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Lyapunov theorems for exponential dichotomies in Hilbert spaces

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For a nonautonomous dynamics defined by a sequence of linear operators, we obtain a complete characterization of the notion of a uniform exponential dichotomy in terms of the existence of appropriate Lyapunov sequences. In sharp contrast to previous results, we consider the case of noninvertible dynamics, thus requiring only the invertibility of operators along the unstable direction. Furthermore, we deal with operators acting on an arbitrary Hilbert space. As a nontrivial application of our work, we study the persistence of uniform exponential behavior under small linear and nonlinear perturbations.

Keywords: Exponential dichotomy; Lyapunov sequence.

Mathematics Subject Classification 2010: 34D09, 37B25

1. Introduction

The notion of an exponential dichotomy, essentially introduced by Perron in [23], plays an important role in a large part of the theory of dynamical systems, such as, for example, in invariant manifold theory. We note that the theory of exponential dichotomies and its applications are very much developed. We refer to the books [10, 14, 15, 25] for details and further references.

Due to the importance of the notion of uniform exponential dichotomy, it is of considerable interest to have any type of the characterization of this notion and this is actually the main theme of our paper. More precisely, for a discrete nonautonomous dynamics obtained from a sequence of linear operators acting on a

Hilbert space, we give a complete characterization of the notion of uniform exponential dichotomy in terms of the existence of appropriate Lyapunov sequences. The main novelty of our work, besides the fact that our results are not restricted to a finite-dimensional case, is that we also don't require the invertibility of operators. We emphasize that cocycles obtained for example from discretization of the evolution family associated to a linear delay equation will in general be noninvertible and defined on an infinite-dimensional space. We then use this characterization to study the persistence of uniform exponential behavior under small linear and nonlinear perturbations. More precisely, we give a short proof of a well-known result that the notion of uniform exponential dichotomy persists under sufficiently small linear perturbations. We also obtain similar results for nonlinear perturbations.

The use of Lyapunov functions in the study of the stability of trajectories in the theories of differential equations and dynamical systems goes back to the seminal work of Lyapunov [20]. Early contributions to the theory are described in books by LaSalle and Lefschetz [17], Hahn [13] and Bhatia and Szegö [9]. The connection between exponential dichotomies and Lyapunov functions was first studied by Maizel [21]. His results were further developed by Coppel in [10, 11]. The first results in the discrete time setting are due to Papaschinopoulos [22]. More recently, Barreira and Valls wrote several important papers in which they have obtained characterization of nonuniform exponential dichotomies in terms of Lyapunov functions both for continuous and discrete time (see [6–8, 4]). We note that the notion of nonuniform dichotomy is weaker and requires less then the notion of uniform exponential dichotomy. However, we emphasize that all the above mentioned works consider only invertible and finite-dimensional dynamics and that to the best of our understanding the methods in those papers cannot be extended to the setting studied in the present paper. This forces us to use completely different arguments that rely heavily on the characterizations of hyperbolic operators on Hilbert spaces presented in [12] and the relationship between dichotomies and the so-called admissibility property (see Theorem 2.1).

Finally, we briefly mention the related results in the theory of smooth dynamical systems. Lewowicz characterized Anosov systems in terms of the existence of Lyapunov functions both for discrete and continuous time (see [18, 19]). Those results have been extended to nonuniformly hyperbolic systems by Katok and Burns [16]. We emphasize that the results of Katok and Burns were also inspired by the work of Wojtkowski in [26] who pointed out that to establish nonvanishing of (some) Lyapunov exponents it is often sufficient to have an invariant family of cones (see also [5]). For more recent results we refer to [1–3].

2. Preliminaries

In this section, we introduce some notation that will be used throughout this paper. Furthermore, we recall the notion of a uniform exponential dichotomy and its characterization in terms of the so-called admissibility property.

