VDFs and novel RSA assumptions

A Note on Low Order Assumptions in RSA groups

István András Seres and Péter Burcsi

Eötvös Loránd University

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Low Order assumption à la Boneh et al. [BBF18]

Definition

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$$Pr\left[u^l = 1,\ u \notin \{1, -1\} : \begin{array}{c} \mathbb{G} \xleftarrow{\$} GGen(\lambda) \\ (u, l) \leftarrow \mathcal{A}(\mathbb{G}) \\ \text{and } l < 2^{poly(\lambda)} \end{array} \right] \leq \mathrm{negl}(\lambda)$$

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- But also weak somewhat as it is not obvious how such an adversary would help in factoring the modulus if $l \neq 2$.
- So, how does this relate to other RSA assumptions?

Adaptive Root assumption [Wes19]

Definition

The *Adaptive Root Assumption* holds for *GGen* if there is no efficient adversary $(\mathcal{A}_0, \mathcal{A}_1)$ that succeeds in the following task. First, $\mathcal{A}_0$ outputs an element $w \in \mathbb{G}$ and some state st. Then, a random prime in $\mathrm{Primes}(\lambda)$ is chosen and $\mathcal{A}_1(w, l, st)$ outputs $w^{1/l} \in \mathbb{G}$. For all efficient $(\mathcal{A}_0, \mathcal{A}_1)$:

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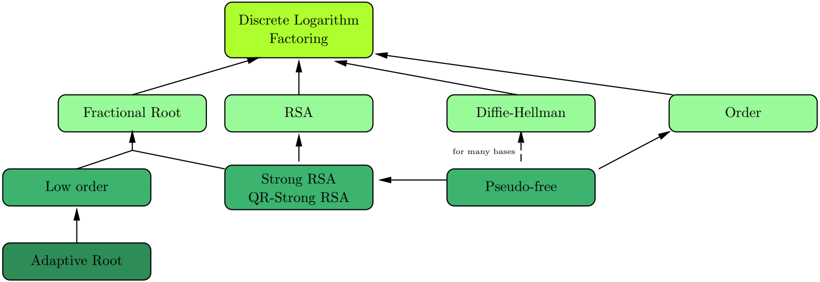
The number of primes in Π_λ should be exponential in λ: it is possible to precompute w using $2^{|\Pi_\lambda|}$ exponentiations.

The RSA Assumption Zoo: a glimpse



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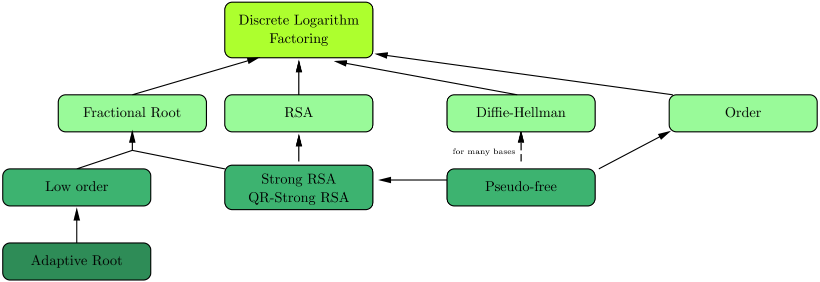

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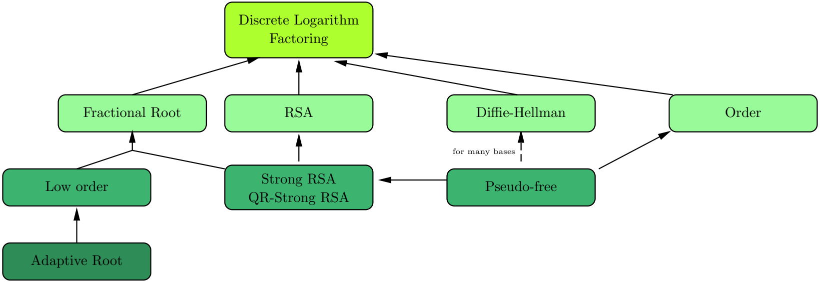

