

Some Positive Radial Solutions of Elliptic Equation with a Gradient–Term*

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Abstract. The aim of this paper is to establish the existence and precise approximation of some positive radial solutions of the equation $\Delta u + \frac{1}{2}x \cdot \nabla u + K(|x|)u^p = 0$, $x \in \mathbb{R}^n$, for every $|x| \geq a > 0$. The errors of the approximations will be defined by the functions which can be sufficiently small for all x , $|x| \geq a$.

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1. Introduction

In this paper we study the problem

$$\begin{aligned} \Delta u + \frac{1}{2}x \cdot \nabla u + K(|x|)u^p &= 0, & x \in \mathbb{R}^n \setminus \{0\}, \\ u(x) &> 0, \\ u(x) &\rightarrow 0 \quad \text{as } x \rightarrow \infty, \end{aligned} \tag{1}$$

where $n \geq 2$, $p > 0$, $K \in C(\mathbb{R}^+, \mathbb{R})$, $\mathbb{R}^+ = (0, \infty)$. We consider radially symmetric solutions u which are functions of the variable $r = |x|$ only, so this problem becomes (in this case $' = d/dr$)

$$\begin{aligned} u'' + \left(\frac{n-1}{r} + \frac{1}{2}r \right) u' + K(r)u^p &= 0, & r \in \mathbb{R}^+, \\ u(r) &> 0, \\ u(r) &\rightarrow 0 \quad \text{as } r \rightarrow \infty. \end{aligned} \tag{2}$$

Many authors have studied the asymptotic behavior of positive radial solutions of the equation of the form (1). For example, in [5] and [8] is proved, respectively,

$$u(r) = O\left(r^{-n} \exp\left(-\frac{r^2}{4}\right)\right) \quad \text{as } r \rightarrow \infty, \quad 1 < p < \frac{n+2}{n-2}, \quad n \geq 3;$$

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$$u(r) = O\left(r^{(n-1)/2} \exp\left(-\frac{r^2}{8}\right)\right) \quad \text{as } r \rightarrow \infty, \quad \frac{n}{2} < \frac{p+1}{p-1},$$

and

$$u(r) = A \exp\left(-\frac{r^2}{4}\right) r^{k-n} [1 + (n-k)(k-2)r^{-2} + o(r^{-2})] \quad \text{as } r \rightarrow \infty.$$

In this paper we shall establish existence of positive radial solutions of (1), $n \geq 2$, satisfying conditions

$$u(r) = O\left(r^{-n} \exp\left(-\frac{r^2}{4}\right)\right) \quad \text{as } r \rightarrow \infty, \quad p > 0, \quad (3)$$

$$u(r) = O(r^{-n-s} \exp(-\beta r^2)) \quad \text{as } r \rightarrow \infty, \quad s \geq 0, \beta \geq \frac{1}{4}, 0 < p < 1, \quad (4)$$

$$u(r) = O(\exp(-\gamma r^h)) \quad \text{as } r \rightarrow \infty, \quad h \geq 2, \gamma > \frac{1}{2h}, 0 < p < 1, \quad (5)$$

and their precise approximations for every $r \geq a > 0$.

2. Results of this paper

Let $a, \alpha, \beta, \gamma, \eta, \theta, h \in \mathbb{R}^+, s \in \mathbb{R}_0^+ = [0, \infty)$.

Theorem 1. *Let*

$$0 < s \leq 2, \quad a > 2, \quad sa^s > 4n, \quad p > 0,$$

$$|K(r)| \leq 2^{-p-1} \alpha^{1-p} (sa^s r^{2-s} - 4n) r^{n(p-1)-2} \exp\left(\frac{1}{4} r^2 (p-1)\right), \quad \forall r \geq a.$$