2.1. *Notation and terminology*

Let X be a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle$. The associated norm will be denoted by $\| \cdot \|$. For two bounded self-adjoint operators A and B on X, we write $A \le B$ if $\langle Ax, x\rangle \le \langle Bx, x\rangle$, for all $x \in X$. Set

$$l^2 = \left\{ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sum_{n=-\infty}^{\infty} \|x_n\|^2 < +\infty \right\}.$$

We note that l^2 is also a Hilbert space with respect to the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n\in\mathbb{Z}} \langle x_n, y_n\rangle, \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}}, \quad \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in l^2.$$

Finally, we recall that a bounded operator acting on a Banach space is *hyperbolic* if its spectrum is disjoint with unit circle $S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

2.2. *Exponential dichotomy*

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear operators on X. The associated *cocycle* is defined by

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *uniform exponential dichotomy* if:

(1) There exist projections $P_m : X \to X$ for each $m \in \mathbb{Z}$ satisfying

$$A_m P_m = P_{m+1} A_m \quad \text{for } m \in \mathbb{Z} \tag{2.1}$$

such that each map $A_m|\mathrm{Ker}\, P_m : \mathrm{Ker}\, P_m \to \mathrm{Ker}\, P_{m+1}$ is invertible;

(2) There exist constants $\lambda, K > 0$ such that for each $x \in X$ and $m \cdot n \in \mathbb{Z}$ we have

$$\|\mathcal{A}(m,n)P_n\| \le Ke^{-\lambda(m-n)} \quad \text{for } m \ge n \tag{2.2}$$

and

$$\|\mathcal{A}(m,n)(I - P_n)\| \le Ke^{-\lambda(n-m)} \quad \text{for } m \le n, \tag{2.3}$$

where

$$\mathcal{A}(m,n) = (\mathcal{A}(n,m)|\mathrm{Ker}\, P_m)^{-1} : \mathrm{Ker}\, P_n \to \mathrm{Ker}\, P_m$$

for $m < n$.

We also recall the following classical result (see [15] for example) that characterizes exponential dichotomies in terms of the so-called admissibility property.

Theorem 2.1. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded operators on X. The following statements are equivalent*:

(1) *The sequence $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy*;

(2) *for each $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in l^2$ there exists a unique $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2$ such that*

$$x_{n+1} - A_n x_n = y_{n+1}, \quad \textit{for each } n \in \mathbb{Z}.$$

3. Main Results

In this section, we obtain a complete characterization of uniform exponential dichotomies in terms of the existence of a suitably chosen sequence $(S_m)_m$ of bounded, self-adjoint and invertible operators. We note that the sequence of maps $(H_m)_m$, where

$$H_m(x) = \langle S_m x, x \rangle, \quad x \in X,\ m \in \mathbb{Z}$$

is sometimes in the literature called a *Lyapunov sequence* (see [7]). We now state our first result.

Theorem 3.1. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ of bounded linear operators in X admits a uniform exponential dichotomy and that there exists $C > 0$ such that*

$$\|A_m\| \le C, \quad \text{for } m \in \mathbb{Z}. \tag{3.1}$$

Then, there exist a sequence $(S_m)_{m\in Z}$ of bounded, self-adjoint and invertible operators on X and $D, \delta > 0$ such that:

(1)

$$\|S_m\| \le D \quad \text{and} \quad \|S_m^{-1}\| \le D, \quad \text{for all } m \in \mathbb{Z}; \tag{3.2}$$

(2)

$$A_m^* S_{m+1} A_m - S_m \le -\delta I \quad \text{and} \quad A_m S_m^{-1} A_m^* - S_{m+1}^{-1} \le -\delta I. \tag{3.3}$$

