Invariant measures and attractors of non-autonomous Frenkel-Kontorova model

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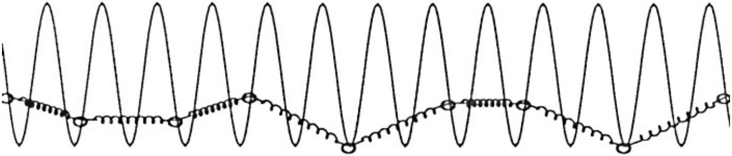
Standard FK model

Standard Frenkel-Kontorova (FK) model is a system of one-dimensional elastically connected chains of particles in a periodic potential.



$$\frac{d}{dt}u_j = u_{j+1} - 2u_j + u_{j-1} + k\sin(2\pi u_j), j \in \mathbb{Z}$$

It is a spatial discretization relative of reaction-diffusion equation.

$$u_t = u_{xx} + f(x, u, u_x)$$

FK model

Let $V(u,v,t)$ be continuous and C^2 in u,v that satisfies

(V1) $V(u+1,v+1,t) = V(u,v,t)$ for $(u,v,t) \in \mathbb{R}^3$

(V2) exists $T > 0$ such that $V(u,v,t+T) = V(u,v,t)$ for $(u,v,t) \in \mathbb{R}^3$

(V3) Twist condition: exists $D > 0$ such that $V_{12}(u,v,t) \leq -D$

(V4) exists $C > 0$ such that $\|D^2_{u,v}(u,v,t)\| \leq C$ for $(u,v,t) \in \mathbb{R}^3$.

We look at the system

$$\frac{d}{dt}u_j = -V_2(u_{j-1},u_j,t) - V_1(u_j,u_{j+1},t) \tag{1}$$

where V satisfies (V1)-(V4) and call it FK model. We will look for solutions for initial conditions $u(0) = u_0 \in K_n$ where

$$K_n = \{u \in \mathbb{R}^{\mathbb{Z}} : \sup_{j\in\mathbb{Z}} |u_{j+1} - u_j| \leq n\}$$

Topology of K_n

On linear space $X = \bigcup_{n\in\mathbb{N}} K_n$ for $\lambda > 0$ we introduce λ-topology introduced from λ-norm

$$\|u\|_\lambda = e^{-\lambda|j|}|u_j|$$

Theorem 1

All λ-topologies are equivalent.

On X and on each K_n we introduce relation R and say that uRv if there exists $m \in \mathbb{Z}$ such that $u_j - v_j = m$ for all $j \in \mathbb{Z}$. Then we put $\mathcal{X} = X/R$ and $\mathcal{K}_n = K_n/R$.

Theorem 2

Space $\mathcal{K}_n$ is compact.

Existence of local solution and strict monotonicity

Theorem 3

For all $u_0 \in K_n$ *there exists* $\delta > 0$ *and unique continuous* $u \in C([-\delta, \delta], K_n)$ *such that it satisfies* (1) *and* $u(0) = u_0$

For configurations u and v we say that $u \leq v$ if $u_j \leq v_j$ for all $j \in \mathbb{Z}$. We say $u < v$ if $u \leq v$ and $u \neq v$ and $u \ll v$ if $u_j < v_j$ for all $j \in \mathbb{Z}$. We say that system is strictly monotone if for all solutions u and v of that system we have that $u(0) < v(0)$ implies $u(t) \ll v(t)$ for all $t > 0$ for which u and v are defined.

Theorem 4

System (1) *is strictly monotone.*

Global existence and semiflow

From monotonicity and local existence it follows global existence.

Theorem 5

For $u_0 \in K_n$ there exists a function $u \in C([0,\infty), K_n)$ such that it satisfies (1) and that $u(0) = u_0$.

From Gronwalls lemma follows:

Theorem 6

For all $u(0), v(0) \in K_n$ exists $M_n > 0$ such that

$$\|u(t) - v(t)\|_\lambda \leq \|u(0) - v(0)\|_\lambda e^{M_n t}$$

From global existence and local uniqueness follows existance of semiflow. We put $\varphi(t, u_0) = u(t)$ where u is solution of (1) such that $u(0) = u_0$.

Theorem 7

Map $\varphi\colon [0,\infty) \times K_n \to K_n$ is semiflow.

Synchronized solutions

We define space shift $(S_{m,n}u)_j = u_{j-m} + n$. We say that solution u of (1) is synchronized if it exists for all times and $S_{m,n}u(t+l) \ll u(t)$ or $S_{m,n}u(t+l) = u(t)$ or $u(t) \ll S_{m,n}u(t+l)$ for all $m, n, l \in \mathbb{Z}$ and $t \in \mathbb{R}$. For configuration u we say that it has mean spacing ω if $\omega = \lim_{|i|\to\infty} u_i/i$. With $\rho(u)$ lets define mean spacing of a configuration u if it exists.

