

Because of $\frac{1}{\alpha} \neq \frac{2}{\alpha+1}$ we see that $-\frac{p(t)}{p'(t)} \neq -\frac{2p'(t)}{p''(t)}$ for t sufficiently large, so $\hat{t}_{2k} \neq t_{2k-1}$ for k_0 sufficiently large. We can take without loss of generality that $t_{2k-1} < \hat{t}_{2k}$. Hence, $\hat{t}_{2k} - t_{2k-1} < \pi/3$ for every $k \in \mathbb{N}$, provided k_0 is sufficiently large. It is easy to see from (19) that $\frac{dr}{dt}(t) > 0$, for all $t \in (t_{2k-1}, \hat{t}_{2k})$. As $\frac{d\varphi}{dt}(t) > 0$ for all t sufficiently large, from $\frac{dr}{dt}(t) = f'(\varphi) \cdot \frac{d\varphi}{dt}(t)$ it follows that $f'(\varphi) > 0$ on the set $\bigcup_{k=1}^{\infty} (\varphi_{2k-1}, \hat{\varphi}_{2k})$, where $\varphi_{2k-1} := \varphi(t_{2k-1})$ and $\hat{\varphi}_{2k} := \varphi(\hat{t}_{2k})$. This implies that the function $f(\varphi)$ is increasing for some φ, so we cannot apply Theorem 2 directly. Notice that if $t \in \bigcup_{k=0}^{\infty} (t_{2k-1}, \hat{t}_{2k})$, then $r'(t) > 0$ and if $t \in \bigcup_{k=0}^{\infty} (\hat{t}_{2k}, t_{2k+1})$, then $r'(t) < 0$.

We would like to prove that for every $k \in \mathbb{N}_0$ there exists a unique $t_{2k} \in (\hat{t}_{2k}, t_{2k+1})$ such that $r(t_{2k}) = r(t_{2k-1})$ and $t_{2k} - t_{2k-1} < \pi/3$ (where we will take k_0 from (20) to be sufficiently large). As $r(\hat{t}_{2k}) > r(t_{2k-1})$, and as the function $r(t)$ is a continuous and strictly decreasing function on the interval $(\hat{t}_{2k}, t_{2k+1})$, it follows that, if such t_{2k} exists, then it is necessary unique, so we only need to prove the existence.

For every $k \in \mathbb{N}_0$ we take $\bar{t}_{2k} := t_{2k-1} + \pi/3$. Observe that $\bar{t}_{2k} \in (\hat{t}_{2k}, t_{2k+1})$, because from (20) follows that $t_{2k+1} - t_{2k-1} > 2\pi/3$ and $\hat{t}_{2k} - t_{2k-1} < \pi/3$. Define $\bar{\varphi}_{2k} := \varphi(\bar{t}_{2k})$ and take φ_{2k-1} as defined before. Using (17), we can take t or equivalently k_0 sufficiently large, such that $(\pi/3+1)/2 \leq \bar{\varphi}_{2k} - \varphi_{2k-1} \leq 2$ for every $k \in \mathbb{N}_0$. (The exact value of the upper bound is not important. We just take a value larger than $\pi/3$. For the lower bound, it is only important that it is between 1 and $\pi/3$, so we take the mean value between these two.)

Now, using Lemma 3, analogously as in *Step 5*, we compute

$$
\begin{aligned}
r(t_{2k-1}) - r(\bar{t}_{2k}) &= r(t(\varphi_{2k-1})) - r(t(\bar{\varphi}_{2k})) \\
&= r(t(\varphi_{2k-1})) - r(t(\varphi_{2k-1} + (\bar{\varphi}_{2k} - \varphi_{2k-1}))) \geq C_2 \varphi_{2k-1}^{-\alpha-1} > 0,
\end{aligned}
$$

for some $C_2 > 0$, provided φ or equivalently k_0 is sufficiently large. From this it follows $r(\bar{t}_{2k}) < r(t_{2k-1})$, and as the function $r(t)$ is of class C^1, strictly decreasing on the interval $(\hat{t}_{2k}, \bar{t}_{2k})$ and $r(\hat{t}_{2k}) > r(t_{2k-1})$, we see that there exist $t_{2k} \in (\hat{t}_{2k}, \bar{t}_{2k})$ such that $r(t_{2k}) = r(t_{2k-1})$ and obviously $t_{2k} - t_{2k-1} < \pi/3$. Using $t_{2k+1} - t_{2k-1} > 2\pi/3$, it follows that $t_{2k+1} - t_{2k} > 2\pi/3 - \pi/3 = \pi/3$. We established that for every $k \in \mathbb{N}_0$ we have $t_{2k+1} > t_{2k} > t_{2k-1}$. Notice that $r'(t_0) \leq 0$ and that the sequence $(t_n)_{n \in \mathbb{N}_0}$, is the same as the sequence from Definition 9, introduced for the function $r(t)$.