Proof. We will divide proof into several parts. We begin by constructing operators S_m. This construction is essentially taken from [6, 22] but we include it for the sake of completeness. Since the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy, there exists projections P_m, $m \in \mathbb{Z}$ satisfying (2.1) and constants $\lambda, K > 0$ such that (2.2) and (2.3) hold. Choose an arbitrary $\rho \in (0, \lambda)$ and set

$$\begin{aligned} S_m = &\sum_{k\ge m} (\mathcal{A}(k,m)P_m)^* \mathcal{A}(k,m)P_m e^{2(\lambda-\rho)(k-m)} \\ &- \sum_{k<m} (\mathcal{A}(k,m)(I-P_m))^* \mathcal{A}(k,m)(I-P_m) e^{2(\lambda-\rho)(m-k)}. \end{aligned}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned} |\langle S_m x, x\rangle| &\le \sum_{k\ge m} \|\mathcal{A}(k,m)P_m x\|^2 e^{2(\lambda-\rho)(k-m)} \\ &\quad + \sum_{k<m} \|\mathcal{A}(k,m)(I-P_m)x\|^2 e^{2(\lambda-\rho)(m-k)} \\ &\le \sum_{k\ge m} K^2 e^{-2\lambda(k-m)} e^{2(\lambda-\rho)(k-m)} \|x\|^2 \\ &\quad + \sum_{k<m} K^2 e^{-2\lambda(m-k)} e^{2(\lambda-\rho)(m-k)} \|x\|^2 \end{aligned}$$

$$= K^2 \left(\sum_{k \geq m} e^{-2\rho(k-m)} + \sum_{k<m} e^{-2\rho(m-k)} \right) \|x\|^2$$

$$= D\|x\|^2$$

for each $m \in \mathbb{Z}$, where

$$D = K^2 \left(\sum_{k \geq m} e^{-2\rho(k-m)} + \sum_{k<m} e^{-2\rho(m-k)} \right) < +\infty.$$

Obviously, S_m is self-adjoint and thus

$$\|S_m\| = \sup_{\|x\|=1} |\langle S_m x, x\rangle| \leq D, \quad \text{for all } m \in \mathbb{Z}.$$

Hence, we have obtained the first inequality in (3.2).

Furthermore, by (2.1) we have that

$$\begin{aligned} A_m^* S_{m+1} A_m &= A_m^* \sum_{k \geq m+1} (\mathcal{A}(k, m+1) P_{m+1})^* \mathcal{A}(k, m+1) P_{m+1} e^{2(\lambda-\rho)(k-m-1)} A_m \\ &\quad - A_m^* \sum_{k<m+1} (\mathcal{A}(k, m+1)(I - P_{m+1}))^* \mathcal{A}(k, m+1) \\ &\quad \times (I - P_{m+1}) e^{2(\lambda-\rho)(m+1-k)} A_m \\ &= \sum_{k \geq m+1} (\mathcal{A}(k, m+1) P_{m+1} A_m)^* \mathcal{A}(k, m+1) P_{m+1} A_m e^{2(\lambda-\rho)(k-m-1)} \\ &\quad - \sum_{k<m+1} (\mathcal{A}(k, m+1)(I - P_{m+1}) A_m)^* \mathcal{A}(k, m+1) \\ &\quad \times (I - P_{m+1}) A_m e^{2(\lambda-\rho)(m+1-k)} \\ &= \sum_{k \geq m+1} (\mathcal{A}(k, m+1) A_m P_m)^* \mathcal{A}(k, m+1) A_m P_m e^{2(\lambda-\rho)(k-m-1)} \\ &\quad - \sum_{k<m+1} (\mathcal{A}(k, m+1) A_m (I - P_m))^* \mathcal{A}(k, m+1) A_m (I - P_m) \\ &\quad \times e^{2(\lambda-\rho)(m+1-k)} \\ &= \sum_{k \geq m+1} (\mathcal{A}(k, m) P_m)^* \mathcal{A}(k, m) P_m e^{2(\lambda-\rho)(k-m-1)} \\ &\quad - \sum_{k<m+1} (\mathcal{A}(k, m)(I - P_m))^* \mathcal{A}(k, m)(I - P_m) e^{2(\lambda-\rho)(m+1-k)} \\ &= e^{-2(\lambda-\rho)} \sum_{k \geq m+1} (\mathcal{A}(k, m) P_m)^* \mathcal{A}(k, m) P_m e^{2(\lambda-\rho)(k-m)} \\ &\quad - e^{2(\lambda-\rho)} \sum_{k<m+1} (\mathcal{A}(k, m)(I - P_m))^* \mathcal{A}(k, m)(I - P_m) e^{2(\lambda-\rho)(m-k)} \end{aligned}$$

