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Retraction Method in the Qualitative Analysis of the Solutions of the Quasilinear Second Order Differential Equation

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Abstract. The quasilinear second order differential equation

$$y'' + P(y,t)y' + Q(y,t)y = F(y,t)$$

where P, Q, $F \in C(\mathbb{R} \times I)$, $I = (a, \infty)$, $a \in \mathbb{R}$, is under consideration. This paper deals with the behaviour, approximation and stability of solutions of this equation. Behaviour of integral curves in neighbourhoods of an arbitrary integral curve is considered. The qualitative analysis theory and topological retraction method are used. The general results and a several appropriate examples are considered and discussed.

AMS subject classification: 34C05

Key words: quasilinear differential equation, behaviour of solutions

1. Introduction

Let us consider the quasilinear second order differential equation

$$y'' + P(y,t)y' + Q(y,t)y = F(y,t), \tag{1}$$

where P, Q, $F \in C(\mathbb{R} \times I)$, $I = (a, \infty)$, $a \in \mathbb{R}$. Let

$$\Gamma = \{(y,t) \in \mathbb{R} \times I \mid y = \psi(t), t \in I\},$$

where $\psi(t) \in C^2(I)$ is an arbitrary integral curve.

In this paper the behaviour of the solutions of equation (1) in the neighbourhood of curve Γ is considered. The notations $\psi_0 = \psi(t_0)$, $y_0 = y(t_0)$, $y'_0 = y'(t_0)$, $t_0 \in I$ are going to be used.

Let r_1, $r_2 \in C^1(I)$, $r_1(t) > 0$, $r_2(t) > 0$ on I. Let us consider the solutions $y(t)$ of equation (1) which satisfy on I, either the conditions

$$|y_0 - \psi_0| \leq r_2(t_0), \quad |y'_0 - \psi'_0| \leq r_1(t_0), \tag{2}$$

or

$$\frac{(y_0 - \psi_0)^2}{r_2^2(t_0)} + \frac{(y'_0 - \psi'_0)^2}{r_1^2(t_0)} \leq 1. \tag{3}$$

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2. Preliminaries

Let

$$y' = x, \tag{4}$$

where $x = x(t)$ is a new unknown function. Equation (1) is transformed into a quasilinear system of equations

$$\begin{cases} x' = -P(y,t)x - Q(y,t)y + F(y,t), \\ y' = x, \\ t' = 1. \end{cases} \tag{5}$$

Let $(\varphi(t), \psi(t), t)$, $t \in I$, where $\varphi(t) = \psi'(t)$, be an arbitrary integral curve of the system (5), and let $\Omega = \mathbb{R}^2 \times I$.

We shall consider the behaviour of the integral curve $(x(t), y(t), t)$ of (5) with respect to the sets

$$\sigma = \big\{(x,y,t) \in \Omega \;\big|\; |x - \varphi(t)| < r_1(t),\ |y - \psi(t)| < r_2(t)\big\}$$

and

$$\omega = \left\{(x,y,t) \in \Omega \;\middle|\; \frac{(x-\varphi(t))^2}{r_1^2(t)} + \frac{(y-\psi(t))^2}{r_2^2(t)} \le 1\right\}.$$

The boundary surfaces of σ and ω are, respectively,

$$X_i = \big\{(x,y,t) \in \operatorname{Cl}\sigma \;\big|\; H_i^1(x,y,t) \equiv (-1)^i(x - \varphi(t)) - r_1(t) = 0\big\}, \quad i = 1,2,$$

$$Y_i = \big\{(x,y,t) \in \operatorname{Cl}\sigma \;\big|\; H_i^2(x,y,t) \equiv (-1)^i(y - \psi(t)) - r_2(t) = 0\big\}, \quad i = 1,2,$$

$$W = \left\{(x,y,t) \in \operatorname{Cl}\omega \;\middle|\; H(x,y,t) \equiv \frac{(x-\varphi(t))^2}{r_1^2(t)} + \frac{(y-\psi(t))^2}{r_2^2(t)} - 1 = 0\right\}.$$

To prove our results we need the following results concerning the applicability of the qualitative analysis theory and topological retraction method of T. Važevski [6].

Let us denote the tangent vector field to an integral curve $(x(t), y(t), t)$ of (5) by T. The vectors ∇H_i^1, ∇H_i^2 and ∇H are the outer normals on surfaces X_i, Y_i and W, respectively. We have

$$T(x,y,t) = \big(-P(y,t)x - Q(y,t)y + F(y,t), x, 1\big),$$

$$\nabla H_i^1(t) = \big((-1)^i, 0, (-1)^{i-1}\varphi' - r_1'\big), \quad i = 1,2,$$

$$\nabla H_i^2(t) = \big(0, (-1)^i, (-1)^{i-1}\psi' - r_2'\big), \quad i = 1,2,$$

$$\frac{1}{2}\nabla H(x,y,t) = \Bigg(\frac{x-\varphi}{r_1^2}, \frac{y-\psi}{r_2^2}, -\frac{(x-\varphi)^2 r_1'}{r_1^3} - \frac{(y-\psi)^2 r_2'}{r_2^3} - \frac{(x-\varphi)\varphi'}{r_1^2} - \frac{(y-\psi)\psi'}{r_2^2}\Bigg).$$

By means of scalar products $\pi_i^1(x,y,t) = (\nabla H_i^1, T)$ on X_i, $\pi_i^2(x,y,t) = (\nabla H_i^2, T)$ on Y_i, and $\pi(x,y,t) = (\frac{1}{2}\nabla H, T)$ on W, we shall establish the existence and behaviour of integral curves of (5) with respect to the sets σ and ω, respectively.

