Cryptanalysis of ITRU

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Summary of the talk

1. Background

2. Main results

3. References
What is ITRU?

ITRU cryptosystem, is a public key cryptosystem and one of the known variants of NTRU (\(N^{th}\) Degree Truncated Polynomial Ring) cryptosystem, which was proposed in 2017 by Gaithuru, Salleh, and Mohamad [1].
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What is NTRU?

1996, Hoffstein, Pipher and Silverman [2] proposed a class of fast public key cryptosystems called NTRU cryptosystem (Classical NTRU cryptosystem). This cryptosystem has the following features:

- It is considered as a lattice-based public key cryptosystem, and it is the first asymmetric cryptosystem based on the polynomial ring $\mathbb{Z}[X]/(X^N-1)$. 

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- Its encryption and decryption procedures rely on a mixing system presented by polynomial algebra combined with a clustering principle based on elementary probability theory.

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- Its encryption and decryption procedures rely on a mixing system presented by polynomial algebra combined with a clustering principle based on elementary probability theory.
- From its lattice-based structure, the security of the NTRU cryptosystem is based on the hardness of solving the Closest Vector Problem (CVP).
The authors proposed this cryptosystem in the way that instead of working in a truncated polynomial ring, ITRU cryptosystem is based on the ring of integers. The parameters and the main steps of ITRU cryptosystem are as follows.

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- The value of $p$ is the small modulus (an integer).
- Random integers $f, g$ and $r$ are chosen such that $f$ is invertible modulo $p$. 

A prime $q$ is fixed satisfying $q > p \cdot r \cdot g + f \cdot m$, where $m$ is the representation of the message in decimal form (assuming that the message is comprised of English alphabetical characters based on ASCII conversion tables, that is the one with $a \rightarrow 97$).
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Structure of ITRU (continued (2))

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- The public key is consisted of \( h \) and \( q \) such that

\[
h \equiv p \cdot F_q \cdot g \pmod{q}.
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The encryption procedure is similar to the one applied in NTRU cryptosystem [2], one generates a random integer $r$ and computes

$$e \equiv r \cdot h + m \pmod{q}.$$
To get the plaintext from the ciphertext one determines

\[ a \equiv f \cdot e \pmod{q}. \]
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Recovering the message is done by computing
\[ F_p \cdot a \pmod{p}. \]
What is ITRU?

Comparison between NTRU and ITRU

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- ITRU has a successful message decryption, while the classical NTRU cryptosystem has a probability of decryption failure of $2^{-145}$. 
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- ITRU has a simple parameter selection algorithm comparing to the one of NTRU.
- ITRU has a successful message decryption, while the classical NTRU cryptosystem has a probability of decryption failure of $2^{-145}$.
- ITRU has a better security than NTRU. In fact, ITRU is based on integer rings as opposed to the lattice structure of the classical NTRU. The security of ITRU is based on the integer factorization problem.
Attacking/ Breaking ITRU cryptosystem

What are the used tools to attack this cryptosystem?

**Used tools to attack ITRU**

- From the construction, ITRU presents a substitution cipher, and one of the best effective attacks against the substitution ciphers in general presented by the frequency analysis technique.

- Therefore, by using a simple frequency analysis this attack is preformed with help of SageMath Software in which the plaintext is completely recovered only from the ciphertext and the public key with no need to have the private key.

We preform the attack with the following Steps:
Step 1: ITRU Cryptosystem Implementation

- To fix $q$ one needs a bound for the largest possible value of the representation, so here if one only uses the letters from 'A' to 'Z' and 'a' to 'z', then the maximum is 122 (ASCII conversion tables).
- In the following SageMath implementation we can use different bounds. With the following example, we use 255 to preform our implementation on the arbitrary message: Cryptanalysis.

**ITRU Implementation Input**

```python
1  s = 'Cryptanalysis'
2  pretty_print('The message is:', s)
3  r = 8
4  p = 1000
```
F = Set([k for k in range(2, 1000) if gcd(k, 1000) == 1])
f = F. random_element()
S = Set([2..1000])
g = S. random_element()
m = [ord(k) for k in s]
pretty_print(' The ASCII code of the message :', m)
q = next_prime(p * r * g + 255 * f)
F_p = (1/f) % p
F_q = (1/f) % q
h = (p * F_q * g) % q
pretty_print(' Large modulus :', q)
pretty_print(' Public key :', h)
pretty_print(' Private key pair :', (f, F_p))
Step 1: ITRU Cryptosystem Implementation (continued 3)

ITRU Implementation Input

```
18 e =[((r * h) + m[i])%q for i in [0..len(m) - 1]]
19 pretty_print(’ The encrypted message :’, e)
20 a = [(f * e[i])%q for i in [0..len(e) - 1]]
21 pretty_print(html (r’$f \cdot e \pmod{q}$ is: \$%s$’%latex(a)))
22 C = [((F_p * a[l])%p for l in [0..len(a) - 1]]
23 pretty_print(html(r’$F_p \cdot a \pmod{q}$ is : \$%s$’% latex(C)))
24 D = [chr(k) for k in C]
25 pretty_print(’ The original message :’, ‘’.join(D))
```

