Stable Algorithm for Calculating with $q$–Splines

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Abstract. We are using a technique to calculate with Chebyshevian splines of order $\leq 4$, based on the known derivative formula for Chebyshevian splines and an Oslo type algorithm, to produce simple formulæ for $qB$-splines developed by Kulkarni and Laurent. Starting with the known fact that local basis for $q$-splines of order 3 and 4 can be evaluated by making positive linear combinations of less smooth, one order higher polynomial B-splines, we deduce a simple and stable algorithm for such splines.

It is an interesting fact in itself, that the coefficients in such linear combinations are discrete Chebyshevian splines, and therefore make a partition of unity. The same is true for $qB$-splines themselves.

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1. Introduction

The notion of $q$-spline has an origin in the beam theory. Consider a simply supported beam with supports $\{(x_i, f_i)\}_{i=0}^{k+1}$: then the deflection of the beam between successive supports is the solution $s(x)$ to the differential equation $[E \cdot I \cdot D^2]s = M$. Here $E$ denotes Young’s modulus of elasticity, $I$ is the cross-sectional moment of inertia, and $M$ is the bending moment. We suppose that $E \cdot I = 1/q$, $q > 0$, where $q$ and, under assumption of weightlessness, $M$, are piecewise linear continuous functions with break points at the supports. Differentiating the above equation twice, we arrive at the two-point boundary value problem on $[x_i, x_{i+1}]$, for $i = 0, \ldots, k$:

$$D^2 1/q D^2 s = 0, \quad s(x_i) = f_i, \quad s(x_{i+1}) = f_{i+1}, \quad s''(x_i) = s''_{i}, \quad s(x_{i+1}) = s''_{i+1},$$

where $s''_{i}$ and $s''_{i+1}$ are chosen so as to ensure that $s \in C^2[x_0, x_{k+1}]$. Such a function $s$ is called a $q$-spline.

The aim is to construct a stable algorithm for calculating with $q$-splines, based on the known derivative formula for Chebyshevian splines and an Oslo type algorithm. To

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this end, we will use one special *Canonical Complete Chebyshev* (CCT)–system, and some general techniques from the *Chebyshevian Spline Theory*. Instead of calculating directly with \(q\)-splines, we propose to write such splines as linear combinations of locally supported ones, which can be expressed as linear combinations of ordinary polynomial B-splines.

Since their introduction by Kulkarni and Laurent [2], \(q\)-splines have been used in various applications in computer aided geometric design.

2. Chebyshev theory preliminaries

Let \(t_1 \leq t_2 \leq t_3 \leq a = t_4 < t_5 < \cdots < t_{k+5} = b \leq t_{k+6} \leq t_{k+7} \leq t_{k+8}\) be an extended partition of the interval \([a, b]\), and let \(q\) be a continuous, piecewise linear function defined by

\[
q(x)_{[t_i, t_{i+1}]} = \frac{q_{i+1} - q_i}{h_i} (x - t_i) + q_i,
\]

where \(h_i = t_{i+1} - t_i\), and \(q_1 > 0\). Consider the CCT–system \(\{u_1, u_2, u_3, u_4\}\):

\[
\begin{align*}
  u_1(x) &= 1, & u_2(x) &= \int_a^x ds_2, \\
  u_3(x) &= \int_a^x ds_2 \int_a^x q(s_3) ds_3, & u_4(x) &= \int_a^x ds_2 \int_a^x q(s_3) ds_3 \int_a^x ds_4.
\end{align*}
\]

We wish to construct a local basis for the spline space spanned piecewisely by these functions, that is, B-splines in \(S(4, m, d\sigma, \Delta)\), where \(m\) is the *multiplicity vector*, \(m = (1, \ldots, 1)^T\), \(d\sigma := (ds_2, q(s_3) ds_3, ds_4)^T\) is the *measure vector*, and \(\Delta = \{t_i\}_{i=1}^{k+8}\) (see [5] for details of the notation). An important role is played by the associated generalized derivatives:

\[
L_{1, d\sigma} = D, \quad L_{2, d\sigma} = \frac{1}{q} D^2, \quad L_{3, d\sigma} = D \frac{1}{q} D^2, \quad L_{4, d\sigma} = D^2 \frac{1}{q} D^2.
\]

To begin with, we focus on the *reduced system* \(\{u_{1,1}, u_{1,2}, u_{1,3}\}\), spanning the space \(S(3, m, d\sigma^{(1)}, \Delta)\), \(d\sigma^{(1)} \equiv (q(s_3) ds_3, ds_4)^T\), on each interval. The CCT–system is:

\[
\begin{align*}
  u_{1,1}(x) &= 1, & u_{1,2}(x) &= \int_a^x q(s_3) ds_3, & u_{1,3}(x) &= \int_a^x q(s_3) ds_3 \int_a^x ds_4.
\end{align*}
\]