As $t_{2k+1} - t_{2k-1} > 2\pi/3$ for every $k \in \mathbb{N}_0$, we conclude that $t_n \to \infty$ as $n \to \infty$, which means that the sequence (t_n) satisfies condition (1)(i). As $t_{2k+1} - t_{2k} > \pi/3$ for every $k \in \mathbb{N}_0$, by taking $\varepsilon = \pi/3$, we see that the sequence (t_n) satisfies condition

(1)(ii). Using (16), we conclude that there exist $C_3, C_4 \in \mathbb{R}$, $C_4 > C_3 > 0$, such that

$$\operatorname*{osc}_{t\in[t_{2k+1},t_{2k+2}]} r(t) = r(\hat{t}_{2k+2}) - r(t_{2k+1}) = \int_{t_{2k+1}}^{\hat{t}_{2k+2}} r'(t)\,dt$$

$$\leq \frac{1}{3} \sup_{t\in[t_{2k+1},\hat{t}_{2k+2}]} r'(t) \leq C_3 t_{2k+1}^{-\alpha-2} \leq C_4 \hat{t}_{2k+2}^{-\alpha-2},$$

for every $k \in \mathbb{N}_0$, which means that the sequence (t_n) satisfies condition (1)(iii). Finally, we conclude that the sequence (t_n) satisfies waviness conditions (1), so that $r(t)$ is a wavy function and Γ' is a wavy spiral near the origin.

Step 9. (The final conclusion.) From the previous steps, we see directly that all assumptions of Theorem 3 are fulfilled. We take $\varepsilon' = (\pi/3+1)/2 < \varepsilon$ and $\theta = \min\{\varepsilon', \pi\} = (\pi/3+1)/2$. Using Theorem 3, we obtain that $\dim_B \Gamma' = 2/(1+\alpha)$.

(ii) To prove that Γ is a wavy spiral near the origin, notice that *Steps 1–8* also hold for $\alpha > 1$. To prove the rectifiability for $\alpha > 1$, from (14), (12) and (16) we have that there exist positive constants C_5, M_1 and C_6 such that for every $t \in [t_0, \infty)$ it holds $r(t) \leq C_5 t^{-\alpha}$, $\varphi'(t) \leq M_1$, $|r'(t)| \leq C_6 t^{-\alpha-1}$. Therefore

$$l(\Gamma) = l(\Gamma') = \int_{t_0}^{\infty} \sqrt{(r(t)\varphi'(t))^2 + (r'(t))^2}\,dt$$

$$\leq \int_{t_0}^{\infty} \sqrt{M_1^2 C_5^2 t^{-2\alpha} + C_6^2 t^{-2\alpha-2}}\,dt \leq M_2(t_0) \int_{t_0}^{\infty} |t|^{-\alpha}\,dt < \infty.$$

□

4. Chirps generated by spirals

Now we state a result which can be regarded as a sort of a converse of Theorem 4, where we obtain the box dimension of a chirp from the corresponding spiral. We begin with a theorem concerning the box dimension of the graph of a generalized (α,β)-chirp.

Theorem 5 (The box dimension and Minkowski content of the graph of a generalized (α,β)-chirp)**.** *Let $y(x) = p(x)S(q(x))$, where $x \in I = (0, c]$ and $c > 0$. Let the functions $p(x)$, $q(x)$ and $S(t)$ satisfy the following assumptions:*

$$p \in C(\bar{I}) \cap C^1(I),\ q \in C^1(I),\ S \in C^1(\mathbb{R}). \tag{21}$$

The function $S(t)$ is assumed to be a $2T$-periodic real function defined on $\mathbb{R}$ such that

$$\begin{cases} S(a) = S(a+T) = 0 \text{ for some } a \in \mathbb{R}, \\ S(t) \neq 0 \text{ for all } t \in (a, a+T) \cup (a+T, a+2T), \end{cases} \tag{22}$$

where T is a positive real number and $S(t)$ alternately changes a sign on intervals $(a+(k-1)T, a+kT)$, for $k \in \mathbb{N}$. Without loss of generality, we take $a = 0$. Let us *suppose that* $0 < \alpha \le \beta$ *and:*

$$p(x) \simeq_1 x^{\alpha} \quad as \quad x \to 0, \qquad q(x) \simeq_1 x^{-\beta} \quad as \quad x \to 0. \tag{23}$$

Then, $y(x)$ is d-dimensional fractal oscillatory near the origin, where $d = 2 - (\alpha+1)/(\beta+1)$. Moreover, $\dim_B(G(y)) = d$ and $G(y)$ is Minkowski nondegenerate.

Theorem 5 is an improved version of [6, Theorems 5 and 6]. Now we do not need any assumptions on the curvature function of $y(x) = p(x)S(q(x))$, as it was needed in [6]. Before proving Theorem 5, we shall cite a new criterion for fractal oscillations of a bounded continuous function and after that we continue with two propositions dealing with the properties of functions p, q and S.

Theorem 6 (Theorem 2.1. from [13]). *Let $y \in C^1((0,T])$ be a bounded function on $(0,T]$. Let $s \in [1,2)$ be a real number and let (a_n) be a decreasing sequence of consecutive zeros of $y(x)$ in $(0,T]$ such that $a_n \to 0$ when $n \to \infty$ and let there exist constants c_1, c_2, ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$ we have:*

$$c_1 \varepsilon^{2-s} \le \sum_{n \ge k(\varepsilon)} \max_{x \in [a_{n+1}, a_n]} |y(x)|(a_n - a_{n+1}), \tag{24}$$

$$a_{k(\varepsilon)} \sup_{x \in (0, a_{k(\varepsilon)}]} |y(x)| + \varepsilon \int_{a_{k(\varepsilon)}}^{a_1} |y'(x)| dx \le c_2 \varepsilon^{2-s}, \tag{25}$$

where $k(\varepsilon)$ is an index function on $(0, \varepsilon_0]$ such that

$$|a_n - a_{n+1}| \le \varepsilon \quad for\ all \quad n \ge k(\varepsilon) \quad and \quad \varepsilon \in (0, \varepsilon_0).$$

Then $y(x)$ is fractal oscillatory near $x = 0$ with $\dim_B G(y) = s$.