$$
\begin{aligned}
&= e^{-2(\lambda-\rho)} \sum_{k\ge m} (\mathcal{A}(k,m)P_m)^* \mathcal{A}(k,m)P_m e^{2(\lambda-\rho)(k-m)} \\
&\quad - e^{-2(\lambda-\rho)} P_m^* P_m - e^{2(\lambda-\rho)} \sum_{k<m} (\mathcal{A}(k,m)(I-P_m))^* \mathcal{A}(k,m) \\
&\quad \times (I-P_m) e^{2(\lambda-\rho)(m-k)} - e^{2(\lambda-\rho)} (I-P_m)^*(I-P_m).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_m^* S_{m+1} A_m - S_m &= (e^{-2(\lambda-\rho)} - 1) \sum_{k\ge m} (\mathcal{A}(k,m)P_m)^* \mathcal{A}(k,m)P_m e^{2(\lambda-\rho)(k-m)} \\
&\quad + (1 - e^{2(\lambda-\rho)}) \sum_{k<m} (\mathcal{A}(k,m)(I-P_m))^* \mathcal{A}(k,m)(I-P_m) \\
&\quad \times e^{2(\lambda-\rho)(m-k)} - e^{-2(\lambda-\rho)} P_m^* P_m - e^{2(\lambda-\rho)} (I-P_m)^*(I-P_m).
\end{aligned}
$$

Since $e^{-2(\lambda-\rho)} - 1 < 0$ and $1 - e^{2(\lambda-\rho)} < 0$, we obtain that

$$
\begin{aligned}
A_m^* S_{m+1} A_m - S_m &\le -e^{-2(\lambda-\rho)} P_m^* P_m - e^{2(\lambda-\rho)} (I-P_m)^*(I-P_m) \\
&\le -e^{-2(\lambda-\rho)} (P_m^* P_m + (I-P_m)^*(I-P_m)).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&2\langle (P_m^* P_m + (I-P_m)^*(I-P_m))x, x\rangle \\
&\quad = 2\|P_m x\|^2 + 2\|(I-P_m)x\|^2 \\
&\quad \ge \|P_m x\|^2 + 2\|P_m x\|\|(I-P_m)x\| + \|(I-P_m)x\|^2 \\
&\quad = (\|P_m x\| + \|(I-P_m)x\|)^2 \\
&\quad \ge \|x\|^2
\end{aligned}
$$

for each $x \in X$ which implies that

$$
-e^{-2(\lambda-\rho)} (P_m^* P_m + (I-P_m)^*(I-P_m)) \le -\frac{1}{2} e^{-2(\lambda-\rho)} I.
$$

Consequently,

$$
A_m^* S_{m+1} A_m - S_m \le -\frac{1}{2} e^{-2(\lambda-\rho)} I, \quad \text{for all } m \in \mathbb{Z}
$$

and we conclude that the first statement in (3.3) holds with $\delta = \frac{1}{2} e^{-2(\lambda-\rho)} > 0$.