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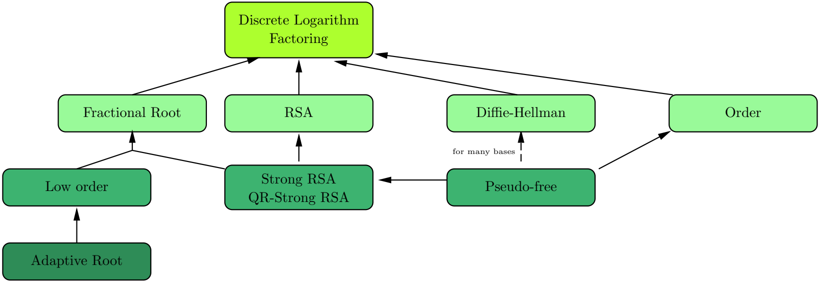

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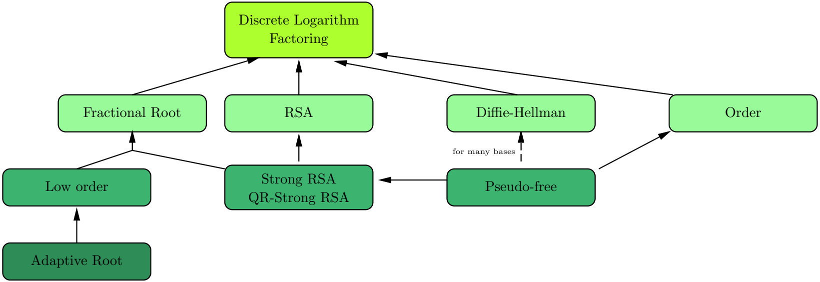

- RSA assumption and Factoring?
- Strong RSA: there are exponential witnesses! [GK16]
- Low Order: is it equivalent to Factoring?
- Adaptive Root: seems like a hard problem but seems like a weak assumption: Interactive and Exponential witnesses!

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- Maybe we'll have a similar result like the one by Oded Regev [Reg09], namely a reduction from LWE to SIS. Reduction from a 1-search problem to an exponential t-search problem!
- Highly recommended reading the position paper on cryptographic assumptions by Goldwasser and Kalai [GK16]!

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Defining VDFs (Verifiable Delay Functions)

A VDF $V = (\mathsf{Setup}, \mathsf{Eval}, \mathsf{Verify})$ is a triple of algorithms:

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Security requirements

- Correctness and SOUNDNESS
- Eval cannot be paralellized! Class groups???
- There should be an exponential gap between the time complexity of Eval and Verify

Example

Iterating a cryptographic hash function is a VDF? How to make it a VDF?

Why people are excited about VDFs?

- Non-interactive timestamping [LSS19]

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- Resource-efficient blockchains [CP19]

Necessity of using a group of unknown order [RSS]

$$\mathcal{L}_{\mathsf{EXP}} = \left\{ \left(\mathbb{G}, g, h, T \right) : h = g^{(2^T)} \in \mathbb{G} \right\}$$

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Currently known groups of unknown order: RSA and class groups!

An algebraic construction

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Great lecture by Dan Boneh! Watch it here:
https://www.youtube.com/watch?v=dN-1q8c50q0

Pietrzak's protocol, The base case of the recursion

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- The verifier checks that $g, h \in \mathbb{G}$ and outputs *reject* if not,
- If $T = 1$ the verifier checks that $h = g^2$ in $\mathbb{G}$, outputs *accept* or *reject*, and stops.

Pietrzak's protocol: The recursion step, ie. if $T > 1$ the prover and verifier do:

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- Both the prover and verifier compute $g_1 \leftarrow g^r v$ and $h_1 \leftarrow v^r h \in \mathbb{G}$.
- The prover and verifier recursively engage in an interactive proof that $(\mathbb{G}, g_1, h_1, T/2) \in \mathcal{L}_{\mathrm{EXP}}$, namely that $h_1 = g_1^{(2^{T/2})} \in \mathbb{G}$.

