Spectral theory under nonuniform hyperbolicity

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(joint work with L. Barreira and C. Valls)

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We first recall the notion of a (uniform) exponential dichotomy and its generalizations. Let \((A_m)_{m\in\mathbb{Z}}\) be a sequence of invertible operators on \(\mathbb{R}^d\). For each \(m, n \in \mathbb{Z}\) we define

\[
\mathcal{A}(m, n) = \begin{cases} 
A_{m-1} \cdots A_n & \text{if } m > n, \\
\text{Id} & \text{if } m = n, \\
A^{-1}_{m-1} \cdots A^{-1}_{n-1} & \text{if } m < n.
\end{cases}
\]

We say that the sequence \((A_m)_{m\in\mathbb{Z}}\) admits a uniform exponential dichotomy if there exist projections \(P_m\) for \(m \in \mathbb{Z}\) satisfying

\[
P_m \mathcal{A}(m, n) = \mathcal{A}(m, n) P_n \quad \text{for } m, n \in \mathbb{Z},
\]

a constant \(\lambda, D > 0\) such that:
\[ \| A(m, n) P_n \| \leq De^{-\lambda(m-n)} \quad \text{for } m \geq n \]

and

\[ \| A(m, n) Q_n \| \leq De^{-\lambda(n-m)} \quad \text{for } m \leq n, \]

where \( Q_n = \text{Id} - P_n \).

Some consequences of the existence of uniform exponential dichotomy:

1. existence and regularity of invariant stable and unstable manifolds;

2. linearization of dynamics;

3. center manifold theory.

We will consider the following notions which lies in between those two notions. We say that the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a *nonuniform exponential dichotomy with an arbitrarily small nonuniform part* if there exist projections \(P_m\) for \(m \in \mathbb{Z}\) satisfying (1), a constant \(\lambda > 0\) and for each \(\varepsilon > 0\) a constant \(D = D(\varepsilon) > 0\) such that

\[
\|A(m, n)P_n\| \leq De^{-\lambda(m-n)+\varepsilon|n|} \quad \text{for } m \geq n
\]
and
\[ \|A(m, n)Q_n\| \leq De^{-\lambda(n-m)+\varepsilon|n|} \quad \text{for } m \leq n, \]

where \( Q_n = \text{Id} - P_n \).

**Proposition**

Assume that the sequence \((A_m)_{m \in \mathbb{Z}}\) admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. Then, we have

\[ P_n(\mathbb{R}^d) = \left\{ v \in \mathbb{R}^d : \sup_{m \geq n} \|A(m, n)v\| < +\infty \right\} \]

and

\[ Q_n(\mathbb{R}^d) = \left\{ v \in \mathbb{R}^d : \sup_{m \leq n} \|A(m, n)v\| < +\infty \right\}, \]

for each \( n \in \mathbb{Z} \).
Assume that $(X, \mathcal{B}, \mu)$ is a Lebesgue measure space and let $f: X \to X$ be an invertible, ergodic measure-preserving transformation. We denote by $GL_d$ be the set of all invertible operators on $\mathbb{R}^d$. A measurable function $\mathcal{A}: X \times \mathbb{Z} \to GL_d$ is called a linear cocycle over $f$ if for every $q \in X$ and $n, m \in \mathbb{Z}$:

1. $\mathcal{A}(q, 0) = \text{Id}$;

2. $\mathcal{A}(q, n + m) = \mathcal{A}(f^n(q), m)\mathcal{A}(q, n)$.

The map $A: X \to GL_d$ defined by $A(q) = \mathcal{A}(q, 1)$ for $q \in X$ is called a generator of cocycle $\mathcal{A}$. 

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Theorem

Assume that

\[ \log^+ \|A\|, \log^+ \|A^{-1}\| \in L^1(X, \mu) \]

and that all Lyapunov exponents of a cocycle \(A\) with respect to \(\mu\) are nonzero. Then for \(\mu\)-almost every \(q \in X\), the sequence \((A_n)_n\) defined by \(A_n = A(f^n(q))\), \(n \in \mathbb{Z}\) admits a nonuniform exponential dichotomy with an arbitrarily small nonuniform part.

Given a sequence \((A_m)_{m \in \mathbb{Z}}\) of invertible operators on \(\mathbb{R}^d\), its nonuniform spectrum is the set \(\Sigma\) of all \(a \in \mathbb{R}\) such that the sequence \((e^{-a}A_m)_{m \in \mathbb{Z}}\) does not admit a nonuniform exponential dichotomy with an arbitrarily small nonuniform part. We are interested in:

1. geometric structure of \(\Sigma\);

2. its connection with the theory of Lyapunov exponents.
In the case of uniform behaviour this has been studied in:


Structure of the spectrum

Theorem

For a sequence \((A_m)_{m \in \mathbb{Z}}\) of invertible operators on \(\mathbb{R}^d\) one of the following alternatives holds:

