Multipartite Secret Sharing

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Secret sharing by groups

- Participants are in disjoint groups

$$P = P_1 \cup P_2 \cup \cdots \cup P_m.$$

  Sometimes we call them *departments*.
- Members of each group play the same role

  any participant can be replaced by any other member from the same group.
- Interesting only if there are few groups and several members in each group.
- Many unsolved problems

  even for the bipartite (two groups) case.

Definitions

- **Access structure**
  is the collection of qualified sets.
- **Complexity**
  is the maximal relative share size; it is at least 1
- **Ideal structures**
  are the ones with minimal complexity 1.
- **κ-ideal structures**
  are where the entropy method gives the lower bound 1 on the complexity (not necessarily ideal).

Theorem (Brickell & Davenport – informal)

κ-ideal access structures and matroids are in a one-to-one correspondence.

The "cap" theorem

Theorem (Csirmaz & Matúš & Padró – informal)

Multipartite κ-ideal structures are the same as "capped" structures.

1. For $m = 1$ "capped" structures are just the threshold ones.
2. Recipe to list / generate / recognize all such structures.
3. For $m = 1$, $m = 2$, and $m = 3$ "capped" structures are linearly representable.

Corollary

We have a complete description of all ideal tripartite access structures.

4. For $m = 4$ there is a κ-ideal structure which is not ideal.

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Capped structures

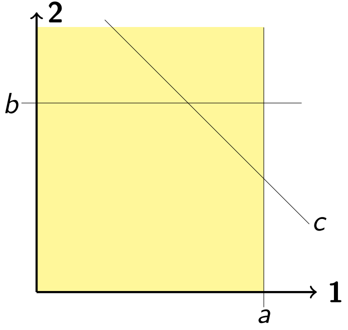
Capped structures

Each subset A of the groups (departments) has a **cap** $f(A)$.

Mnemonic: the power of the coalition A of some departments is limited to $f(A)$ counts.

Example:

Departments: $\{\mathbf{1},\mathbf{2}\}$; $f(\mathbf{1})=a$, $f(\mathbf{2})=b$, $f(\mathbf{12})=c$:



cap a for department $\mathbf{1}$

Capped structures

Each subset A of the groups (departments) has a **cap** $f(A)$.

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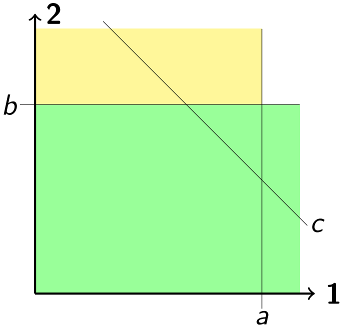


cap b for department $\mathbf{2}$

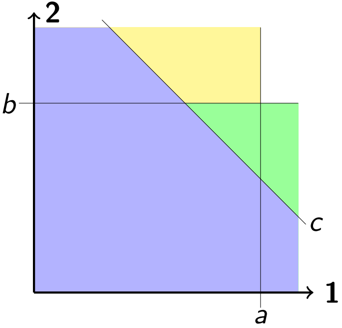
Capped structures

Each subset A of the groups (departments) has a **cap** $f(A)$.

Mnemonic: the power of the coalition A of some departments is limited to $f(A)$ counts.

Example:

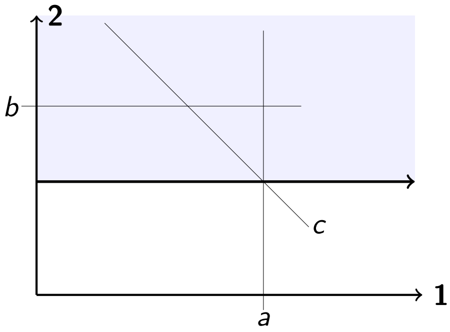
Departments: $\{\mathbf{1},\mathbf{2}\}$; $f(\mathbf{1})=a$, $f(\mathbf{2})=b$, $f(\mathbf{12})=c$:



cap c for both departments

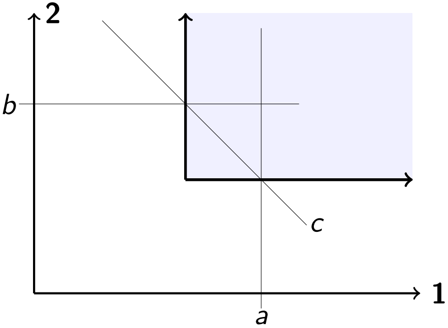
Hitting the cap c

As $f(\mathbf{1}) = a$, there must be at least $c{-}a$ members from group $\mathbf{2}$.



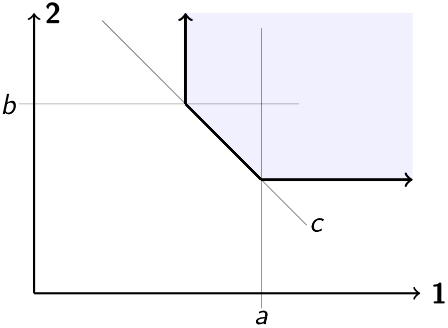
Hitting the cap c

As $f(\mathbf{1}) = a$, there must be at least $c - a$ members from group $\mathbf{2}$.
As $f(\mathbf{2}) = b$, there must be at least $c - b$ members from group $\mathbf{1}$.



