Spectral theory under nonuniform hyperbolicity

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(joint work with L. Barreira and C. Valls)

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Preliminaries

We first recall the notion of a (uniform) exponential dichotomy and its generalizations. Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible operators on $\mathbb{R}^d$. For each $m, n \in \mathbb{Z}$ we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *uniform exponential dichotomy* if there exist projections P_m for $m \in \mathbb{Z}$ satisfying

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n \quad \text{for } m,n \in \mathbb{Z}, \tag{1}$$

a constant $\lambda, D > 0$ such that:

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$.

Some consequences of the existence of uniform exponential dichotomy:

1. existence and regularity of invariant stable and unstable manifolds;
2. linearization of dynamics;
3. center manifold theory.

Recently, Barreira and Valls introduced the concept of a *nonuniform exponential dichotomy* and obtain generalizations of those results. We refer to L. Barreira and C. Valls, *Stability of Nonautonomous Differential Equations*, Springer, 2008.

We will consider the following notions which lies in between those two notions. We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a *nonuniform exponential dichotomy with an arbitrarily small nonuniform part* if there exist projections P_m for $m \in \mathbb{Z}$ satisfying (1), a constant $\lambda > 0$ and for each $\varepsilon > 0$ a constant $D = D(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)P_n\| \le De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \ge n$$

and

$$\|\mathcal{A}(m,n)Q_n\| \le De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \le n,$$

where $Q_n = \mathrm{Id} - P_n$.

Proposition

Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Then, we have

$$P_n(\mathbb{R}^d) = \left\{ v \in \mathbb{R}^d : \sup_{m\ge n} \|\mathcal{A}(m,n)v\| < +\infty \right\}$$

and

$$Q_n(\mathbb{R}^d) = \left\{ v \in \mathbb{R}^d : \sup_{m\le n} \|\mathcal{A}(m,n)v\| < +\infty \right\},$$

for each $n \in \mathbb{Z}$.

Context of ergodic theory

Assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue measure space and let $f\colon X \to X$ be an invertible, ergodic measure-preserving transformation. We denote by GL_d be the set of all invertible operators on $\mathbb{R}^d$. A measurable function $\mathcal{A}\colon X \times \mathbb{Z} \to GL_d$ is called a *linear cocycle over f* if for every $q \in X$ and $n, m \in \mathbb{Z}$:

1. $\mathcal{A}(q, 0) = \mathrm{Id}$;
2. $\mathcal{A}(q, n+m) = \mathcal{A}(f^n(q), m)\mathcal{A}(q, n)$.

The map $A\colon X \to GL_d$ defined by $A(q) = \mathcal{A}(q, 1)$ for $q \in X$ is called a *generator* of cocyle $\mathcal{A}$.

Theorem

Assume that

$$\log^+\|A\|,\ \log^+\|A^{-1}\| \in L^1(X,\mu)$$

and that all Lyapunov exponents of a cocycle $\mathcal{A}$ *with respect to* μ *are nonzero. Then for* μ*-almost every* $q \in X$*, the sequence* $(A_n)_n$ *defined by* $A_n = A(f^n(q))$*,* $n \in \mathbb{Z}$ *admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part.*

We refer to L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents*, Cambridge University Press, 2007.

Nonuniform spectrum

Given a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible operators on $\mathbb{R}^d$, its *nonuniform spectrum* is the set Σ of all $a \in \mathbb{R}$ such that the sequence $(e^{-a}A_m)_{m\in\mathbb{Z}}$ does not admit a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. We are interested in:

1. geometric structure of Σ;
2. its connection with the theory of Lyapunov exponents.

In the case of uniform behaviour this has been studied in:

1. R. Sacker and G. Sell, *A spectral theory for linear differential systems*, J. Differential Equations **27** (1978), 320–358.
2. S. Siegmund, *Dichotomy spectrum for nonautonomous differential equations*, J. Dynam. Differential Equations **14** (2002), 243–258.
3. B. Aulbach and S. Siegmund, *A spectral theory for nonautonomous difference equations*, in New Trends in Difference Equations (Temuco, 2000), Taylor & Francis, 2002, pp. 45–55.

Structure of the spectrum

Theorem

For a sequence $(A_m)_{m\in\mathbb{Z}}$ of invertible operators on $\mathbb{R}^d$ one of the following alternatives holds:

1. $\Sigma = \emptyset$;
2. $\Sigma = \mathbb{R}$;
3. $\Sigma = I_1 \cup [a_2, b_2] \cup \cdots \cup [a_{k-1}, b_{k-1}] \cup I_k$, *where* $I_1 = [a_1, b_1]$ *or* $I_1 = (-\infty, b_1]$ *and* $I_k = [a_k, b_k]$ *or* $I_k = [a_k, +\infty)$ *for some finite numbers*

$$a_1 \le b_1 < a_2 \le b_2 < \cdots < a_k \le b_k \quad \text{and} \quad k \le d.$$

Examples

1. Let $d = 1$ and $A_m = e^{m^3}$ for $m \in \mathbb{Z}$. It is easy to verify that $\Sigma = \emptyset$.

2. Let $d = 1$ and take $w > c > 0$. For each $n \in \mathbb{Z}$, let

$$A_n = e^{-w+c(n+1)\cos(n+1)-cn\cos n - c\sin(n+1)+c\sin n}.$$

   One can verify that $\Sigma = \mathbb{R}$.