Theorem 8

For all $t \in \mathbb{R}$ configuration $u(t)$ where u is a synchronized solution has mean spacing and $\rho(u(t))$ is constant.

Theorem 9

For every $\omega \in \mathbb{R}$ there exists syncronized solution u such that $\omega = \rho(u(t))$ for all $t \in \mathbb{Z}$.

S, T-invariant measures

Definition 1

We say that Borel probability measure μ on $\mathcal{X}$ is S-invariant if $\mu(A) = \mu(S^{-1}(A))$ for all measurable A where $S\colon \mathcal{X} \to \mathcal{X}$ is given with $S = S_1$ where $(S_m u)_j = u_{j-m}$.

Definition 2

We say that Borel probability measure μ on $\mathcal{X}$ is T-invariant if $\mu(A) = \mu(T^{-1}A)$ for all measurable A where $Tu = \varphi(T, u)$ where φ is semiflow and $T > 0$ is period from (V2). We put $T_n = T_{n-1} \circ T$ for $n > 1$.

Spatio-temporal attractor

Let u be a configuration. We define weak ω-limit set $\tilde{\omega}(u)$ as the smallest closed set such that for every open neighbourhood U we have

$$\frac{1}{N+1}\frac{1}{2N+1}\sum_{n=0}^{N}\sum_{m=-N}^{N} 1_U(S_m T_n u) \to 1$$

when $N \to \infty$.

We define spatio-temporal attractor $\mathcal{A}$ dynamics (1) as

$$\mathcal{A} = \mathrm{Cl}\left(\bigcup_{u\in\mathcal{X}} \tilde{\omega}(u)\right)$$

Let $\mathcal{X}_\rho$ be set of all elements from $\mathcal{X}$ that have mean spacing ρ. Set $\mathcal{X}_\rho$ is compact. We put $\mathcal{A}_\rho = \mathcal{A} \cap \mathcal{X}_\rho$.

Configuration is with probalility one drawn to attractor

Teorem 1

Spatio-temporal attractor $\mathcal{A}$ coinsides with union of supports of S, T-invariant measures. Space $\mathcal{A}_\rho$ is not empty.

Theorem 10

Let μ be S-invariant measure, $p_N = \frac{1}{N+1}\sum_{j=0}^{N}\delta_j$, $u \in \mathcal{X}$ and let U be arbitrary open neighbourhood of attractor $\mathcal{A}$ then

$$\mu \times p_N(\{(u,n) \in \mathcal{X} \times \{0,\dots,N\} : T_n u \in U\}) \to 1, \text{ for } N \to \infty.$$

Proof.

It is equal to $\nu^N(U) = \frac{1}{N+1}\sum_{n=0}^{N} T_n^*\mu(U)$, where $T_n^*\mu$ is pushforward μ with respect to T_n. Measure ν^N has a weakly convergent subsequence that converges weakly to ν. But then ν is S, T-invariant so its support is contained in $\mathcal{A}$, but then $\nu(U) = 1$. □

Transversal and nontransversal intersections

Let w be a configuration. We say that w has zero on spot j if exists $x \in [0,1)$ such that $w_j + (w_{j+1} - w_j)x = 0$. If $w_j = 0$ we say that zero on spot j is singular if $w_{j-1} = 0$ or $w_{j+1} = 0$ or $w_{j-1}w_{j+1} > 0$.
We say that u and v intersect if $u - v$ has zero, and we say that they intersect nontransversal if $u - v$ has singular zero. If u and v intersect but not nontransversal we say that they intersect transversal.

Definition 3

For attractor $\mathcal{A}$ we say that it is transversal in no two $u, v \in \mathcal{A}$ intersect nontransversal.

Teorem 2

If attractor $\mathcal{A}$ is transversal then map $\pi\colon \mathcal{A} \to \mathbb{S}^1 \times \mathbb{R}$ is injective where $\pi(u) = (u_0, u_1 - u_0)$.

Counting zeros

We define $z_i(w)$ to be 1 if w has zero on spot i, otherwise 0. Let $u, v \in \mathcal{K}_n$ then we define

$$z_i(u, v) = \sum_{r \in \mathbb{Z}} z_i(u - v + r)$$

Lema 1

We have $z_i(u, v) \leq 2n + 1$ for all $i \in \mathbb{Z}$.