We now define the operator $T : l^2 \to l^2$ by

$$
(T\mathbf{x})_n = A_{n-1} x_{n-1}, \quad \text{for } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.
$$

It follows readily from (3.1) that T is well defined and bounded linear operator. In the following two auxiliary results we establish some properties of operator T. □

Lemma 3.1. *$T^* : l^2 \to l^2$ is given by*

$$(T^*\mathbf{x})_n = A_n^* x_{n+1}, \quad \textit{for all } n \in \mathbb{Z} \textit{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.$$

Proof of the lemma. We define an operator $S : l^2 \to l^2$ by

$$(S\mathbf{x})_n = A_n^* x_{n+1}, \quad \text{for all } n \in \mathbb{Z} \text{ and } \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.$$

Then, for every $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in l^2$ we have that

$$\begin{aligned}\langle S\mathbf{x}, \mathbf{y}\rangle &= \sum_{n\in\mathbb{Z}} \langle (S\mathbf{x})_n, y_n\rangle = \sum_{n\in\mathbb{Z}} \langle A_n^* x_{n+1}, y_n\rangle \\ &= \sum_{n\in\mathbb{Z}} \langle x_{n+1}, A_n y_n\rangle = \sum_{n\in\mathbb{Z}} \langle x_{n+1}, (T\mathbf{y})_{n+1}\rangle = \langle \mathbf{x}, T\mathbf{y}\rangle\end{aligned}$$

which yields that $S = T^*$. □

Lemma 3.2. *T is hyperbolic operator.*

Proof of the lemma. Take $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Since the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy, we have that the sequence $(\frac{1}{\lambda} A_m)_{m\in\mathbb{Z}}$ also admits a uniform exponential dichotomy. Thus, it follows from Theorem 2.1 that the operator

$$\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \mapsto \left(x_n - \frac{1}{\lambda} A_{n-1} x_{n-1}\right)_{n\in\mathbb{Z}}$$

is an invertible linear operator on l^2. Hence, the operator

$$\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \mapsto (\lambda x_n - A_{n-1} x_{n-1})_{n\in\mathbb{Z}}$$

is also an invertible on l^2 and therefore $\lambda \notin \sigma(T)$. We conclude that $\sigma(T) \cap S^1 = \emptyset$ and thus T is hyperbolic. □

We now define $W : l^2 \to l^2$ by

$$(W\mathbf{x})_n = S_n x_n, \quad n \in \mathbb{Z}, \ \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2.$$

Since $\|S_m\| \le D$ for all $m \in \mathbb{Z}$, we have that W is well-defined and bounded linear operator. Furthermore, since $A_m^* S_{m+1} A_m - S_m \le -\delta I$ for each $m \in \mathbb{Z}$, it follows from Lemma 3.1 that $T^*WT - W \le -\delta I$. Then, Theorem 7.1′ from [12] implies that W is invertible. The final ingredient of the proof is the following lemma.

Lemma 3.3. *S_m is invertible for all $m \in \mathbb{Z}$ and $m \mapsto \|S_m^{-1}\|$ is a bounded function.*

Proof of lemma. We divide the proof into several parts. We first prove that operators S_m are injective. Assume that $S_m v = 0$ form some $v \in X$. Define $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in l^2$ by $x_m = v$ and $x_n = 0$ for all $n \neq m$. Then, $W\mathbf{x} = 0$ and thus the invertibility of W implies that $\mathbf{x} = 0$. Hence, $v = 0$.

Now we establish surjectivity of operators S_m. Take $v \in X$ and define $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in l^2$ by $y_m = v$ and $y_n = 0$ for all $n \neq m$. Since W is invertible, there exists $\mathbf{x} \in l^2$ such that $W\mathbf{x} = \mathbf{y}$. Thus, $(Wx)_m = y_m$, whence $S_m x_m = y_m = v$ and S_m is surjective.

Moreover, using the notation from the previous step, we have that $S_m^{-1}v = (W^{-1}\mathbf{y})_m$, thus

$$\|S_m^{-1}v\| = \|(W^{-1}\mathbf{y})_m\| \le \|W^{-1}\mathbf{y}\| \le \|W^{-1}\| \cdot \|\mathbf{y}\| = \|W^{-1}\| \cdot \|v\|.$$

Hence, $\|S_m^{-1}\| \le \|W^{-1}\|$ for all $m \in \mathbb{Z}$ and the proof of the lemma is complete. □

By Lemma 3.3, we conclude that the second inequality in (3.2) holds. Furthermore, it follows from [12, Theorem 7.3] that there exists $\delta' > 0$ such that $TW^{-1}T^* - W^{-1} \le -\delta' I$ and therefore we have that

$$A_m S_m^{-1} A_m^* - S_{m+1}^{-1} \le -\delta' I, \quad \text{for all } m \in \mathbb{Z}$$

and the proof of (3.3) and of the theorem is completed.