We remark that the claim of Theorem 6 is true if we substitute a_1, appearing in (25), by a_{k_0}, where k_0 is a fixed positive integer.

Proposition 1. *Assume that the functions $p(x)$ and $q(x)$ satisfy conditions* (21), (23)*. Then there exist $\delta_0 > 0$ and positive constants C_1 and C_2 such that:*

$$C_1 x^{\alpha} \le p(x) \le C_2 x^{\alpha}, \qquad C_1 x^{\alpha-1} \le p'(x) \le C_2 x^{\alpha-1}, \tag{26}$$

$$C_1 x^{-\beta} \le q(x) \le C_2 x^{-\beta}, \qquad C_1 x^{-\beta-1} \le -q'(x) \le C_2 x^{-\beta-1}, \tag{27}$$

for all $x \in (0, \delta_0]$. Furthermore, there exists the inverse function q^{-1} of the function q defined on $[m_0, \infty)$, where $m_0 = q(\delta_0)$, and it holds:

$$q^{-1}(t) \simeq_1 t^{-1/\beta} \quad as \quad t \to \infty, \tag{28}$$

$$C_1 t^{-\frac{1}{\beta}-1}(t-s) \le q^{-1}(s) - q^{-1}(t) \le C_2 s^{-\frac{1}{\beta}-1}(t-s), \quad m_0 \le s < t. \tag{29}$$

Proof. Inequalities (26) and (27) follow directly from (23) by the definition. The function $q|_{(0,\delta_0]}$ is a positive and decreasing function, and its inverse function is defined on $[m_0, \infty)$. Relation (28) follows from (27), applying the well known formula for a derivative of the inverse function. Then, exploiting the mean value theorem and (28), we get (29). □

Proposition 2. *For any function $S(t)$ satisfying (22), and for any function $q(x)$ with properties (21) and (23), we have:*

(i) $S(kT) = 0,\ k \in \mathbb{N}$.

(ii) *Let $a_k = q^{-1}(kT)$ and $s_k = q^{-1}(t_0 + kT),\ k \in \mathbb{N}$, where $t_0 \in (0,T)$ is arbitrary. Then there exist $k_0 \in \mathbb{N}$ and $c_0 > 0$ such that $a_k \in (0,\delta_0]$, $y(a_k) = 0$, $s_k \in (a_{k+1}, a_k)$ for all $k \geq k_0$, $a_k \searrow 0$ as $k \to \infty$, $a_k \simeq k^{-1/\beta}$ as $k \to \infty$, and*

$$\max_{x\in[a_{k+1},a_k]} |y(x)| \geq c_0(k+1)^{-\alpha/\beta} \quad \text{for all } k \geq k_0,\ c_0 > 0. \tag{30}$$

(iii) *There exist $\varepsilon_0 > 0$ and a function $k : (0,\varepsilon_0) \to \mathbb{N}$ such that*

$$\frac{1}{T}\left(\frac{\varepsilon}{TC_2}\right)^{-\frac{\beta}{\beta+1}} \leq k(\varepsilon) \leq \frac{2}{T}\left(\frac{\varepsilon}{TC_2}\right)^{-\frac{\beta}{\beta+1}}. \tag{31}$$

In particular, $C_1T((k+1)T)^{-\frac{1}{\beta}-1} \leq a_k - a_{k+1} \leq \varepsilon$, for all $k \geq k(\varepsilon)$ and $\varepsilon \in (0,\varepsilon_0)$.

Proof. The claim in (i) is evident. To prove (ii), it suffices to take $k_0 \in \mathbb{N}$ such that $k_0T \geq m_0$. We shall prove inequality (30) only, because the other properties are easy consequences of Proposition 1. From (23) we obtain that $p(x)$ is a positive and increasing function near $x = 0$, and we have

$$\max_{x\in[a_{k+1},a_k]} |y(x)| \geq p(s_k)|S(q(s_k))| \geq cp(a_{k+1}) \geq c_1(a_{k+1})^{\alpha} \geq c_0(k+1)^{-\frac{\alpha}{\beta}},$$

for all $k \geq k_0$, where $c = \min\{|S(t_0)|, |S(t_0+T)|\}$, $c_1 = cC_1$ and $c_0 = cC_1^2$ are positive constants. Now we prove (iii). Let $\varepsilon > 0$ and let $k(\varepsilon) \in \mathbb{N}$ be such that

$$k(\varepsilon) \geq \frac{1}{T}\left(\frac{\varepsilon}{TC_2}\right)^{-\frac{\beta}{\beta+1}} = c\varepsilon^{-\frac{\beta}{\beta+1}}, \quad c = T^{-1}(TC_2)^{\frac{\beta}{\beta+1}}.$$