Then the equation (1) has at least one positive radial solution $u(r)$ satisfying the conditions

$$\begin{aligned} |u(r) - \varphi(r)| &< a^s r^{-s} \varphi(r), \\ |u'(r) - \varphi'(r)| &< a^s \left(\frac{n+s}{r} + \frac{r}{2}\right) r^{-s} \varphi(r), \end{aligned}$$

for all $r \geq a$, where

$$\varphi(r) = \alpha r^{-n} \exp\left(-\frac{r^2}{4}\right). \quad (6)$$

Theorem 2. *Let*

$$0 < \theta < 1, \quad 0 < p < 1, \quad (7)$$

$$(1-\theta)^{1-p} \kappa(r) < -K(r) < (1+\theta)^{1-p} \kappa(r), \quad \forall r \geq a.$$

(i) *If*

$$\beta \geq \frac{1}{4}, \quad s \geq 0, \quad (8)$$

$$\kappa(r) = \left\{ (n+s)(2+s) + \left[2\beta(n+2s) - \frac{1}{2}(n+s) \right] r^2 + \beta(4\beta-1)r^4 \right\} \\ \times \alpha^{1-p} r^{(n+s)(p-1)-2} \exp(\beta r^2(p-1)), \quad r \geq a, \quad (9)$$

then the equation (1), for $0 < p < 1$, has at least one positive radial solution $u(r)$ satisfying the conditions

$$|u(r) - \varphi(r)| < \theta \varphi(r), \quad (10) \\ |u'(r) - \varphi'(r)| < \theta \left(2\beta r + \frac{n+s}{r} \right) \varphi(r),$$

for all $r \geq a$, where

$$\varphi(r) = \alpha r^{-n-s} \exp(-\beta r^2). \quad (11)$$

(ii) If

$$\gamma > \frac{1}{2h}, \quad h \geq 2, \quad a > 1, \quad 2\gamma h a^h - a^2 \geq 2(n+h-2), \quad (12)$$

$$\kappa(r) = \gamma h \left(\gamma h r^h - \frac{1}{2} r^2 + 2 - n - h \right) \alpha^{1-p} r^{h-2} \exp(\gamma r^h(p-1)), \quad r \geq a, \quad (13)$$

then the equation (1), for $0 < p < 1$, has at least one positive radial solution $u(r)$ satisfying the condition (10), and

$$|u'(r) - \varphi'(r)| < \theta \gamma h r^{h-1} \varphi(r),$$

for all $r \geq a$, where

$$\varphi(r) = \alpha \exp(-\gamma r^h). \quad (14)$$

Theorem 3. Let

$$0 < s < 2, \quad \eta > 0, \quad \eta s a^{2-s} + 1 > (1 + 4n\eta a^{-s})^p, \quad p > 0,$$

$$\kappa(r) < -K(r) \leq (1 + \eta s r^{2-s})(1 + 4n\eta r^{-s})^{-p} \kappa(r), \quad \forall r \geq a,$$

where

$$\kappa(r) = 2n\alpha^{1-p} r^{n(p-1)-2} \exp\left(\frac{1}{4} r^2(p-1)\right).$$

Then the equation (1) has at least one positive radial solution $u(r)$ satisfying the conditions

$$\varphi(r) < u(r) < (1 + 4n\eta r^{-s})\varphi(r), \\ \varphi'(r) - 4n\eta \left(\frac{n+s}{r} + \frac{r}{2} \right) r^{-s} \varphi(r) < u'(r) < \varphi'(r),$$

for all $r \geq a$, where function $\varphi(r)$ is defined by (6).

Theorem 4. *Let*

$$\eta > 0, \quad 0 < p < 1,$$

$$\kappa(r) < -K(r) < (1 + \eta)^{1-p} \kappa(r), \quad \forall r \geq a.$$

(i) *If (8) holds true and if function $\kappa(r)$ is defined by (9), then the equation (1), for $0 < p < 1$, has at least one positive radial solution $u(r)$ satisfying the conditions*

$$\varphi(r) < u(r) < (1 + \eta)\varphi(r), \quad (15)$$

$$(1 + \eta)\varphi'(r) < u'(r) < \varphi'(r), \quad (16)$$

for all $r \geq a$, where function $\varphi(r)$ is defined by (11).