1. \(\Sigma = \emptyset\);
2. \(\Sigma = \mathbb{R}\);
3. \(\Sigma = l_1 \cup [a_2, b_2] \cup \cdots \cup [a_{k-1}, b_{k-1}] \cup l_k\), where \(l_1 = [a_1, b_1]\)
   or \(l_1 = (-\infty, b_1]\) and \(l_k = [a_k, b_k]\) or \(l_k = [a_k, +\infty)\) for some finite numbers

\[ a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_k \leq b_k \quad \text{and} \quad k \leq d. \]
Examples

1. Let $d = 1$ and $A_m = e^{m^3}$ for $m \in \mathbb{Z}$. It is easy to verify that $\Sigma = \emptyset$.

2. Let $d = 1$ and take $w > c > 0$. For each $n \in \mathbb{Z}$, let

$$A_n = e^{-w+c(n+1)\cos(n+1)-cn\cos n-c\sin(n+1)+c\sin n}.$$ 

One can verify that $\Sigma = \mathbb{R}$.

3. Let $d = 1$ and take $w > b > 0$. For each $n \in \mathbb{Z}$, let

$$A_n = \begin{cases} 
  e^{-w+b+\sqrt{1+|n|}\cos(n+1)-\sqrt{|n|}\cos n}, & n \geq 0, \\
  e^{-w-b+\sqrt{1+|n|}\cos(n+1)-\sqrt{|n|}\cos n}, & n < 0.
\end{cases}$$

We have that $\Sigma = [e^{-w-b}, e^{-w+b}]$. 

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Lyapunov regularity

We say that a sequence \((A_n)_{n \in \mathbb{Z}}\) of invertible operators on \(\mathbb{R}^d\) is \textit{Lyapunov regular} if there exist a decomposition

\[
\mathbb{R}^d = \bigoplus_{i=1}^{s} E_i
\]

and real numbers \(\lambda_1 < \cdots < \lambda_s\) such that:

1. if \(i = 1, \ldots, s\) and \(v \in E_i \setminus \{0\}\), then

\[
\lim_{n \to \pm \infty} \frac{1}{n} \log \|A(n, 0)v\| = \lambda_i;
\]

2. \[
\lim_{n \to \pm \infty} \frac{1}{n} \log |\det A(n, 0)| = \sum_{i=1}^{s} \lambda_i \dim E_i.
\]
Theorem

If the sequence \((A_n)_{n \in \mathbb{Z}}\) is Lyapunov regular, then

\[ \Sigma = \{ \lambda_1, \ldots, \lambda_s \} . \]

Proposition

Assume that there exists \(d > 0\) and for each \(\varepsilon > 0\) a constant \(K = K(\varepsilon) > 0\) such that

\[ \| A(m, n) \| \leq Ke^{d|m-n|+\varepsilon|n|}, \quad \text{for } m, n \in \mathbb{Z}. \] (2)

Then, \(\Sigma\) is compact and nonempty.

From now on we assume that (2) holds.
Lyapunov exponents and nonuniform spectrum

**Theorem**

For each \( v \in \mathbb{R}^d \setminus \{0\} \), the numbers

\[
\liminf_{n \to \infty} \frac{1}{n} \log \| A(n, 0)v \| \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} \log \| A(n, 0)v \|
\]

belong to the same connected component of \( \Sigma \).

Let \( f_n : \mathbb{R}^d \to \mathbb{R}^d, \ n \geq 0 \) be a sequence of continuous maps. We consider the nonlinear dynamics

\[
x_{n+1} = A_n x_n + f_n(x_n), \quad n \geq 0.
\]
Theorem

Assume that there exists a sequence \((\gamma_k)_{k \geq 0}\) such that:

1. \[ \| f_k(x) \| \leq \gamma_k \| x \|, \quad \text{for } x \in \mathbb{R}^d; \] (4)

2. \[ \limsup_{k \to \infty} \frac{1}{k} \log \gamma_k < 0. \] (5)

Then, for each solution \((x(n))_{n \geq 0}\) of (3) the numbers

\[ \liminf_{n \to \infty} \frac{1}{n} \log \| x(n) \| \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} \log \| x(n) \| \]

belong to the same connected component of \(\Sigma\).
Corollary

Assume that the sequence \((A_n)_{n \in \mathbb{Z}}\) is Lyapunov regular and that conditions (4) and (5) hold. Then, for each solution \((x(n))_{n \geq 0}\) of (3) there exists \(i \in \{1, \ldots, s\}\) such that

\[
\liminf_{n \to \infty} \frac{1}{n} \log \|x(n)\| = \limsup_{n \to \infty} \frac{1}{n} \log \|x(n)\| = \lambda_i.
\]

Results of this type have a long history:


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Further developments:

1. noninvertible dynamics;
2. one-sided dynamics;
3. case of compact operators acting on Banach space;
4. spectral theory for nonuniformly hyperbolic sets;
5. continuous time (ODE, PDE, DDE).