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Collocation with High Order Tension Splines*

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Abstract. Tension spline of order k is a function that, for a given partition $x_0 < \cdots < x_n$, on each interval $[x_i, x_{i+1}]$, satisfies differential equation $(D^k - (p_i^2/h_i^2)D^{k-2})u = 0$, where p_i's are prescribed nonnegative real numbers.

Many articles deal with tension splines of order four applied to collocation methods for solving singularly perturbed boundary value problem

$$(\mathcal{L}u)(x) = \varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x), \quad 0 \le x \le 1,$$

with boundary conditions

$$u(0) = u_0, \quad u(1) = u_1.$$

Accuracy of considered approximations is $\mathcal{O}(h)$ or $\mathcal{O}(h^2)$ for small perturbation parameter ε, depending on the choice of collocation points.

Here we present an algorithm for a collocation method with high order tension splines. Our objective is to obtain approximations of higher order of accuracy for the solution of singularly perturbed differential equation.

AMS subject classification: 65L60, 65L10

Key words: tension spline, collocation method, boundary value problems

1. Preliminaries

We consider numerical solution of the singularly perturbed two-point boundary value problem for the ordinary differential equation (ODE)

$$(\mathcal{L}u)(x) := \varepsilon u''(x) + b(x)u'(x) + c(x)u(x) = f(x) \tag{1}$$

$$u(0) = u_0, \quad u(1) = u_1. \tag{2}$$

We assume that the perturbation parameter ε satisfies $0 < \varepsilon \ll 1$, and the coefficient c satisfies $-c(x) \ge c_{min} > 0$ for $x \in [0,1]$. These conditions guarantee existence and uniqueness of a solution to the problem (1)–(2).

In this paper we will consider numerical solution of the problem (1)–(2) by collocation methods with tension splines of order k.

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Tension spline of order k is a function that, for a given partition $x_0 < \cdots < x_n$, on each interval $[x_i, x_{i+1}]$, satisfies differential equation

$$\left(D^k - \frac{p_i^2}{h_i^2} D^{k-2}\right) u = 0,$$

where p_i's are prescribed nonnegative real numbers (so-called "tension parameters"), and $h_i = x_{i+1} - x_i$.

Almost all papers dealing with tension splines consider tension splines of order four. Applications of such splines are mostly shape preserving approximation [13, 14], and numerical solution of singularly perturbed two-point boundary value problem for ODE [3, 10]. Higher order tension splines are described in several papers [4, 7], but no application is given.

Our aim in this paper is to show that collocation methods with higher order tension splines can increase accuracy of numerical approximation of (1)–(2). Due to the requirements of the numerical testing, tension splines of order six or eight will be referred to as "higher order tension splines".

2. B-splines

As in the case of polynomial splines, we use B-spline representation for tension splines [4, 5, 6, 7, 9, 11].

Suppose m and n are positive integers and $m \geq 2$. Let $t_1 \leq t_2 \leq \cdots \leq t_{n+m}$ be a nondecreasing sequence of knots, and for each nonempty interval (t_i, t_{i+1}) let $\rho_i = p_i/h_i$ be a given nonnegative number. Tension B-splines of order k are defined recursively [4, 12]

$$B_{j,k} := \frac{1}{\sigma_{j,k-1}} \int_{t_j}^{x} B_{j,k-1}(y)\, dy - \frac{1}{\sigma_{j+1,k-1}} \int_{t_{j+1}}^{x} B_{j+1,k-1}(y)\, dy,$$

for $j = m - k + 1, \ldots, n$, where

$$\sigma_{j,k-1} := \int_{t_j}^{t_{j+k-1}} B_{j,k-1}(y)\, dy.$$

The recursion starts with functions $B_{i,2}$ defined by

$$B_{i,2}(x) := \begin{cases} \dfrac{\sinh(\rho_i(x - t_i))}{\sinh(\rho_i \Delta_i)}, & t_i \leq x < t_{i+1}, \\ \dfrac{\sinh(\rho_{i+1}(t_{i+2} - x))}{\sinh(\rho_{i+1}\Delta_{i+1})}, & t_{i+1} \leq x < t_{i+2}, \\ 0, & \text{elsewhere}, \end{cases}$$

where

$$\Delta_i = t_{i+1} - t_i.$$

In [4], it is proved that B-splines $B_{i,k}$ have the following properties:

(i) $\operatorname{supp} B_{i,k} = [t_i, t_{i+k}]$ (local support property),

(ii) $B_{i,k}(x) > 0$ for $t_i < x < t_{i+k}$ (positivity),

(iii) $\sum_i B_{i,k} \equiv 1$, $k \geq 3$, (partition of unity),

(iv) $B_{i,k}$ has $k-1-l$ continuous derivatives at a knot of multiplicity l.

Moreover, B-splines $B_{j,k}$, $j = m-k+1, \ldots, n$, are linearly independent and form the basis for the vector space $\mathcal{T}_k = \operatorname{span}\{B_{m-k+1,k}, \ldots, B_{n,k}\}$. $\mathcal{T}_k$ is a span of tension splines of order k with smoothness at knots t_i defined by multiplicity of the knots.

3. Collocation method

Here we consider collocation methods by tension splines of order k which are in $C^{k-2}(0,1)$. Assume that a partition Π of the interval $[0,1]$ is given by

$$0 = x_0 < x_1 < \cdots < x_n = 1,$$

and the nonnegative real numbers p_i are prescribed. By $\mathcal{T}_k$ we denote the space of tension splines of order k with the knot sequence t_i, $i = 1-k, \ldots, n+k-1$, defined by

$$t_i = \begin{cases} x_0, & i = 1-k, \ldots, 0, \\ x_i, & i = 1, \ldots, n-1, \\ x_n, & i = n, \ldots, n+k-1, \end{cases}$$

and ρ_i, $i = 1-k, \ldots, n+k-2$, given by

$$\rho_i = \begin{cases} \dfrac{p_i}{t_{i+1}-t_i}, & t_i < t_{i+1}, \\ 0, & t_i = t_{i+1}. \end{cases}$$

B-splines $B_{j,k}$, $j = 1-k, \ldots, n$, form the basis for $\mathcal{T}_k$, and from property (iv) we conclude that $\mathcal{T}_k \subset C^{k-2}(0,1)$, and $\dim \mathcal{T}_k = n+k-1$.

Further, suppose that x_i, $i = 0, \ldots, n$, are the collocation points. Spline $s \in \mathcal{T}_k$ is the collocation spline for the problem (1)–(2) if it satisfies collocation equations

$$(\mathcal{L}s)(x_i) = f(x_i), \quad i = 0, \ldots, n, \tag{3}$$

$$s(0) = u_0 \quad \text{and} \quad s(1) = u_1. \tag{4}$$

These $n+3$ conditions are sufficient for the unique determination of tension spline $s \in \mathcal{T}_4$ of order four, but to determine $s \in \mathcal{T}_6$ (or $s \in \mathcal{T}_8$) we need two (four) additional conditions [1]. For $s \in \mathcal{T}_6$ additional two equations are

$$\left(\frac{d}{dx}\mathcal{L}s\right)(0) = \frac{df}{dx}(0) \quad \text{and} \quad \left(\frac{d}{dx}\mathcal{L}s\right)(1) = \frac{df}{dx}(1). \tag{5}$$

For $s \in \mathcal{T}_8$, additional four equations are given by (5) and

$$\left(\frac{d^2}{dx^2}\mathcal{L}s\right)(0) = \frac{d^2 f}{dx^2}(0) \quad \text{and} \quad \left(\frac{d^2}{dx^2}\mathcal{L}s\right)(1) = \frac{d^2 f}{dx^2}(1). \tag{6}$$

If the collocation spline s is represented in the B-spline basis,

$$s(x) = \sum_{i=1-k}^{n-1} c_i \, B_{i,k}(x),$$

collocation equations (3)–(6) yield the matrix equation $Ac = f$. Matrix A has at most k nonzero elements in each row (this follows from the local support property (i) of $B_{i,k}$). Indeed, matrix A is a band matrix with bandwidth k.