Next consider less smooth B-splines \(\tilde{T}_j^3\) from the space \(S(3, \tilde{m}, d\sigma^{(1)}, \Delta)\), with the multiplicity vector \(\tilde{m} = (2, \ldots, 2)^T\) on the same knot sequence. For the fixed index \(i\), we denote the points in the new extended partition as \(t_i = \tilde{t}_{r-1} < \tilde{t}_r < \tilde{t}_{r+1}\), and polynomial B-splines on this partition simply as \(\tilde{B}_j^3\). It is easily seen from the definition of the basis (1) that we can write \(\tilde{T}_j^3\) as

\[
\tilde{T}_{r-1}^3(x) = \sum_{j=s=3}^r a_{r-1,j} \tilde{B}_j^3(x),
\]

(2)
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\[ \tilde{T}_r^3(x) = \sum_{j=s-3}^{s+3} \alpha_{r,j} \tilde{B}_j^4(x), \]  

(3)

with $\tilde{B}_j^4 \in S(4, \tilde{m}, d\lambda, \Delta)$, where $\tilde{m} = (3, \ldots, 3)^T$ on the same knot sequence $\Delta$, and $d\lambda$ is the measure vector determined by Lebesgue measures only. Points in this partition will be denoted as $t_i = t_{s-2} = t_{s-1} = t_s < t_{s+1}$.

We will use the following general theorem, which is a generalization to Chebyshevian splines of the derivative formula for polynomial B-splines [1, 4]:

**Theorem 1.** Let $L_{1, d\sigma}$ be the first generalized derivative with respect to CCT–system $S(n, d\sigma)$, and let the multiplicity vector $m = (m_1, \ldots, m_k)^T$ satisfy $m_i < n - 1$ for $i = 1, \ldots, k$. Then for $x \in [a, b]$ and $i = 1, \ldots, n + \sum_{i=1}^k m_i$, the following derivative formula holds:

\[ L_{1, d\sigma} T_{i, d\sigma}^n(x) = \frac{T_{i+1, d\sigma}^{n-1}(x)}{C_{n-1}(i+1)} - \frac{T_{i, d\sigma}^{n-1}(x)}{C_{n-1}(i+1)}, \]

(4)

where

\[ C_{n-1}(i) := \int_{t_i}^{t_{i+n-1}} T_{i, d\sigma}^{n-1} \, d\sigma_2, \]

(5)

with measure vectors

\[ d\sigma = (d\sigma_2(\delta), \ldots, d\sigma_n(\delta))^T \in \mathbb{R}^{n-1}, \quad d\sigma^{(1)} := (d\sigma_3(\delta), \ldots, d\sigma_n(\delta))^T \in \mathbb{R}^{n-2}, \]

for all measurable $\delta$.

3. Construction of the local basis for $q$-spline spaces

It is obvious from (1) and Theorem 1 that

\[ \tilde{L}_1 \tilde{T}_r^3 = \frac{\tilde{B}_r^2(x)}{C_2(r-1)} - \frac{\tilde{B}_r^2(x)}{C_2(r)}, \]

where $\tilde{L}_1 := L_{1, d\sigma^{(1)}} = \frac{1}{q} D$ is the generalized derivative for the reduced CCT–system, and

\[ \tilde{C}_2(j) = \int_{t_j}^{t_{j+2}} \tilde{B}_2(t) \, dt. \]

In particular,

\[ \tilde{C}_2(r-1) = \frac{(2q_i + q_{i+1})h_i}{6}, \quad \tilde{C}_2(r) = \frac{(q_i + 2q_{i+1})h_i}{6}. \]

From the simple properties of B-splines:

\[ \tilde{T}_r^3(t_i) = 0, \quad \tilde{L}_1 \tilde{T}_r^3(t_i^+) = \frac{1}{C_2(r-1)}, \]

\[ \tilde{T}_r^3(t_{i+1}) = 0, \quad \tilde{L}_1 \tilde{T}_r^3(t_{i+1}^-) = \frac{1}{C_2(r)}. \]
we get the coefficients in (2):

\[ a_{r-1,s-3} = a_{r-1,s} = 0, \]
\[ a_{r-1,s-2} = \frac{2q_i}{2q_i + q_{i+1}}, \quad a_{r-1,s-1} = \frac{2q_{i+1}}{q_i + 2q_{i+1}}, \]

whence

\[ \tilde{T}^3_{r-1}(x) = \frac{2q_i}{2q_i + q_{i+1}} \hat{B}^4_{s-2}(x) + \frac{2q_{i+1}}{q_i + 2q_{i+1}} \hat{B}^4_{s-1}(x). \] (6)