Let ε_0' be such that for all $0 < \varepsilon \leq \varepsilon_0'$ it holds $k(\varepsilon)T \geq m_0 = q(\delta_0)$. Further, for all $\varepsilon < c^{\frac{\beta+1}{\beta}}$ we have $2c\varepsilon^{-\frac{\beta}{\beta+1}} - c\varepsilon^{-\frac{\beta}{\beta+1}} > 1$. So, there exists $k(\varepsilon) \in \mathbb{N}$ such that

$$1 < c\varepsilon^{-\frac{\beta}{\beta+1}} \leq k(\varepsilon) \leq 2c\varepsilon^{-\frac{\beta}{\beta+1}}, \quad \text{for all} \quad \varepsilon < c^{\frac{\beta+1}{\beta}}.$$

Let us take $\varepsilon_0 = \min\{\varepsilon_0', c^{\frac{\beta+1}{\beta}}\}$. Then, we can find $k(\varepsilon) \in \mathbb{N}$ such that

$$c\varepsilon^{-\frac{\beta}{\beta+1}} \leq k(\varepsilon) \leq 2c\varepsilon^{-\frac{\beta}{\beta+1}}, \quad k(\varepsilon)T \geq m_0 \quad \text{for all} \quad \varepsilon \in (0,\varepsilon_0).$$

Using (29), then for all $k \geq k(\varepsilon)$ and $\varepsilon \in (0,\varepsilon_0)$ it holds

$$C_1T((k+1)T)^{-\frac{1}{\beta}-1} \leq a_k - a_{k+1} \leq \varepsilon.$$

$\square$

Proof of Theorem 5. First we check inequality (24). By Proposition 2 we have

$$\sum_{k\geq k(\varepsilon)} \max_{x\in[a_{k+1},a_k]} |y(x)|(a_k-a_{k+1}) \geq c \sum_{k=k(\varepsilon)+1}^{\infty} (k+1)^{-\frac{\alpha+\beta+1}{\beta}} = c \sum_{k=k(\varepsilon)}^{\infty} k^{-\frac{\alpha+\beta+1}{\beta}} = ca,$$

where the series $a = \sum_{k=k(\varepsilon)}^{\infty} k^{-\frac{\alpha+\beta+1}{\beta}}$ is convergent, because of $\frac{\alpha+\beta+1}{\beta} > 1$. Then, using the inequality $(\frac{1}{k(\varepsilon)})^{\frac{\alpha+\beta+1}{\beta}-1} < 1$, the integral test for convergence and (31), we obtain that

$$\sum_{k\geq k(\varepsilon)} \max_{x\in[a_{k+1},a_k]} |y(x)|(a_k-a_{k+1}) \geq ca \geq c_1(\frac{1}{k(\varepsilon)})^{\frac{\alpha+\beta+1}{\beta}-1} \geq c_1\varepsilon^{\frac{\alpha+1}{\beta+1}} = c_1\varepsilon^{2-(2-\frac{\alpha+1}{\beta+1})},$$

for all $\varepsilon \in (0,\varepsilon_0)$. By [13, Lemma 2.1.], this implies that $0 < \mathcal{M}_*^d(G(y))$ and $\underline{\dim}_B G(y) \geq d$, where $G(y)$ is the graph of the function y and $d = 2-(\alpha+1)/(\beta+1)$. Now we check inequality (25). From (23) it follows that

$$|y'(x)| = |p'(x)S(q(x)) + p(x)q'(x)S'(q(x))| \leq cx^{\alpha-\beta-1},$$

which holds near $x = 0$, where $c = \max\{\max_{x\in[0,2T]}|S(t)|, \max_{x\in[0,2T]}|S'(t)|\}$. By Proposition 2 we have that

$$a_{k(\varepsilon)} \sup_{x\in(0,a_{k(\varepsilon)}]} |y(x)| + \varepsilon\int_{a_{k(\varepsilon)}}^{a_{k_0}} |y'(x)|dx \leq c\varepsilon^{\frac{\alpha+1}{\beta+1}} + \varepsilon[a_{k_0}^{\alpha-\beta} + a_{k(\varepsilon)}^{\alpha-\beta}] \leq c_2\varepsilon^{\frac{\alpha+1}{\beta+1}},$$

for all $\varepsilon \in (0,\varepsilon_0)$. By [13, Lemma 2.2.] it follows that $\mathcal{M}^{*d}(G(y)) < \infty$ and $\overline{\dim}_B G(y) \leq d = 2-(\alpha+1)/(\beta+1)$. Finally, combining the obtained results, we conclude that the graph $G(y)$ is Minkowski nondegenerate, and $\dim_B G(y) = 2-(\alpha+1)/(\beta+1) = d$. □

Now we can state a spiral-chirp comparison result.

Theorem 7 (The spiral-chirp comparison)**.** *Let $\alpha \in (0,1)$. Assume that $x : [t_0,\infty) \to \mathbb{R}$, where $t_0 > 0$, is a function of class C^2, such that the planar curve $\Gamma = \{(x(t),\dot{x}(t)) : t \in [t_0,\infty)\}$ is a spiral $r = f(\varphi)$, $\varphi \in (\varphi_0,\infty)$, $\varphi_0 > 0$, in polar coordinates, near the origin, where*

$$f(\varphi) \simeq_1 \varphi^{-\alpha}, \text{ as } \varphi \to \infty.$$

Let $\varphi = \varphi(t)$ be a function of class C^1 defined by $\tan\varphi(t) = \frac{\dot{x}(t)}{x(t)}$, such that $\dot{\varphi}(t) \simeq 1$, as $t \to \infty$. Define $X(\tau) = x(1/\tau)$. Then, $X = X(\tau)$ is an $(\alpha,1)$-chirp-like function, and

$$\dim_{osc}(x) := \dim_B G(X) = (3-\alpha)/2,$$

where $G(X)$ is the graph of the function X. Furthermore, $G(X)$ is Minkowski nondegenerate.