In the previously mentioned collocation methods we always use the following choice of tension parameters p_i given in [8]:

$$p_i := \max\{\tilde{p}_i^-, \tilde{p}_i^+\},$$

where

$$\tilde{p}_i^- = \frac{h_i}{2\varepsilon}\left(|b(x_i)| + \sqrt{|b(x_i)|^2 + 4\varepsilon|c(x_i)|}\,\right)$$

and

$$\tilde{p}_i^+ = \frac{h_i}{2\varepsilon}\left(|b(x_{i+1})| + \sqrt{|b(x_{i+1})|^2 + 4\varepsilon|c(x_{i+1})|}\,\right).$$

4. Numerical experiments

We illustrate the collocation methods for different tension splines on two examples known to posses exact solutions.

Example 1. *The first example is a singularly perturbed boundary value problem*

$$\varepsilon y'' - \big[1 + x(1-x)\big]y = f(x), \quad 0 < x < 1, \tag{7}$$

$$y(0) = y(1) = 0, \tag{8}$$

where

$$f(x) = -e^{-x/\sqrt{\varepsilon}}\big[2\sqrt{\varepsilon} - x(1-x)^2\big] - e^{-(1-x)/\sqrt{\varepsilon}}\big[2\sqrt{\varepsilon} - x^2(1-x)\big] - 1 - x(1-x).$$

The exact solution of (7)–(8) *is*

$$y(x) = 1 + (x-1)e^{-x/\sqrt{\varepsilon}} - xe^{-(1-x)/\sqrt{\varepsilon}}.$$

Example 2. *The second example is* (cf. [3])

$$\varepsilon y'' - (x^2 + \sqrt{\varepsilon})y = -(x^2 + \sqrt{\varepsilon})(1 + \sin \pi x) - \varepsilon\pi^2 \sin \pi x, \quad 0 < x < 1, \tag{9}$$

$$y(0) = y(1) = 0. \tag{10}$$

The exact solution of this problem is

$$y(x) = 1 + \sin \pi x - \frac{1}{\operatorname{erf}\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)} \left\{ \left(1 - e^{-1/(2\sqrt{\varepsilon})}\right) e^{-x^2/(2\sqrt{\varepsilon})} W\left(\frac{x}{\sqrt[4]{\varepsilon}}\right) + \left[1 - e^{-1/(2\sqrt{\varepsilon})} W\left(\frac{1}{\sqrt[4]{\varepsilon}}\right)\right] e^{-(1-x^2)/(2\sqrt{\varepsilon})} \right\},$$

where

$$W(z) = e^{z^2} \operatorname{erfc}(z).$$

We have solved the problems by choosing an equidistant partition and dividing the interval $[0, 1]$ into 2^n subintervals, $n = 1, \ldots, 13$, and for different perturbation parameters $\varepsilon = 2^{-k}$, $k = 2, \ldots, 20$. For the calculation of the collocation error e_n, we evaluated the error at 10 equidistant points in each of 2^n subintervals. For evaluation of tension splines of higher order we use algorithms described in [2]. All calculations were done in FORTRAN 77 in double precision arithmetic.

In Figure 1 we compare errors of the solutions of Example 1 by tension splines and polynomial splines. Note that polynomial splines are tension splines with tension parameters equal to zero. We denote the space of collocation polynomial splines of order k with $\mathcal{P}_k$. We discuss the collocation with splines of order four, six and eight. It is evident that the collocation with tension splines has the same order of convergence as the collocation with polynomial splines, but tension splines have a smaller error than polynomial. Also, we can see that collocation with higher order splines has a smaller error than collocation with spline of order four. In Figure 2 we can see the same behaviour of error for the solution of Example 2.

In Figure 3, we compare numerical order of convergence r_n of approximate solution of Example 1 for tension splines of different order. The same comparison for Example 2 is shown in Figure 4. Numerical order of convergence is calculated according to

$$r_n := \log_2 \frac{e_{n-1}}{e_n},$$

where e_n is the collocation error for the partition with 2^n subintervals.