To calculate \( \tilde{T}^3_r \), we use the equations

\[ \tilde{T}^3_r(t_i) = \bar{L}_1 \tilde{T}^3_r(t_i) = \tilde{T}^3_r(t_{i+2}) = 0, \]
\[ \tilde{T}^3_r(t_{i+1}) = 1, \quad {\bar{L}_1} \tilde{T}^3_r(t_{i+1}) = \frac{1}{C_2(r)}, \quad L_1 \tilde{T}^3_r(t_{i+1}) = -\frac{1}{C_2(r+1)}, \]

to get the coefficients in (3):

\[ a_{r,s-3} = a_{r,s-2} = a_{r,s+2} = a_{r,s+3} = 0, \]
\[ a_{r,s} = 1, \quad a_{r,s-1} = \frac{q_i}{2q_i + q_{i+1}}, \quad a_{r,s+1} = \frac{q_{i+2}}{2q_{i+1} + q_{i+2}}, \]

and, finally

\[ \tilde{T}^3_r(x) = \frac{q_i}{q_i + 2q_{i+1}} \hat{B}^2_{s-1}(x) + \hat{B}_s^4(x) + \frac{q_{i+2}}{2q_{i+1} + q_{i+2}} \hat{B}^4_{s+1}(x). \] (7)

By integrating (4) in Theorem 1, we can further calculate splines of higher order. We start with the equation

\[ \tilde{T}^3_{r-1}(x) = \frac{1}{C_3(r-1)} \int_{t_{r-1}}^x \tilde{T}^3_{r-1}(t) \, dt - \frac{1}{C_3(r)} \int_{t_r}^x \tilde{T}^3_{r}(t) \, dt. \] (8)

It is easy to see from (6) and (7) that

\[ C_3(r-1) = \frac{h_i}{4} \left[ \frac{2q_i}{2q_i + q_{i+1}} + \frac{2q_{i+1}}{q_i + 2q_{i+1}} \right], \]
\[ C_3(r) = \frac{1}{4} \left[ \frac{q_i h_i}{q_i + 2q_{i+1}} + h_i + h_{i+1} + \frac{q_{i+2}h_{i+1}}{2q_{i+1} + q_{i+2}} \right]. \]

From (8), by using (6), (7), and the well known recurrence for integrals of the polynomial B-splines

\[ \int_{-\infty}^x B^r_i(t) \, dt = \frac{t_{i+n} - t_i}{n} \sum_{j=1}^{i+n-1} B^r_j(x), \]
where \( \{t_i\} \) is now any extended partition, we obtain (by looking separately at \( x \) from each of the subintervals \([t_i, t_{i+1}]\) and \([t_{i+1}, t_{i+2}]\)), that

\[
\tilde{T}_{r-1}^4(x) = \frac{1}{C_3(r-1) 2q_i + q_{i+1}} \frac{h_i}{4} \tilde{B}_{s-2}^5(x)
\]

\[
+ \frac{1}{C_3(r)} \left( \frac{h_i + h_{i+1}}{4} + \frac{q_{i+2}}{2q_{i+1} + q_{i+2}} \frac{h_{i+1}}{4} \right) \tilde{B}_{s-1}^5(x)
\]

\[
+ \frac{1}{C_3(r)} \frac{q_{i+2}}{2q_{i+1} + q_{i+2}} \frac{h_{i+1}}{4} \tilde{B}_s^5(x).
\]

In the same way,

\[
\tilde{T}_r^4(x) = \frac{1}{C_3(r+1)} \frac{q_i}{4} \tilde{B}_{s-1}^5(x)
\]

\[
+ \frac{1}{C_3(r+1)} \left( \frac{q_i}{4} + \frac{h_i + h_{i+1}}{4} \right) \tilde{B}_s^5(x)
\]

\[
+ \frac{1}{C_3(r+1)} \frac{2q_{i+2}}{h_{i+1}} \frac{q_{i+2}}{4} \tilde{B}_{s+1}^5(x).
\]

The following lemma and theorem are connecting general T-splines of orders 3 and 4 with less smooth ones, which are simpler to calculate, and (in the case of \( q \)-splines) have already been constructed by the explicit formulæ. Proofs are omitted and may be found in [4].