3. Let $d = 1$ and take $w > b > 0$. For each $n \in \mathbb{Z}$, let

$$A_n = \begin{cases} e^{-w+b+\sqrt{1+|n|}\cos(n+1)-\sqrt{|n|}\cos n}, & n \geq 0, \\ e^{-w-b+\sqrt{1+|n|}\cos(n+1)-\sqrt{|n|}\cos n}, & n < 0. \end{cases}$$

   We have that $\Sigma = [e^{-w-b}, e^{-w+b}]$.

Lyapunov regularity

We say that a sequence $(A_n)_{n\in\mathbb{Z}}$ of invertible operators on $\mathbb{R}^d$ is *Lyapunov regular* if there exist a decomposition

$$\mathbb{R}^d = \bigoplus_{i=1}^{s} E_i$$

and real numbers $\lambda_1 < \cdots < \lambda_s$ such that:

1. if $i = 1, \ldots, s$ and $v \in E_i \setminus \{0\}$, then
$$\lim_{n\to\pm\infty} \frac{1}{n} \log\|\mathcal{A}(n,0)v\| = \lambda_i;$$
2. $$\lim_{n\to\pm\infty} \frac{1}{n} \log|\det \mathcal{A}(n,0)| = \sum_{i=1}^{s} \lambda_i \dim E_i.$$

Theorem

If the sequence $(A_n)_{n\in\mathbb{Z}}$ is Lyapunov regular, then

$$\Sigma = \{\lambda_1, \dots, \lambda_s\}.$$

Proposition

Assume that there exists $d > 0$ and for each $\varepsilon > 0$ a constant $K = K(\varepsilon) > 0$ such that

$$\|\mathcal{A}(m,n)\| \le Ke^{d|m-n|+\varepsilon|n|}, \quad \text{for } m,n \in \mathbb{Z}. \tag{2}$$

Then, Σ is compact and nonempty.

From now on we assume that (2) holds.

Lyapunov exponents and nonuniform spectrum

Theorem

For each $v \in \mathbb{R}^d \setminus \{0\}$, the numbers

$$\liminf_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(n,0)v\| \text{ and } \limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{A}(n,0)v\|$$

belong to the same connected component of Σ.

Let $f_n\colon \mathbb{R}^d \to \mathbb{R}^d$, $n \geq 0$ be a sequence of continuous maps. We consider the nonlinear dynamics

$$x_{n+1} = A_n x_n + f_n(x_n), \quad n \geq 0. \tag{3}$$

Theorem

Assume that there exists a sequence $(\gamma_k)_{k\ge 0}$ *such that:*

1. $$\|f_k(x)\| \le \gamma_k \|x\|, \quad \textit{for } x \in \mathbb{R}^d; \tag{4}$$

2. $$\limsup_{k\to\infty} \frac{1}{k} \log \gamma_k < 0. \tag{5}$$

Then, for each solution $(x(n))_{n\ge 0}$ *of* (3) *the numbers*

$$\liminf_{n\to\infty} \frac{1}{n} \log\|x(n)\| \textit{ and } \limsup_{n\to\infty} \frac{1}{n} \log\|x(n)\|$$

belong to the same connected component of Σ.

Corollary

Assume that the sequence $(A_n)_{n\in\mathbb{Z}}$ *is Lyapunov regular and that conditions* (4) *and* (5) *hold. Then, for each solution* $(x(n))_{n\geq 0}$ *of* (3) *there exists* $i \in \{1, \ldots, s\}$ *such that*

$$\liminf_{n\to\infty} \frac{1}{n} \log\|x(n)\| = \limsup_{n\to\infty} \frac{1}{n} \log\|x(n)\| = \lambda_i.$$

Results of this type have a long history:

P. Hartman and A. Wintner, *Asymptotic integrations of linear differential equations*, Amer. J. Math. **77** (1955), 45–86.

C. Coffman, *Asymptotic behavior of solutions of ordinary difference equations*, Trans. Amer. Math. Soc. **110** (1964) 22–51.

M. Pituk, *A Perron type theorem for functional differential equations*, J. Math. Anal. Appl. **316** (2006), 24–41.

Further developments:

1. noninvertible dynamics;
2. one-sided dynamics;
3. case of compact operators acting on Banach space;
4. spectral theory for nonuniformly hyperbolic sets;
5. continuous time (ODE, PDE, DDE).