We now establish the converse result. Due to the power of deep results we use, the proof is surprisingly easy.

Theorem 3.2. *Assume that $(A_m)_{m\in\mathbb{Z}}$ is the sequence of bounded linear operators with the property that there exists $C > 0$ such that (3.1) holds. Furthermore, suppose that there exists a sequence $(S_m)_{m\in\mathbb{Z}}$ of bounded, self-adjoint and invertible operators on X and constants $D, \delta > 0$ satisfying (3.2) and (3.3). Then, $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy.*

Proof. Let T and W be defined as in the proof of Theorem 3.1. We note that W is invertible. Furthermore, it follows from (3.3) that there exists $\delta > 0$ such that $T^*WT - W \le -\delta I$ and $TW^{-1}T^* - W^{-1} \le -\delta I$. By [12, Theorem 7.3] we have that T is hyperbolic. In particular, $1 \notin \sigma(T)$ and thus the operator

$$\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \mapsto (x_n - A_{n-1}x_{n-1})_{n\in\mathbb{Z}}$$

is invertible on l^2. It follows from Theorem 2.1 that $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy. □

We end this section with the comparison of our results with the former work relating Lyapunov functions and exponential dichotomies. In [22] it is proved that under the condition (3.1), the sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators acting on $\mathbb{R}^d$ admits a uniform exponential dichotomy if and only if there exist two bounded sequences $(H_m)_{m\in\mathbb{Z}}$ and $(G_m)_{m\in\mathbb{Z}}$ of self-adjoint operators on $\mathbb{R}^d$ such that:

$$A_m^* H_{m+1} A_m - H_m \le -\delta I, \tag{3.4}$$

$$A_m^{-1} G_{m+1} (A_m^*)^{-1} - G_m \le -\delta I \tag{3.5}$$

for some $\delta > 0$. When compared with our work, we note that the condition (3.4) is the same as (3.2). On the other hand, (3.5) is different from (3.3) and it involves inverses of operators A_n which in our setting may not exist. Moreover, the existence of two sequences of self-adjoint operators is required in contrast while we require the existence of a single sequence of self-adjoint operators. More recently, Barreira and Valls [6] proved that the sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible linear operators acting on $\mathbb{R}^d$ admits a uniform exponential dichotomy if and only if there exist a bounded sequence $(S_m)_{m\in\mathbb{Z}}$ of invertible and self-adjoint operators satisfying (3.2) and

$$H_{m+1}(A_m x) - H_m(x) \leq -\delta |H_m(x)| \tag{3.6}$$

for some $\delta > 0$, where $H_m(x) = \langle S_m x, x\rangle$. Thus, instead of (3.3) they require (3.6). We emphasize that the results in [22, 6] deal only with invertible operators acting on a finite-dimensional space and that to the best of our understanding those methods cannot be extended to our setting. Consequently, our methods are completely different from theirs.

4. Robustness of Exponential Dichotomies

In this section, we use the results obtained in the previous section to establish in a simple manner the stability of uniform exponential dichotomies under sufficiently small linear perturbations.

Theorem 4.1. *Let $(A_m)_{m\in\mathbb{Z}}$ and $(B_m)_{m\in\mathbb{Z}}$ be sequences of bounded linear operators on X such that*:

(1) *The sequence $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy and there exists $C > 0$ such that* (3.1) *holds*;
(2) *there exists $\rho > 0$ such that*

$$\|A_m - B_m\| \leq \rho, \quad \text{for } m \in \mathbb{Z}. \tag{4.1}$$

If ρ is sufficiently small, then the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy.