Definition 4

Let μ^1 and μ^2 be two S-invariant measures. We put

$$Z(\mu^1, \mu^2) = \int z_0(u, v) d\mu^1(u) d\mu^2(v)$$

Number of zeros in nonincreasing

Singular zero w is of infinite degree if exists $i \in \mathbb{Z}$ such that $w_i \neq 0$, but $w_{i+1} = w_{i+2} = \cdots = 0$ or exists $i \in \mathbb{Z}$ such that $w_i \neq 0$, but $w_{i-1} = w_{i-2} = \cdots = 0$. If zero is not of infinite degree then it is of finite degree.

Teorem 3

Let μ^1 and μ^2 be two S-invariant measures and let $\mu^1(t) = \varphi_*^t \mu^1$ and $\mu^2(t) = \varphi_*^t \mu^2$ be pushforwards. Then

(M1) *Function $t \mapsto Z(\mu^1(t), \mu^2(t))$ is nonincreasing.*

(M2) *If for some t_0 there exist $u \in \operatorname{supp}(\mu^1(t_0))$ and $v \in \operatorname{supp}(\mu^2(t_0))$ such that $u - v$ has singular zero of finite degree then $t \mapsto Z(\mu^1(t), \mu^2(t))$ is strictly decreasing at $t = t_0$.*

We define $Z(\mu) = Z(\mu, \mu)$.

We say for S, T-invariant measure μ that it is synchronized if no two configuration in its support intersect that is $Z(\mu) = 0$.

Definition 5

We say that (1) is in depinned phase for $\omega \in \mathbb{R}$ if there exists synchronized measure μ with ω for its mean spacing that $p_0 \colon \mathrm{supp}\mu \to \mathbb{S}^1$ is surjection, otherwise it is in pinned phase. Where $p_0(u) = u_0$.

We define $\mathcal{S}_\rho$ as union of supports of synchronized measures on $\mathcal{X}_\rho$.

Ratchet system

We will observe special case of dynamics (1) so called Ratchet dynamics

$$V(u, v, t) = W(u - v) + F(u)K(t)$$

$$\frac{d}{dt}u_j = W'(u_{j+1} - u_j) - W'(u_j - u_{j-1}) + F'(u_j)K(t), \quad j \in \mathbb{Z} \qquad (2)$$

where

(R0) W, F, K are real-analityc, F and K are not constants,

(R1) exists $\delta > 0$ such that $W'(x) \geq \delta > 0$ for all $x \in \mathbb{R}$,

(R2) $F(u+1) = F(u)$ for all $u \in \mathbb{R}$,

(R3) exists fundamental period $T > 0$ such that $K(t+T) = K(t)$ for all $t \in \mathbb{R}$.

Teorem 4

Attractor of Ratchet dynamics is transversal.

Teorem 5

In depinned phase we have $\mathcal{A}_\rho = \mathcal{S}_\rho$.

Teorem 6

For Ratchet dynamics we have that ρ *is in depinned phase if and only if*

(i) *no configuration from* $\mathcal{A}_\rho$ *intersects configuration from* $\mathcal{A}$ *more than once.*

(ii) $p_0(\mathcal{A}_\rho)$ *is dense or open set.*

Transport

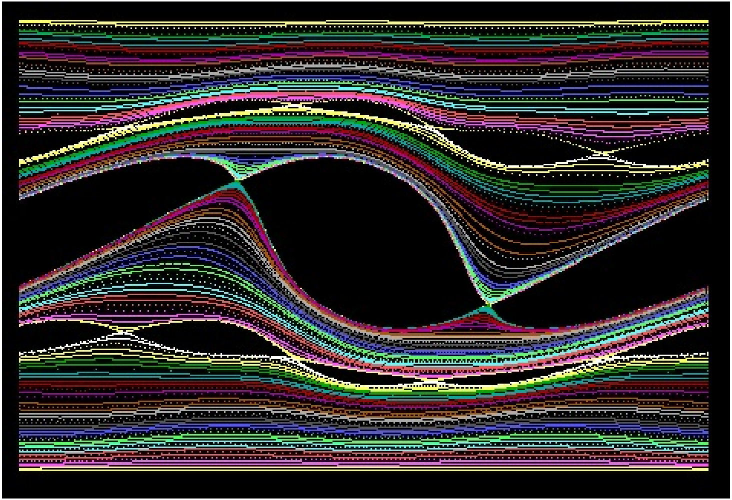
We say that transport is not zero if there exists S, T-invariant measure μ such that

$$0 \neq \overline{v} = \int ((T_1 u)_0 - u_0) d\mu(u)$$

Teorem 7

For Ratchet dynamics in depinned phase $\rho \in \mathbb{R}$ transport is not zero if T_1 is not constant function.

Attractor



$$\frac{d}{dt}u_j = u_{j+1} - 2u_j + u_{j-1} + \sin(2\pi u_j)\cos(2\pi t)$$

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