(ii) *If (12) holds true and if function $\kappa(r)$ is defined by (13), then the equation (1), for $0 < p < 1$, has at least one positive radial solution $u(r)$ satisfying the conditions (15) and (16), $\forall r \geq a$, where function $\varphi(r)$ is defined by (14).*

Theorem 5. *Let*

$$0 < s \leq 2, \quad a \geq 2\sqrt{\frac{2n}{s}}, \quad p > 0,$$

$$|K(r)| < 2n\alpha^{1-p} r^{n(p-1)-2} \exp\left(\frac{1}{4} r^2(p-1)\right), \quad \forall r \geq a.$$

Then the equation (1) has at least one positive radial solution $u(r)$ satisfying the conditions

$$(1 - a^s r^{-s})\varphi(r) < u(r) < \varphi(r),$$

$$\varphi'(r) < u'(r) < \varphi'(r) + a^s \left(\frac{n+s}{r} + \frac{r}{2}\right) r^{-s} \varphi(r),$$

for all $r \geq a$, where function $\varphi(r)$ is defined by (6).

Theorem 6. *Let (7) hold true and*

$$(1 - \theta)^{1-p} \kappa(r) < -K(r) < \kappa(r), \quad \forall r \geq a.$$

(i) *If (8) holds true and if function $\kappa(r)$ is defined by (9), then the equation (1), for $0 < p < 1$, has at least one positive radial solution $u(r)$ satisfying the conditions*

$$(1 - \theta)\varphi(r) < u(r) < \varphi(r), \quad (17)$$

$$\varphi'(r) < u'(r) < (1 - \theta)\varphi'(r), \quad (18)$$

for all $r \geq a$, where function $\varphi(r)$ is defined by (11).

(ii) *If (12) holds true and if function $\kappa(r)$ is defined by (13), then the equation (1), for $0 < p < 1$, has at least one positive radial solution $u(r)$ satisfying the conditions (17) and (18), $\forall r \geq a$, where function $\varphi(r)$ is defined by (14).*

3. Proof of Theorems 1–6

We shall study the equation (1) or (2) by means of the equivalent system

$$u' = v, \quad v' = -\left(\frac{n-1}{r} + \frac{1}{2}r\right)v - K(r)u^p, \quad r > 0. \quad (19)$$

According to known theorems, the Cauchy problem for the system (19) has the unique solution in $\Omega = \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+$.

Let us consider the behavior of integral curves $(u(r), v(r), r)$ of (19) with respect to the set Ω and the set of the form

$$\omega = \{(u, v, r) \in \Omega \mid \varphi_1(r) < u < \varphi_2(r), \psi_1(r) < v < \psi_2(r), r > a\},$$

where φ_i and ψ_i are functions such that $\varphi_i, \psi_i \in C^1(\mathbb{R}^+, \mathbb{R})$, and

$$0 < \varphi_1(r) < \varphi_2(r), \quad \psi_1(r) < \psi_2(r), \quad r > a.$$

The boundary surfaces of ω are

$$U_i = \{T \in \text{Cl}\omega \mid \Phi_i := (-1)^i(u - \varphi_i(r)) = 0\},$$

$$V_i = \{T \in \text{Cl}\omega \mid \Psi_i := (-1)^i(v - \psi_i(r)) = 0\}, \quad i = 1, 2.$$

Let us denote the tangent vector field to an integral curve $(u(r), v(r), r)$ of (19) by X , i.e.,

$$X = \left(v, -\left(\frac{n-1}{r} + \frac{1}{2}r\right)v - Ku^p, 1\right).$$

The vectors $\nabla\Phi_i$ and $\nabla\Psi_i$, $i = 1, 2$, are the external normals on surfaces U_i and V_i , respectively,

$$\nabla\Phi_i = ((-1)^i, 0, (-1)^{i+1}\varphi'_i(r)), \quad \nabla\Psi_i = (0, (-1)^i, (-1)^{i+1}\psi'_i(r)), \quad i = 1, 2.$$

By means of scalar products $P_i = (\nabla\Phi_i, X)$ on U_i , and $Q_i = (\nabla\Psi_i, X)$ on V_i , $i = 1, 2$, we shall establish the behavior of integral curves of the system (19) with respect to sets ω and Ω .