Proof. We first note that it follows from (3.1) and (4.1) that $\|B_m\| \leq C + \rho$ for all $m \in \mathbb{Z}$. By Theorem 3.1, there exists a sequence $(S_m)_m$ of bounded, self-adjoint and invertible operators and constants $D, \delta > 0$ such that (3.2) and (3.3) hold. Furthermore, we have

$$\begin{aligned}
&\langle S_{m+1}B_m x, B_m x\rangle - \langle S_m x, x\rangle \\
&\quad = \langle S_{m+1}(B_m - A_m)x, (B_m - A_m)x\rangle + \langle S_{m+1}A_m x, (B_m - A_m)x\rangle \\
&\qquad + \langle S_{m+1}(B_m - A_m)x, A_m x\rangle + \langle S_{m+1}A_m x, A_m x\rangle - \langle S_m x, x\rangle
\end{aligned} \tag{4.2}$$

for each $x \in X$ and $m \in \mathbb{Z}$. On the other hand, it follows from (3.2) and (4.1) that

$$\begin{aligned}
\langle S_{m+1}(B_m - A_m)x, (B_m - A_m)x\rangle &\leq \|S_{m+1}(B_m - A_m)x\| \cdot \|(B_m - A_m)x\| \\
&\leq D\rho^2 \langle x, x\rangle.
\end{aligned}$$

Similarly, by (3.1), (3.2) and (4.1), we have

$$\langle S_{m+1}A_m x, (B_m - A_m)x\rangle \le DC\rho\langle x, x\rangle \quad \text{and}$$
$$\langle S_{m+1}(B_m - A_m)x, A_m x\rangle \le DC\rho\langle x, x\rangle.$$

Thus, it follows from (3.3) and (4.2) that

$$\langle S_{m+1}B_m x, B_m x\rangle - \langle S_m x, x\rangle \le (D\rho^2 + 2DC\rho - \delta)\langle x, x\rangle$$

for each $m \in \mathbb{Z}$ and $x \in X$. Setting $r = -D\rho^2 - 2DC\rho + \delta$, we note that for ρ sufficiently small, we have that $r > 0$ and

$$B_m^* S_{m+1} B_m - S_m \le -rI, \quad \text{for } m \in \mathbb{Z}.$$

Similarly, one can show that

$$B_m S_m^{-1} B_m^* - S_{m+1}^{-1} \le -rI, \quad \text{for } m \in \mathbb{Z}.$$

By Theorem 3.2, the sequence $(B_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy. □

The version of Theorem 4.1 was first established in [15] for dichotomies on an arbitrary Banach space with the proof based on Theorem 2.1. For further references regarding the robustness problem we refer to [6, 24].

5. Nonlinear Perturbations and Lyapunov Sequences

We consider the nonlinear dynamics

$$x_{m+1} = A_m x_m + f_m(x_m), \tag{5.1}$$

where $f_m : X \to X$, $m \in \mathbb{Z}$ are continuous functions. We are going to show that if the linear part of the Eq. (5.1) admits an exponential dichotomy and if the nonlinear perturbation is sufficiently small that then each solution of (5.1) has the property that the associated sequence obtained by projecting the solution on the stable subspace of our dichotomy is uniformly exponentially stable. The precise statement is given below. Our arguments rely on the use of Lyapunov sequences.

Theorem 5.1. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a uniform exponential dichotomy with projections P_m and that the sequence $(f_m)_{m\in\mathbb{Z}}$ satisfies*:

(1) *There exists $\rho > 0$ such that*

$$\|f_m(x)\| \le \rho\|x\|, \quad \textit{for } m \in \mathbb{Z} \textit{ and } x \in X; \tag{5.2}$$

(2)

$$P_{m+1}f_m(x) = f_m(P_m x), \quad \textit{for } m \in \mathbb{Z} \textit{ and } x \in X. \tag{5.3}$$

Then for sufficiently small ρ, there exists $L > 0$ and $\eta \in (0,1)$ such that

$$\|P_n x_n\| \le L\eta^{n-m}\|P_m x_m\| \tag{5.4}$$

for $m \ge n$ and every solution $(x_m)_{m\in\mathbb{Z}}$ of (5.1).