For the collocation method using tension splines of order four, it is proved that the resulting approximation converges linearly for small perturbation parameter ε, and quadratically for large [10], as we can see from Figure 3 and Figure 4. For the collocation method using higher order tension splines we found in our numerical experiments that the rate of convergence is $k - 2$ for large ε, and $(k - 2)/2$ for small ε. Note that results for $\varepsilon = 2^{-9}$, and number of subintervals greater than 2^9, are contaminated by roundoff errors.

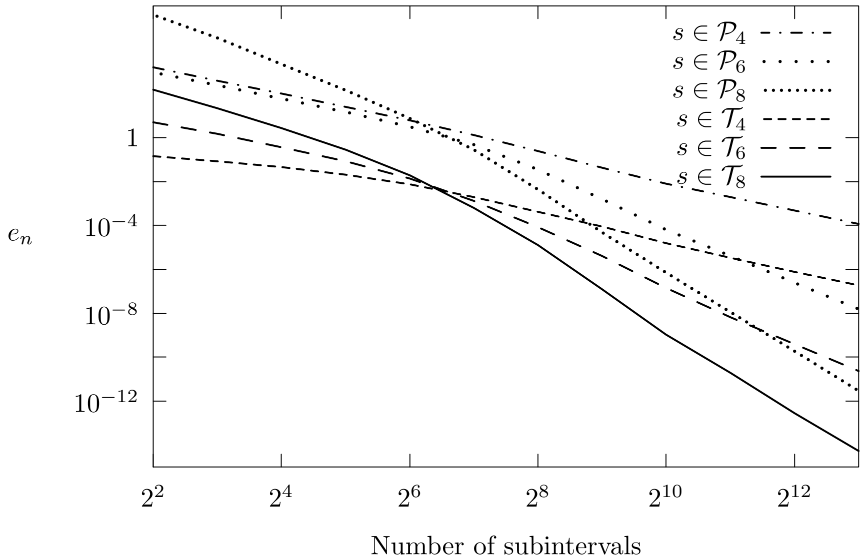

Figure 1. Collocation error to the solution of Example 1 by tension splines from $\mathcal{T}_k$ and polynomial splines from $\mathcal{P}_k$, $k = 4, 6, 8$, depending on the number of subintervals, for perturbation parameter $\varepsilon = 2^{-19}$.

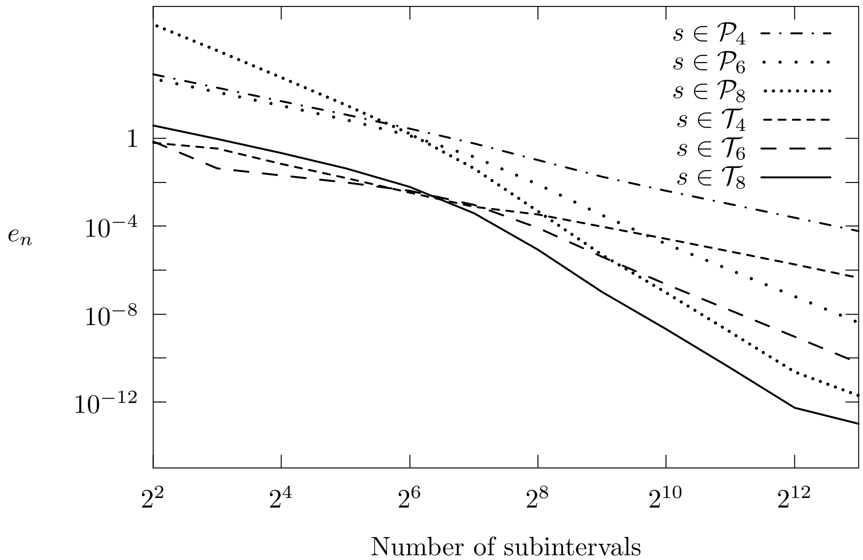

Figure 2. Collocation error to the solution of Example 2 by tension splines from $\mathcal{T}_k$ and polynomial splines from $\mathcal{P}_k$, $k = 4, 6, 8$, depending on the number of subintervals, for perturbation parameter $\varepsilon = 2^{-9}$.