For proofs of Theorems 1–6 we shall consider the case

$$\psi_1(r) = \varphi'_2(r), \quad \psi_2(r) = \varphi'_1(r).$$

Then we have

$$\begin{aligned} P_1 &\equiv 0 \quad \text{on } L = U_1 \cap V_2, \\ P_1 &= -v + \varphi'_1 > -\psi_2 + \varphi'_1 \equiv 0 \quad \text{on } U_1 \setminus L, \\ P_2 &\equiv 0 \quad \text{on } M = U_2 \cap V_1, \\ P_2 &= v - \varphi'_2 > \psi_1 - \varphi'_2 \equiv 0 \quad \text{on } U_2 \setminus M, \end{aligned}$$

$$\begin{aligned} u' &= v = \psi_2 = \varphi_1' && \text{on } L, \\ u'' &= v' = -\left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi_1' - K\varphi_1^p > \varphi_1'' && \text{on } L, \end{aligned} \quad (20)$$

$$\begin{aligned} u' &= v = \psi_1 = \varphi_2' && \text{on } M, \\ u'' &= v' = -\left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi_2' - K\varphi_2^p < \varphi_2'' && \text{on } M, \end{aligned} \quad (21)$$

$$Q_1 = \left(\frac{n-1}{r} + \frac{1}{2}r\right)v + Ku^p + \psi_1' = \left(\frac{n-1}{r} + \frac{1}{2}r\right)\psi_1 + \psi_1' + Ku^p \quad \text{on } V_1,$$

$$Q_2 = -\left(\frac{n-1}{r} + \frac{1}{2}r\right)v - Ku^p - \psi_2' = -\left(\frac{n-1}{r} + \frac{1}{2}r\right)\psi_2 - \psi_2' - Ku^p \quad \text{on } V_2.$$

The estimates (20) and (21) follow from the corresponding estimates $Q_2 > 0$ on V_2 and $Q_1 > 0$ on V_1 , respectively.

Now, in case of Theorem 1 we have

$$\varphi_i(r) = \varphi(r) + (-1)^i \rho(r), \quad \psi_i(r) = \varphi'(r) - (-1)^i \rho'(r), \quad i = 1, 2, \quad (22)$$

where $\varphi(r) = \alpha r^{-n} \exp(-r^2/4)$, $\rho(r) = a^s r^{-s} \varphi(r)$, and

$$Q_1 \geq \varphi'' + \rho'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)(\varphi' + \rho') - |K|(2\varphi)^p > 0 \quad \text{on } V_1,$$

$$Q_2 \geq -\varphi'' + \rho'' - \left(\frac{n-1}{r} + \frac{1}{2}r\right)(\varphi' - \rho') - |K|(2\varphi)^p > 0 \quad \text{on } V_2.$$

In case of Theorem 2 we have (22), with $\rho(r) = \theta\varphi(r)$ (where $\varphi(r) = \alpha r^{-n-s} \exp(-\beta r^2)$ in case (i), and $\varphi(r) = \alpha \exp(-\gamma r^h)$ in case (ii)), and

$$Q_1 \geq (1 + \theta) \left[\varphi'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' \right] + K(1 + \theta)^p \varphi^p > 0 \quad \text{on } V_1,$$

$$Q_2 \geq -(1 - \theta) \left[\varphi'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' \right] - K(1 - \theta)^p \varphi^p > 0 \quad \text{on } V_2.$$

Consequently, $E = U_1 \cup U_2 \cup V_1 \cup V_2$ is a set of points of strict exit of integral curves of (19) with respect to sets Ω and ω . Moreover, the Cauchy problem for the system (19) has the unique solution in Ω . Hence, according to the retraction method, the system (19) has at least one solution $(u(r), v(r))$ which satisfies the conditions

$$\varphi_1(r) < u(r) < \varphi_2(r), \quad \psi_1(r) < v(r) < \psi_2(r), \quad \forall r \geq a, \quad (23)$$

or

$$|u(r) - \varphi(r)| < \rho(r), \quad |v(r) - \varphi'(r)| < -\rho'(r), \quad \forall r \geq a,$$

and that means that Theorems 1 and 2 hold true.