Proof. Let us write the function $X(\tau)$ in the form $X(\tau) = p(\tau)\cos q(\tau)$, with $\tau \in (0, \frac{1}{t_0}]$, where $p(\tau) = f(\varphi(\frac{1}{\tau}))$, $q(\tau) = \varphi(\frac{1}{\tau})$.

The function $p(\tau)$ is increasing near $\tau = 0$ since $\frac{1}{\tau}$ is decreasing, $\varphi(t)$ is increasing and $f(\varphi)$ is decreasing near $\varphi = \infty$. Furthermore, $p \in C([0, 1/t_0])$ since $p(0) = \lim_{\tau\to 0} f(\varphi(1/\tau)) = 0$, by noting that $\dot\varphi \simeq 1$ implies $\varphi(t) \to \infty$ as $t \to \infty$. Now, the claim follows from Theorem 5. We only have to check that its assumptions are satisfied with $S(q) = \cos q$ and $\beta = 1$. The functions φ, p and q have the following properties: $\varphi(t) \simeq t$ as $t \to \infty$, that is, $\varphi(\frac{1}{\tau}) \simeq \frac{1}{\tau}$ as $\tau \to 0$, and $p(\tau) \simeq_1 \tau^\alpha$ as $\tau \to 0$, $q(\tau) \simeq_1 \frac{1}{\tau}$ as $\tau \to 0$, $q^{-1}(t) \simeq \frac{1}{t}$ as $t \to \infty$. The function q is decreasing near the origin, thus q^{-1} exists for t sufficiently large. We see that all the conditions of Theorem 5 are fulfilled. □

Remark 2. *Theorem 7 is a new version of [15, Theorem 4]. If we compare Theorems 4 and 7 in terms of their conditions, then we see that Theorem 7 requires derivatives of lower order than Theorem 4. Phase-plane analysis already provides the information about the first derivative.*

The following result shows that rectifiable spirals generate rectifiable chirp-like functions.

Theorem 8 (Rectifiability of a chirp generated by a rectifiable spiral). *Let $\alpha > 1$. Assume that $x : [t_0, \infty) \to \mathbb{R}$, with $t_0 > 0$, is a function of class C^2 such that the planar curve $\Gamma = \{(x(t), \dot x(t)) : t \in [t_0, \infty)\}$ is a rectifiable spiral $r = f(\varphi)$, with $\varphi \in (\varphi_0, \infty)$, $\varphi_0 > 0$, in polar coordinates, near the origin, where*

$$f(\varphi) \simeq_1 \varphi^{-\alpha}, \text{ as } \varphi \to \infty, \quad |f''(\varphi)| \le C\varphi^{-\alpha-2} \quad \text{and} \quad \dot\varphi(t) \simeq 1 \text{ as } t \to \infty.$$

Let $\varphi = \varphi(t)$ be a function of class C^1 defined by $\tan\varphi(t) = \frac{\dot x(t)}{x(t)}$, such that $\dot\varphi(t) \simeq 1$, as $t \to \infty$. Define $X(\tau) = x(1/\tau)$.

Then $X = X(\tau)$ is an $(\alpha, 1)$-chirp-like rectifiable function near the origin.

In order to prove the theorem we shall use the following two lemmas.

Lemma 4. *Let $F, G \in C^1(I)$, where I is an open interval in $\mathbb{R}$, and assume that $\inf F' > \sup G'$. Then, the equation $F(z) = G(z)$ has at most one solution.*

Proof. Suppose that there are two different solutions z_1 and z_2. Then applying the mean-value theorem to $F(z_1) - F(z_2) = G(z_1) - G(z_2)$, we obtain that there exist $\tilde z_1$ and $\tilde z_2$ such that $F'(\tilde z_1) = G'(\tilde z_2)$. Therefore, $\inf F' \le \sup G'$. This contradicts the condition $\inf F' > \sup G'$. □

Lemma 5. *Let $F \in C^1(0, \infty)$ be such that $F(z) \sim az$ as $z \to \infty$ for some $a < 0$. Assume that $\inf F' > -\infty$. Then, there exists a nonnegative integer k_0 such that for each $k \ge k_0$ the equation $\cot z = F(z)$ possesses the unique solution in $J_k = (k\pi, (k+1)\pi)$.*

Proof. Since $F(z)$ is continuous and $F(z) \sim az$ as $z \to \infty$, and $\cot z$ restricted to J_k is a continuous function onto $\mathbb{R}$, it follows that the equation $\cot z = F(z)$ possesses at least one solution z_k on each interval J_k. We have to show that the solution is unique on each J_k for all k sufficiently large.