Hitting the cap c

As $f(\mathbf{1}) = a$, there must be at least $c-a$ members from group $\mathbf{2}$.
As $f(\mathbf{2}) = b$, there must be at least $c-b$ members from group $\mathbf{1}$.
And at least c members from the two groups together.



The cap function f

Participants are in m disjoint groups (departments)

$$P = P_1 \cup P_2 \cup \cdots \cup P_m.$$

For each subset A of the groups $f(A)$ is the "cap" of A so that

1. $f(\emptyset) = 0$, otherwise $f(A)$ is a positive integer,
2. f is monotonic: $f(A) \le f(A \cup B)$,
3. f is submodular:

$$f(A) + f(B) \ge f(A \cap B) + f(A \cup B).$$

Otherwise there is no way to hit the the cap $f(A \cup B)$.

Capped structures

In secret sharing a capped access structure is defined by

- the set of participants P who are in m disjoint groups:

$$P = P_1 \cup P_2 \cup \cdots \cup P_m,$$

- the cap function $f(A)$ defined for each subset of the groups,
- an upward closed collection of group subsets:

$$\mathcal{A} = \{A_1, A_2. \ldots, A_t\}$$

(if $B \supset A_i$, then B is also in $\mathcal{A}$).

Definition ( **Capped access structure)**

A subset of participants is qualified if and only if they hit the cap $f(A_i)$ for some $A_i \in \mathcal{A}$.

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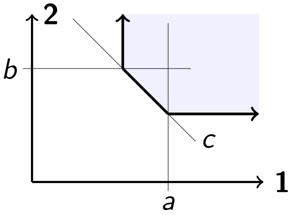
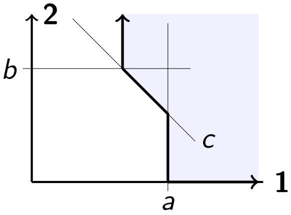
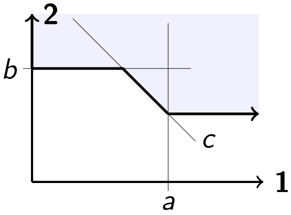
Case of two departments 1 and 2



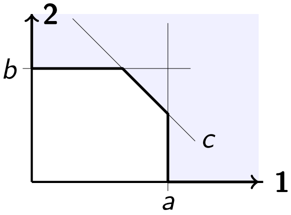
$\mathcal{A} = \{\mathbf{12}\}$



$\mathcal{A} = \{\mathbf{1}, \mathbf{12}\}$



$\mathcal{A} = \{\mathbf{2}, \mathbf{12}\}$



$\mathcal{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{12}\}$

Case of three departments $\mathbf{1}, \mathbf{2}, \mathbf{3}$

Seven cap values:

$$f(\mathbf{123})$$

$$f(\mathbf{12}) \quad f(\mathbf{13}) \quad f(\mathbf{23})$$

$$f(\mathbf{1}) \quad f(\mathbf{2}) \quad f(\mathbf{3})$$

numerous possibilities for $\mathcal{A}$, e.g.,

$$\mathcal{A} = \{\mathbf{1}, \mathbf{12}, \mathbf{13}, \mathbf{123}\},$$

each yielding an ideal structure.

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The C-M-P theorem, main points

- Σ is a κ-ideal multipartite structure with partition π.
- The matroid M corresponds to Σ (Brickell-Davenport thm).
- Factor M by the partition to get $N = M/\pi$, an integer polymatroid on the partition groups.
  Note: the ranks of N define the  values!
- M can be recovered from N uniquely (due to the multipartite symmetry).
- The secret defines a one-point extension of M (and of N) and it has rank 1. Qualified subsets are those whose rank is not increased by this extension.
- Such a one-point extension is characterized by a *modular cut* in the factor polymatroid N: this is the collection of all flats whose ranks do not increase – the collection $\mathcal{A}$ in the examples.

Tripartite κ-ideal structures are linear

- In the tripartite case the factor polymatroid N is integer and it is on three points. Such polymatroids are known to be linear.
- **If** the one-point extension of N (by the secret) is linear, **then** M is linear. There are arbitrary large vector space representations and one can choose many "generic" elements.
- An integer polymatroid on a, b, c, d is linearly representable if and only if it satisfies all instances of the Ingleton inequality
$$0 \le \text{ING}(a, b, c, d) = f(ab) + f(ac) + f(ad) + f(bc) + f(bd) - \\ -f(a) - f(b) - f(abc) - f(abd) - f(cd).$$
- In any polymatroid, $2 \cdot \text{ING}(a, b, c, d) + f(s) \ge 0$ where s is any of a, b, c, d.
- The one-point extension $N \cup \{s\}$ is integer with $f(s) = 1$. Thus $\text{ING}(a, b, c, d)$ is integer and at least $-1/2$, thus non-negative.

Contents

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Thank your for your attention