Let us consider now Theorems 3 and 4 analogously. In case of Theorem 3 we have

$$\varphi_i(r) = \varphi(r) - (1-i)\rho(r), \quad \psi_i(r) = \varphi'(r) + (2-i)\rho'(r), \quad i = 1, 2, \quad (24)$$

where $\varphi(r) = \alpha r^{-n} \exp(-r^2/4)$, $\rho(r) = 4n\eta r^{-s}\varphi(r)$, and

$$\begin{aligned} Q_1 &\geq \varphi'' + \rho'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)(\varphi' + \rho') + K(\varphi + \rho)^p > 0 \quad \text{on } V_1, \\ Q_2 &\geq -\varphi'' - \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' - K\varphi^p > 0 \quad \text{on } V_2. \end{aligned} \quad (25)$$

In case of Theorem 4 we have (24), with $\rho(r) = \eta\varphi(r)$ (the function φ is defined in Theorem 4), and

$$Q_1 \geq (1+\eta) \left[\varphi'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' \right] + K(1+\eta)^p \varphi^p > 0 \quad \text{on } V_1,$$

and (25). Now we can note that the corresponding set $E = U_1 \cup U_2 \cup V_1 \cup V_2$ is a set of points of strict exit of integral curves of (19) with respect to sets Ω and ω . Hence, the system (19) has at least one solution $(u(r), v(r))$ satisfying the conditions (23), with functions φ_i and ψ_i defined by (24). This means that Theorems 3 and 4 hold true.

Finally, in case of Theorems 5 and 6 we set

$$\varphi_i(r) = \varphi(r) - (2-i)\rho(r), \quad \psi_i(r) = \varphi'(r) + (1-i)\rho'(r), \quad i = 1, 2,$$

where $\rho(r) = a^s r^{-s}\varphi(r)$ in case of Theorem 5, and $\rho(r) = \theta\varphi(r)$ in case of Theorem 6. (The function φ is defined in Theorems 5 and 6.) Moreover, it is sufficient to notice that, in case of Theorem 5,

$$\begin{aligned} Q_1 &\geq \varphi'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' - |K|\varphi^p > 0 \quad \text{on } V_1, \\ Q_2 &\geq -\varphi'' + \rho'' - \left(\frac{n-1}{r} + \frac{1}{2}r\right)(\varphi' - \rho') - |K|\varphi^p > 0 \quad \text{on } V_2, \end{aligned}$$

and, in case of Theorem 6,

$$\begin{aligned} Q_1 &\geq \varphi'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' + K\varphi^p > 0 \quad \text{on } V_1, \\ Q_2 &\geq -(1-\theta) \left[\varphi'' + \left(\frac{n-1}{r} + \frac{1}{2}r\right)\varphi' \right] - K(1-\theta)^p \varphi^p > 0 \quad \text{on } V_2. \end{aligned}$$

This completes the proof. ■

4. Remark on the approximation of solutions

In this paper we established existence of some positive radial solutions of (1) satisfying conditions (3), (4) and (5), and obtained precise estimates of behavior of these solutions for every $r \geq a > 0$. In Theorems 1–6 we have given the complete answer to the approximation of solutions $u(r)$ whose existence has been established. The errors of the approximations for solutions $u(r)$ and the first derivative $u'(r)$ are defined by the function $\rho(r)$ which can be sufficiently small $\forall r \geq a$. The function ρ is of the form

$$\rho(r) = \mu r^{-n-s} \exp(-\beta r^2), \quad s \geq 0, \beta \geq \frac{1}{4},$$

or

$$\rho(r) = \mu \exp(-\gamma r^h), \quad \gamma > \frac{1}{2h}, h \geq 2.$$

These functions tend to zero as $r \rightarrow \infty$ and can be sufficiently small $\forall r \geq a > 0$, because parameter $\mu > 0$ can be arbitrarily small.

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