Since $m = \inf F' > -\infty$, there exists $s_0 \in (\pi/2, \pi)$ sufficiently close to π such that $\cot'(s_0) = -(\sin s_0)^{-2} < m$. The condition $F(z) \sim az$ implies that, given any fixed $b \in (a, 0)$, there exists $M = M(b) > 0$ such that $F(z) < bz$ for all $z \geq M$. Let us fix any such b.

Let k_0 be a nonnegative integer such that $b(k_0\pi) < \cot s_0$. It suffices to take $k_0 > (b\pi)^{-1} \cot s_0$. Taking k_0 even larger, we can achieve that $k_0\pi \geq M$. Hence, for $z \geq k_0\pi$ we have $F(z) < bz$. In particular,

$$F(z) < bz \leq b(k_0\pi) < \cot s_0.$$

Since for $z \geq k_0\pi$ we have $F(z) < \cot s_0$, while $\cot z \geq \cot s_0$ for each $z \in J_k \setminus I_k$, where $I_k = (k\pi + s_0, (k+1)\pi)$, then all the solutions of equation $F(z) = \cot z$ for $z \geq k_0\pi$ are contained in $\cup_{k \geq k_0} I_k$.

Let us define $G(z) = \cot z$, and consider the equation $F(z) = G(z)$ on I_k for any $k \geq k_0$. We have

$$\sup_{I_k} G' = \cot'(k_0\pi + s_0) = -(\sin s_0)^{-2} < \inf_{(0,\infty)} F' \leq \inf_{I_k} F'.$$

The unique solvability of $F(z) = G(z)$ on I_k then follows from Lemma 4. The equation is uniquely solvable on J_k as well, since there are no solutions in $J_k \setminus I_k$. $\square$

Remark 3. *The condition $F(z) \sim az$ as $z \to \infty$ in Lemma 5 can be weakened. It suffices to assume that $F(z) < bz$ for some $b < 0$ and for all z sufficiently large.*

Remark 4. *The condition $\inf F' > -\infty$ in Lemma 5 cannot be dropped. To see this, we construct a function $y = F(z)$ by means of a sequence of lines $y = b_n z$, where $a < b_n < 0$ and $b_n \to a$ as $n \to \infty$. We first construct a continuous function F_0 such that on $J'_k = (k\pi, (k+1)\pi]$,*

$$F_0(z) = \begin{cases} b_k z, & \text{for } z \in (k\pi, z_k], \\ \cot z, & \text{for } z \in (z_k, v_k], \\ b_{k+1} z, & \text{for } z \in (v_k, (k+1)\pi], \end{cases}$$

where z_k and v_k are the respective solutions of the equations $\cot z = b_k z$ and $\cot b_{k+1} v = b_{k+1} v$ in J_k. The function F_0 is of class C^1 everywhere in $(0, \infty)$ except at the points z_k and v_k. We can perform its smoothing in sufficiently small neighborhoods of these points, in order to get a function $F \in C^1(0, \infty)$. It is clear that $F(z) \sim az$ as $z \to \infty$ and $\inf F' = -\infty$. But $F(z) = \cot z$ possesses infinitely many solutions on each interval I_k.

Remark 5. *Assume that $F(z) = f(z)/f'(z)$, where $f \in C^2(0, \infty)$. (a) The condition $\inf F' > -\infty$ is equivalent to $f(z) f''(z) \leq C[f'(z)]^2$, where C is a positive constant. (b) The condition $F(z) < bz$ for z sufficiently large, where b is a negative constant (see Remark 3), is satisfied if for all z sufficiently large we have $f(z) \geq az^{-\alpha}$ and $f'(z) \geq a_1 z^{-\alpha-1}$, where $a > 0$ and $a_1 < 0$ are constants. It suffices to take $b \in (a/a_1, 0)$.*

A variation of Lemma 5 is the following lemma.

Lemma 6. *Let $F \in C^1(0,\infty)$ be such that $F(z) \sim az$ as $z \to \infty$ for some $a > 0$. Assume that $\sup F' < \infty$. Then there exists a nonnegative integer k_0 such that for each $k \geq k_0$ the equation $\tan z = F(z)$ possesses the unique solution in $J_k = ((k-1/2)\pi, (k+1/2)\pi)$.*

Remark 6. *The condition $F(z) \sim az$ as $z \to \infty$ for $a > 0$ in Lemma 6 can be weakened by assuming that $F(z) > az$ for some $a > 0$ and for all z sufficiently large. If $F(z)$ has the form $F(z) = \frac{f(z)}{f'(z)}$, where $f \in C^2(0,\infty)$, the condition $\sup F' < \infty$ is equivalent to $f(z)f''(z) \geq C[f'(z)]^2$, where C is a positive constant. Also, in that case, the condition $F(z) > az$ for z sufficiently large is satisfied if for all z sufficiently large we have $f(z) \geq a_1 z^{-\alpha}$ and $f'(z) \leq a_2 z^{-\alpha-1}$, where a_1 and a_2 are positive constants. It suffices to take $a \in (0, \frac{a_1}{a_2})$.*