**Lemma 1.** Let \( T_{i,d\sigma}^3 \in S(3, m, d\sigma^{(1)}, \Delta) \) be a Chebyshevian B-spline of order 3 associated with the multiplicity vector \( m = (1, \ldots, 1)^T \), and let us assume that \( \tilde{T}_{i,d\sigma}^3 \in S(3, \tilde{m}, d\sigma^{(1)}, \Delta) \) are B-splines associated with multiplicity vector \( \tilde{m} = (2, \ldots, 2)^T \) on the same knot sequence. If \( \{t_i, \ldots, t_{k+6}\} \) and \( \{\tilde{t}_i, \ldots, \tilde{t}_{k+6}\} \) are the associated extended partitions, and \( r \) an index such that \( t_r < \tilde{t}_{r+1} \), then for \( i = 1, \ldots, k+3 \):

\[
T_{i,d\sigma}^3 = T_{i,d\sigma}^3(t_{i+1}) \tilde{T}_{r+1,d\sigma}^3(t_r) + T_{i,d\sigma}^3(t_{i+1}) T_{r+1,d\sigma}^3(t_r) + T_{i,d\sigma}^3(t_{i+1}) T_{r+2,d\sigma}^3(t_r).
\]

**Theorem 2.** Let \( T_{i,d\sigma}^4 \in S(4, m, d\sigma, \Delta) \), \( \tilde{T}_{i,d\sigma}^4 \in S(4, \tilde{m}, d\sigma, \Delta) \), the multiplicity vectors \( m, \tilde{m} \) being as in Lemma 1. Then positive \( \delta_i^4(j) \) exist such that

\[
T_{i,d\sigma}^4 = \sum_{j=r}^{r+3} \delta_i^4(j) \tilde{T}_{j,d\sigma}^4,
\]

where \( r = r_i \) satisfies \( t_r = \tilde{t}_{r+1} \). Let the extended partitions be \( \{t_1, \ldots, t_{k+6}\} \) and \( \{\tilde{t}_1, \ldots, \tilde{t}_{k+8}\} \). Then \( \delta_i^4(j), j = r, \ldots, r+3 \), are determined by the formulæ:

\[
\delta_i^4(r) = \frac{T_{i,d\sigma}^3(t_{i+1}) \tilde{C}(r)}{T_{i,d\sigma}^3(t_{i+1}) \tilde{C}(r) + \tilde{C}(r+1) + T_{i,d\sigma}^3(t_{i+2}) \tilde{C}(r+2)},
\]

\[
\delta_i^4(r+1) = \frac{T_{i,d\sigma}^3(t_{i+1}) \tilde{C}(r) + \tilde{C}(r+1)}{T_{i,d\sigma}^3(t_{i+1}) \tilde{C}(r) + \tilde{C}(r+1) + T_{i,d\sigma}^3(t_{i+2}) \tilde{C}(r+2)}.
\]
\[ \delta_4^r(r + 2) = \frac{T^3_{i+1, d\sigma_1}(t_{i+3}) \tilde{C}(r + 4) + C(r + 3)}{T^3_{i+1, d\sigma_1}(t_{i+2}) \tilde{C}(r + 2) + C(r + 3) + T^3_{i+1, d\sigma_1}(t_{i+3}) \tilde{C}(r + 4)}, \]

\[ \delta_4^r(r + 3) = \frac{T^3_{i+1, d\sigma_1}(t_{i+3}) \tilde{C}(r + 4)}{T^3_{i+1, d\sigma_1}(t_{i+2}) \tilde{C}(r + 2) + C(r + 3) + T^3_{i+1, d\sigma_1}(t_{i+3}) \tilde{C}(r + 4)}, \]

where, as in (5)

\[ \tilde{C}(i) = \int_{\text{supp}} T^3_{i, d\sigma_1} \, d\sigma. \]

To use Lemma 1 and Theorem 2, it remains to calculate \( T^3_3(t_{i+1}) \) and \( T^3_3(t_{i+2}) \).

The derivative formula in Theorem 1 implies

\[ T^3_3(x) = \frac{1}{C_2(i)} \int_{t_i}^x B^3_1(t) q(t) \, dt - \frac{1}{C_2(i+1)} \int_{t_{i+1}}^x B^3_{i+1}(t) q(t) \, dt, \]

where

\[ C_2(i) = \int_{t_i}^{t_{i+2}} B^3_1(t) q(t) \, dt = \frac{1}{6} \left[ (q_i + 2q_{i+1})h_i + (2q_{i+1} + q_{i+2})h_{i+1} \right]. \]

One finds easily that

\[ T^3_3(t_{i+1}) = \frac{h_i(q_i + 2q_{i+1})}{6C^2_2(i)}, \quad T^3_3(t_{i+2}) = \frac{h_{i+2}(2q_{i+1} + q_{i+3})}{6C^2_2(i+1)}, \]

and we have everything that is needed for the evaluation of \( T^4_3 \) by means of Theorem 2.

4. Conclusion

We have constructed formulae for calculating with \( q \)-splines as linear combinations of polynomial B-splines. Moreover, all the coefficients involved are positive, and thus we have to calculate scalar products of positive quantities only, guaranteeing numerical stability of such an algorithm.

References