Proof. Consider operators S_m, $m \in \mathbb{Z}$ given by Theorem 3.1. We define a sequence of functions H_m, $m \in \mathbb{Z}$ by

$$H_m(x) = \langle S_m x, x\rangle, \quad x \in X.$$

Furthermore, let $u_m = P_m x_m$, $m \in \mathbb{Z}$. In the proof of Theorem 3.1, we have showed that

$$\begin{aligned}
A_m^* S_{m+1} A_m = {} & e^{-2(\lambda-\rho)} \sum_{k\ge m} (\mathcal{A}(k,m)P_m)^* \mathcal{A}(k,m) P_m e^{2(\lambda-\rho)(k-m)} \\
& - e^{-2(\lambda-\rho)}(P_m)^* P_m - e^{2(\lambda-\rho)} \sum_{k<m} (\mathcal{A}(k,m)(I-P_m))^* \mathcal{A}(k,m) \\
& \times (I-P_m) e^{2(\lambda-\rho)(m-k)} - e^{2(\lambda-\rho)}(I-P_m)^*(I-P_m).
\end{aligned}$$

In particular, this implies that

$$A_m^* S_{m+1} A_m \le e^{-2(\lambda-\rho)} S_m, \quad \text{for every } m \in \mathbb{Z}. \tag{5.5}$$

Using (5.3), we have

$$\begin{aligned}
H_{m+1}(u_{m+1}) &= \langle S_{m+1} u_{m+1}, u_{m+1}\rangle \\
&= \langle S_{m+1} P_{m+1} x_{m+1}, P_{m+1} x_{m+1}\rangle \\
&= \langle S_{m+1} P_{m+1}(A_m x_m + f_m(x_m)), P_{m+1}(A_m x_m + f_m(x_m))\rangle \\
&= \langle S_{m+1}(A_m u_m + f_m(u_m)), A_m u_m + f_m(u_m)\rangle \\
&= \langle A_m^* S_{m+1} A_m u_m, u_m\rangle + \langle S_{m+1} A_m u_m, f_m(u_m)\rangle \\
&\quad + \langle S_{m+1} f_m(u_m), A_m u_m\rangle + \langle S_{m+1} f_m(u_m), f_m(u_m)\rangle.
\end{aligned}$$

By (3.1), (3.2), (5.2) and (5.5),

$$\begin{aligned}
H_{m+1}(u_{m+1}) &\le e^{-2(\lambda-\rho)} H_m(u_m) + 2\|S_{m+1}\| \cdot \|A_m u_m\| \cdot \|f_m(u_m)\| \\
&\quad + \|S_{m+1}\| \cdot \|f_m(u_m)\|^2 \\
&\le e^{-2(\lambda-\rho)} H_m(u_m) + 2DC\rho\|u_m\|^2 + D\rho^2\|u_m\|^2.
\end{aligned}$$

Noting that $H_n(z) \ge \|z\|^2$ for $z \in \operatorname{Im} P_n$ and that $u_n \in \operatorname{Im} P_n$, we conclude that

$$H_{m+1}(u_{m+1}) \le \eta^2 H_m(u_m), \tag{5.6}$$

where

$$\eta^2 = e^{-2(\lambda-\rho)} + 2DC\rho + D\rho^2.$$

By choosing ρ sufficiently small, we can achive that $\eta \in (0,1)$. Iterating (5.6), we obtain that

$$H_n(u_n) \le \eta^{2(n-m)} H_m(u_m), \quad \text{for } n \ge m.$$

Since $\|u_n\|^2 \le H_n(u_n)$ and $H_m(u_m) \le D\|u_m\|^2$, we conclude that (5.4) holds with $L = \sqrt{D}$. □

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