Proof of Theorem 8. We can write the function $X(\tau)$ in the form $X(\tau) = p(\tau)\cos q(\tau)$, where $p(\tau) = f(\varphi(1/\tau)) \simeq \tau^\alpha$, $p'(\tau) \simeq \tau^{\alpha-1}$, $q(\tau) = \varphi(1/\tau) \simeq \tau^{-1}$, $q'(\tau) \simeq -\tau^{-2}$ as $\tau \to 0$. It follows that X is an $(\alpha, 1)$-chirp-like function. Using the assumptions of the theorem, for the function $F(t) := \frac{pq'}{p'}(q^{-1}(t)) = \frac{f(t)}{f'(t)}$ we have that $F(t) \simeq -t$ as $t \to \infty$, and $\frac{f(t)f''(t)}{[f'(t)]^2} < C$, for t sufficiently large, $C > 0$. Then there exists $k_0 \in \mathbb{N}$ such that the equation $\cot q(t) = F(q(t)) = \frac{p(\tau)q'(\tau)}{p'(\tau)}$ has the unique solution $s_k \in (a_{k+1}, a_k)$ where $a_{k+1} = q^{-1}((2k+1)\frac{\pi}{2}$ and $a_k = q^{-1}((2k-1)\frac{\pi}{2})$ for all $k \geq k_0$; see Lemma 5 and Remark 3. These solutions are just the points of local extrema of $X(\tau)$ on (a_{k+1}, a_k), $k \geq k_0$. The sequence $(a_k)_{k\geq 1}$ of zero-points of X on $(0, 1/t_0]$ is decreasing. Hence the sequence (s_k) of consecutive points of local extrema of X is also decreasing. We have that $a_k = q^{-1}((2k-1)\frac{\pi}{2}) \simeq k^{-1}$ as $k \to \infty$. So the same is true also for s_k, i.e., $s_k \simeq k^{-1}$ as $k \to \infty$, and we also have that $|X(s_k)| \leq p(s_k) \leq Cs_k^\alpha \leq C_1 k^{-\alpha}$. This implies that

$$\sum_{k=k_0}^{\infty} |X(s_k)| \leq C_1 \sum_{k=k_0}^{\infty} k^{-\alpha} < \infty \tag{32}$$

for $\alpha > 1$. The length of the graph $G(X)$ is defined by

$$\text{length}(G(X)) := \sup \sum_{i=1}^{m} \|(t_i, X(t_i)) - (t_{i-1}, X(t_{i-1}))\|_2,$$

where the supremum is taken over all partitions $0 = t_0 < t_1 < \ldots < t_m = 1/t_0$ of the interval $[0, 1/t_0]$ and where $\|.\|_2$ denotes the Euclidean norm in $\mathbb{R}^2$. Using [12, Lemma 3.1.], it follows that $\text{length}(G(X)) \leq 2\sum_k |X(s_k)| + 1/t_0$. Then X is rectifiable due to (32). □

5. Concluding remarks

1. Chirps and spirals. In Section 3 of this article, we considered the spirals generated by chirps, while the chirps generated by spirals are studied in Section 4. Using the box dimension we establish a connection between oscillatority of the graph of a function and oscillatority of the corresponding curve in the phase plane. The main results are contained in Theorems 4 and 7. Theorem 4 could be applied to solutions of the Bessel equation of order ν, as well as to some of its generalizations; see [8]. Applications of Theorem 7 include the study of a weak focus of planar autonomous systems, that is, the case when the singularity has pure imaginary eigenvalues. This type of singularities generates spiral trajectories of power type, i.e., $r = \varphi^{-\alpha}$, where $\alpha \in (0, 1)$; see [23].

2. Limit cycles born from foci. The relationship between chirps and spirals is important in the study of limit cycles. The standard qualitative approach to nonlinear differential equations includes the study of the corresponding systems. Through phase plane oscillatority we obtain information of the oscillatority of the graph of a solution. The number of the limit cycles that can be generated by a weak focus is directly related to the box dimension of any trajectory of the system; see [23, 25]. It has been proven for a weak focus that the nontrivial jump of the value of the box dimension of a spiral trajectory, from 1 to 4/3, corresponds to the classic Hopf bifurcation; see [23]. The degenerate Hopf bifurcation or Hopf-Takens bifurcation can reach an even larger box dimension of a trajectory, which is related to the multiplicity of the focus. The result was obtained using the Takens normal form (see [19]) and the Poincaré map of the weak focus.

We find it interesting to examine the connection between the phase dimension of Bessel functions, which is equal to 4/3, and the maximal number of limit cycles that can be generated by a small perturbation of the Bessel equation. By analogy with the Hopf bifurcation, we expect this number to be equal to 1.

The Poincaré map corresponding to a weak focus is known to be analytic, while the Poincaré map near a general nilpotent or degenerate focus is not analytic, and the logarithmic terms show up in the asymptotic expansion; see Roussarie [18]. In that case, the Poincaré map has different asymptotics, showing the characteristic directions by the method of blow-up; see Han and Romanovski [4]. The nilpotent focus has two different asymptotics, so that we can relate that focus with two chirps with different asymptotics. The degenerate focus appears in a generalized Bessel equation for $\nu \neq 0$; see [8].

3. Oscillatory integrals. Nonrectifiable spirals can be generated using oscillatory integrals, viewed as complex functions of the real variable, like in the case of the Fresnel integral and the clothoid. The corresponding two chirps are graphs of the real and imaginary parts of the oscillatory integral. The box dimension of the image of an oscillatory integral and the box dimension of the corresponding chirps are related to the asymptotics of the integral, which is essentially connected to the type of the singular point of the phase function of the integral; see Arnold [1]. All of these notions are strongly related to the Newton diagrams, the resolution of singularities, the notion of the multiplicity of a singularity and the classification of singularities through the normal forms. Also, the study of bifurcations of the parametric families and the caustic surfaces could be a very interesting direction for further study by this approach.