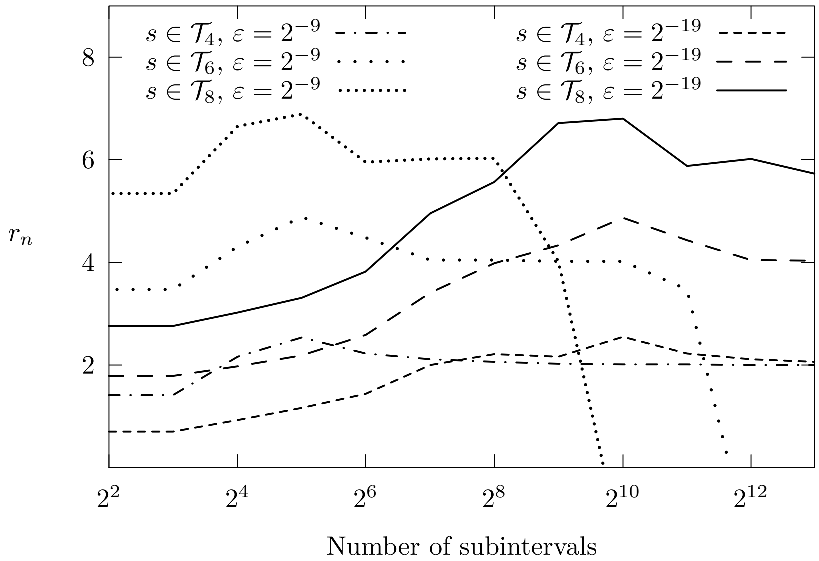

Figure 3. Numerical order of convergence r_n for Example 1 using collocation by tension splines from $\mathcal{T}_k$, $k = 4, 6, 8$, for perturbation parameters $\varepsilon = 2^{-9}$ and $\varepsilon = 2^{-19}$.

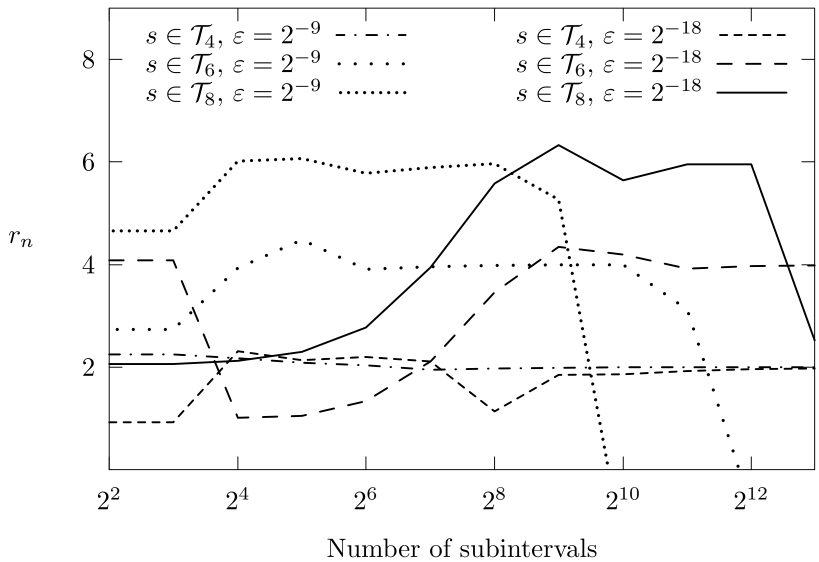

Figure 4. Numerical order of convergence r_n for Example 2 using collocation by tension splines from $\mathcal{T}_k$, $k = 4, 6, 8$, for perturbation parameters $\varepsilon = 2^{-9}$ and $\varepsilon = 2^{-18}$.

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