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Breaking soundness of Pietrzak's VDF and non-necessity of the exponential LO assumption

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$$u^{(2^{T/2})} = u^{r+1} \iff r + 1 \equiv 2^{T/2} \mod l$$

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Remark: there are non-negligible $\phi(N)$ having factors in $2^{\Theta(poly(\lambda))}$. [BS13, Wei01]

Superpolynomial LO assumptions are sufficient

Definition

The *Subexponential Low Order assumption*. For any probabilistic polynomial time adversary $\mathcal{A}$, and for any $0 < \epsilon$, finding any element of subexponentially low order is hard:

$$\Pr\left[u^l = 1,\ u \notin \{1,-1\} : \begin{array}{c} \mathbb{G} \xleftarrow{\$} GGen(\lambda) \\ (u,l) \leftarrow \mathcal{A}(\mathbb{G}) \\ \text{and } l < 2^{\log^{1+\epsilon}(\lambda)} \end{array} \right] \leq \mathrm{negl}(\lambda) \quad (1)$$

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In the Forking-lemma we have that breaking Pietrzak's soundness yields an adversary against $f(\lambda)$-LO with success probability $(\epsilon^2/t) - (\epsilon/f(\lambda))$.
It seems that $f(\lambda)$ needs to be superpolynomial.

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It is a simple but technical proof which applies the general forking lemma [BN06].
Why can't we assume even weaker assumptions, like polynomial LO?
In the Forking-lemma we have that breaking Pietrzak's soundness yields an adversary against $f(\lambda)$-LO with success probability $(\epsilon^2/t) - (\epsilon/f(\lambda))$.
It seems that $f(\lambda)$ needs to be superpolynomial.
Maybe with other techniques we can prove even the sufficiency of polynomial LO assumptions?

Factoring$\equiv$ superpolynomial LOs???

Theorem

Let $\mathfrak{B}$ be a fixed integer. The Factoring assumption is reducible in polynomial time to the Low Order assumption for RSA-moduli when $\phi(N)$ has no prime factor between $\mathfrak{B}$ and $2^{poly(\lambda)}$ and $gcd(p-1, q-1) = 2$.

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We devise an efficient adversary $\mathcal{B}$ who can factor non-negligible fraction of random RSA moduli by using $\mathcal{A}$ as a subroutine.

Main idea of the reduction

Given (u, l) pair such that $u^l \equiv 1 \mod N$ and $2 \leq l \leq 2^{poly(\lambda)} \wedge u \neq -1$.

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Note that, whenever $ord_p(u) \neq ord_q(u)$, adversary $\mathcal{B}$ could factor $N = pq$ if l was smooth enough.The reason being that, adversary $\mathcal{B}$ raises u to the power of $\frac{l}{r}$ for all prime factors r of l, until modulo one prime factor of N, but not the other the order of u divides $\frac{l}{r}$, which can be detected by $0 < \gcd(u^{\frac{l}{r}} - 1 \bmod N, N) < N$, hence factoring the modulus N.

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Hence, towards our goal one thing that we need to show is that $ord_p(u) \neq ord_q(u)$ with non-negligible probability.

Partial reduction II. $\Pr[ord_p(u) \neq ord_q(u)] =?$

$$\Pr[gcd(\frac{p-1}{2}, \frac{q-1}{2}) = 1 | p, q \in_R \mathcal{O}(2^{poly(\lambda)})] \approx \frac{1}{\zeta(2)} = \frac{6}{\pi^2}$$

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So with constant probability $gcd(p-1, q-1) = 2$. Let's examine these two cases.

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Partial Reduction III. LO-smooth Integers [Wei01]

We need to establish the fraction of primes up to $N = \mathcal{O}(2^{s(\lambda)})$, that do not have prime factors in $(\mathfrak{B}, 2^{poly(\lambda)}]$.

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$$\Pr[\mathcal{B} \quad breaks \quad Factoring] \geq \frac{6}{\pi^2} q(\lambda)\mathcal{P}_{los}^2(\lambda)$$

A cool application of a HVZK by Goldberg et al [GRSB19]

Lemma

The map $x \to x^e \mod N$ is a permutation of Z_N^* if and only if $gcd(e, \phi(N)) = 1$.

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Let p_n denote the largest prime smaller than $\mathfrak{B}$. Then let $e = \prod_{i=1}^{n} p_i$, where p_i is the ith odd prime.
In a typical parameter setting $\lambda = 80, \mathcal{B} = 2^{10}$ the proof is $6.4KB$.
Generally the proof contains $\approx \lambda / \log 3$ group elements and requires the same amount of modular exponentiations.

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- If you could answer any of these questions then you can claim some cash! For more info, see https://rsa.cash

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Thanks!

Questions?