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References

[1] V. I. Arnol′d, S. M. Guseĭn-Zade, A. N. Varchenko, *Singularities of differentiable maps, Vol. II*, Monographs in Mathematics, Vol. 83, Birkhäuser Boston Inc., Boston, MA, 1988.

[2] Y. Dupain, M. Mendès France, and C. Tricot, *Dimensions des spirales*, Bull. Soc. Math. France **111**(1983), 193–201.

[3] K. Falconer, *Fractal geometry, Mathematical foundations and applications*, John Wiley & Sons Ltd., Chichester, 1990.

[4] M. Han, V. G. Romanovski, *Limit cycle bifurcations from a nilpotent focus or center of planar systems*, Abstr. Appl. Anal. **2012**(2012), Article ID 720830, 28 pages.

[5] L. Horvat Dmitrović, *Box dimension and bifurcations of one-dimensional discrete dynamical systems*, Discrete Contin. Dyn. Syst. **32**(2012), 1287–1307.

[6] L. Korkut, M. Resman, *Oscillations of chirp-like functions*, Georgian Math. J. **19**(2012), 705–720.

[7] L. Korkut, D. Vlah, D. Žubrinić, V. Županović, *Generalized Fresnel integrals and fractal properties of related spirals*, Appl. Math. Comput. **206**(2008), 236–244.

[8] L. Korkut, D. Vlah, V. Županović, *Fractal properties of Bessel functions*, Appl. Math. Comput. **283**(2016), 55–69.

[9] P. Mardešić, M. Resman, V. Županović, *Multiplicity of fixed points and growth of ε-neighborhoods of orbits*, J. Differential Equations **253**(2012), 2493–2514.

[10] J. P. Milišić, D. Žubrinić, V. Županović, *Fractal analysis of Hopf bifurcation for a class of completely integrable nonlinear Schrödinger Cauchy problems*, Electron. J. Qual. Theory Differ. Equ. **2010**(2010), No. 60, 32 pages.

[11] M. Pašić, *Fractal oscillations for a class of second order linear differential equations of Euler type* J. Math. Anal. Appl. **341**(2008), 211–223.

[12] M. Pašić, A. Raguž, *Rectifiable oscillations and singular behaviour of solutions of second-order linear differential equations*, Int. J. Math. Anal. **2**(2008), 477–490.

[13] M. Pašić, S. Tanaka, *Fractal oscillations of self-adjoint and damped linear differential equations of second-order*, Appl. Math. Comput. **218**(2011), 2281–2293.

[14] M. Pašić, J. S. W. Wong, *Rectifiable oscillations in second-order half-linear differential equations*, Ann. Mat. Pura Appl. **188**(2009), 517–541.

[15] M. Pašić, D. Žubrinić, V. Županović, *Oscillatory and phase dimensions of solutions of some second-order differential equations*, Bull. Sci. Math. **133**(2009), 859–874.

[16] G. Radunović, D. Žubrinić, V. Županović, *Fractal analysis of Hopf bifurcation at infinity*, International Journal of Bifurcation and Chaos **22**(2012):1230043, 15 pages, DOI: 10.1142/S0218127412300431.

[17] M. Resman, *Epsilon-neighborhoods of orbits and formal classification of parabolic diffeomorphisms*, Discrete Contin. Dyn. Syst. **33**(2013), 3767–3790.

[18] R. Roussarie, *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, Progress in Mathematics, Vol. 164, Birkhäuser Verlag, Basel, 1998.

[19] F. Takens, *Unfoldings of certain singularities of vectorfields: generalized Hopf bifurcations*, J. Differential Equations **14**(1973), 476–493.

[20] C. Tricot, *Curves and fractal dimension*, Springer-Verlag, New York, 1995. (With a foreword by Michel Mendès France, Translated from the 1993 French original.)

[21] J. S. W. Wong, *On rectifiable oscillation of Euler type second order linear differential equations*, Electron. J. Qual. Theory Differ. Equ. **2007**(2007), No. 20, 12 pages.
[22] H. Wu, W. Li, *Isochronous properties in fractal analysis of some planar vector fields*, Bull. Sci. Math. **134**(2010), 857–873.
[23] D. Žubrinić, V. Županović, *Fractal analysis of spiral trajectories of some planar vector fields*, Bull. Sci. Math. **129**(2005), 457–485.
[24] D. Žubrinić, V. Županović, *Fractal analysis of spiral trajectories of some vector fields in* $\mathbb{R}^3$, C. R. Math. Acad. Sci. Paris **342**(2006), 959–963.
[25] D. Žubrinić, V. Županović, *Poincaré map in fractal analysis of spiral trajectories of planar vector fields*, Bull. Belg. Math. Soc. Simon Stevin **15** (2008), 947–960.
[26] V. Županović, D. Žubrinić, *Fractal dimensions in dynamics*, in: *Encyclopedia of Mathematical Physics, Volume 2*, (J.-P. Françoise, G. Naber and S. Tsou, Eds.), Elsevier, Oxford, 2006, 394–402.