The output is as follows :
The message is: Cryptanalysis
The ASCII code of the message: [67, 114, 121, 112, 116, 97, 110, 97, 108, 121, 115, 105, 115]
Large modulus: 6186617
Public key: 180058
Private key pair: (73, 137)
The encrypted message: [1440531, 1440578, 1440585, 1440576, 1440580, 1440561, 1440574, 1440572, 1440585, 1440576, 1440580, 1440569, 1440579]
$f \cdot e \pmod{q}$ is: [6172891, 6176322, 6176833, 6176176, 6176468, 6175081, 6176030, 6175081, 6175884, 6176833, 6176395, 6175665, 6176395]
$F_p \cdot a \pmod{p}$ is: [67, 114, 121, 112, 116, 97, 110, 97, 108, 121, 115, 105, 115]
The original message: Cryptanalysis
We may perform our implementation with the bound $4999$ as the largest possible value of the representations on the arbitrary message: Implementation of ITRU cryptosystem.

Output

The message is: Implementation of ITRU cryptosystem
Large modulus: 3212849
Public key: 3160038
Private key pair: (177, 113)
The encrypted message: [2790434, 2790470, 2790473, 2790469, 2790462, 2790470, 2790462, 2790471, 2790477, 2790458, 2790477, 2790466, 2790472, 2790472, 2790393, 2790472]
Output

\[ \begin{align*}
2790463, & \quad 2790393, \quad 2790434, \quad 2790445, \quad 2790443, \quad 2790446, \\
2790393, & \quad 2790460, \quad 2790475, \quad 2790482, \quad 2790473, \quad 2790477, \\
2790472, & \quad 2790476, \quad 2790482, \quad 2790476, \quad 2790477, \quad 2790462, \\
2790470 & \quad \end{align*} \]

\[ f \cdot e \pmod{q} = [2340921, 2347293, 2347824, 2347116, 2345877, \\
2347293, 2345877, 2347470, 2348532, 2345169, 2348532, \\
2346585, 2347647, 2347470, 2333664, 2347647, 2346054, \\
2333664, 2340921, 2342868, 2342514, 2343045, 2333664, \\
2345523, 2348178, 2349417, 2347824, 2348532, 2347647, \\
2348355, 2349417, 2348355, 2348532, 2345877, 2347293] \]

\[ F_p \cdot a \pmod{p} = [73, 109, 112, 108, 101, 109, 101, 110, 116, 97, \\
116, 105, 111, 110, 32, 111, 102, 32, 73, 84, 82, 85, 32, 99, 114, \\

The original message: Implementation of ITRU cryptosystem
Performing the attack: This attack technique can be applied on any encrypted message using the ITRU cryptosystem, let us preform this technique on the following paragraph from the article describing ITRU cryptosystem [1] (without spaces):

‘The goal of this study is to present a variant of NTRU which is based on the ring of integers as opposed to using the polynomial ring with integer coefficients. We show that NTRU based on the ring of integers (ITRU), has a simple parameter selection algorithm, invertibility and successful message decryption. We describe a parameter selection algorithm and also provide an implementation of ITRU using an example. ITRU is shown to have successful message decryption, which provides more assurance of security in comparison to NTRU.’
Step 2: ITRU Plaintext Recovery

Remarks:

- If this paragraph is encrypted with the large modulus $q = 8170933$ and the public key $h = 3942626$ (this key is created in case of having 4999 as an upper bound for the representations), then the ciphertext starts as

  7028293, 7028313, 7028310, 7028312, 7028320, 7028306, 7028317, 7028320, 7028311, 7028325,....

- In fact, there are 32 different numbers appearing in the ciphertext; these are between 7028249 and 7028330.

- A simple frequency analysis with the SageMath function: frequency_distribution() provides the following data:
Obtained data:

\[
\begin{align*}
&(7028249, \ 0.00223713646532438), 
&(7028250, \ 0.00223713646532438), 
&(7028253, \ 0.00671140939597315), 
&(7028255, \ 0.00894854586129754), 
&(7028282, \ 0.00671140939597315), 
&(7028287, \ 0.00671140939597315), 
&(7028291, \ 0.0134228187919463), 
&(7028293, \ 0.0156599552572707), 
&(7028294, \ 0.0134228187919463), 
&(7028296, \ 0.00447427293064877), 
&(7028306, \ 0.0693512304250559), 
&(7028307, \ 0.00894854586129754), 
&(7028308, \ 0.0357941834451902), 
&(7028309, \ 0.0246085011185682), 
&(7028310, \ 0.109619686800895), 
&(7028311, \ 0.0223713646532438), 
&(7028312, \ 0.0290827740492170), 
&(7028313, \ 0.0380313199105145), 
&(7028314, \ 0.0850111856823266), 
&(7028317, \ 0.0313199105145414), 
&(7028318, \ 0.0290827740492170), 
&(7028319, \ 0.0648769574944072), 
&(7028320, \ 0.0738255033557047), 
&(7028321, \ 0.0313199105145414),
\end{align*}
\]
We see that the number 7028310 appears the most in the ciphertext. Therefore, 7028310 represents either 'e', 'a' or 't'. If it is 'e' = 101, then we apply the formula 

$$c_i - 7028209,$$

where $c_i$ represents the ciphertext blocks in the ASCII character code for all $i$. Thus, we get a sequence of numbers starting with 

84, 104, 101, 103, 111, 97, 108, 111, 102,....

which corresponds to the original plaintext.
References:


Thank you for your attention!