Let us denote by $S^p(I)$, $p \in \{0,1,2\}$, a class of solutions $(x(t), y(t), t)$ of the system (5) defined on I, which depends on p parameters. We shall simply say that the class of solutions $S^p(I)$ belongs to the set η ($\eta = \omega$ or $\eta = \sigma$) if graphs of functions in $S^p(I)$ are contained in η. In that case we shall write $S^p(I) \subset \eta$. For $p = 0$ we have the notation $S^0(I)$, which means that there exists at least one solution $(x(t), y(t), t)$ on I of the system (5), whose graph belongs to the set η.

The results of this paper are based on the following Lemmas (see [4, 5]).

Lemma 1. *If, for the system* (5)*, the scalar product* $\pi < 0$ *on* W ($\pi_i^k < 0$ *on* $\partial\sigma = X_1 \cup X_2 \cup Y_1 \cup Y_2$, $i = 1,2$, $k = 1,2$)*, then the system* (5) *has a class of solutions* $S^2(I)$ *belonging to the set* ω *for all* $t \in I$*, i.e.,* $S^2(I) \subset \omega$ ($S^2(I) \subset \sigma$)*.*

Lemma 2. *If, for the system* (5)*, the scalar product* $\pi > 0$ *on* W ($\pi_i^k > 0$ *on* $\partial\sigma = X_1 \cup X_2 \cup Y_1 \cup Y_2$, $i = 1,2$, $k = 1,2$)*, then the system* (5) *has at least one solution on* I *whose graph belongs to the set* ω *for all* $t \in I$*, i.e.,* $S^0(I) \subset \omega$ ($S^0(I) \subset \sigma$)*.*

Lemma 3. *If, for the system* (5)*, the scalar product* $\pi_i^1 < 0$ *on* $X_1 \cup X_2$*, and* $\pi_i^2 > 0$ *on* $Y_1 \cup Y_2$ *(or reversely), then the system* (5) *has a class of solutions* $S^1(I)$ *belonging to the set* σ *for all* $t \in I$*, i.e.,* $S^1(I) \subset \sigma$*.*

3. Main results

Theorem 1. *Let* $P(y,t)$*,* $Q(y,t)$*,* $F(y,t) \in C(\mathbb{R} \times I)$ *satisfy the conditions:*

$$|P(y_1,t) - P(y_2,t)| < L_1|y_1 - y_2|, \quad (y_1,t),(y_2,t) \in \mathbb{R} \times I, \tag{6}$$

$$|Q(y_1,t) - Q(y_2,t)| < L_2|y_1 - y_2|, \quad (y_1,t),(y_2,t) \in \mathbb{R} \times I, \tag{7}$$

$$|F(y_1,t) - F(y_2,t)| < L_3|y_1 - y_2|, \quad (y_1,t),(y_2,t) \in \mathbb{R} \times I, \tag{8}$$

and let r_1*,* $r_2 \in C^1(I)$*,* $r_1(t) > 0$*,* $r_2(t) > 0$*. Then:*

(i) *If the conditions*

$$\big(L_1|\varphi| + L_2|\psi| + L_3 + |Q(y,t)|\big) r_2 < r_1' + P(y,t) r_1, \tag{9}$$

$$r_1 < r_2', \tag{10}$$

are satisfied on $\operatorname{Cl}\sigma$*, then all solutions* $y(t)$ *of the problem* (1)*,* (2) *satisfy the conditions*

$$|y(t) - \psi(t)| < r_2(t), \quad |y'(t) - \psi'(t)| < r_1(t), \quad \text{for} \quad t > t_0. \tag{11}$$

(ii) *If the conditions*

$$\big(L_1|\varphi| + L_2|\psi| + L_3 + |Q(y,t)|\big) r_2 < -r_1' - P(y,t) r_1, \tag{12}$$

$$r_1 < -r_2', \tag{13}$$

are satisfied on Cl σ*, then at least one solution of the problem* (1), (2) *satisfies the conditions* (11).

(*iii*) *If the conditions* (9) *and* (13), *or* (10) *and* (12) *are satisfied on* Cl σ*, then the problem* (1), (2) *has one-parameter class of solutions satisfying the conditions* (11).

Proof. We shall consider the equation (1) through the equivalent system (5). Let us consider the integral curves of the system (5) with respect to the set σ. For the scalar products $\pi_i^1(x,y,t)$ on X_i and $\pi_i^2(x,y,t)$ on Y_i, we have:

$$\begin{aligned}\pi_i^1(x,y,t) &= (-1)^i\big[-P(y,t)x - Q(y,t)y + F(y,t)\big] + (-1)^{i-1}\varphi' - r_1' \\ &= (-1)^i\big[-P(y,t)(x-\varphi) - Q(y,t)(y-\psi) + F(y,t) \\ &\qquad\qquad - P(y,t)\varphi - Q(y,t)\psi - \varphi'\big] - r_1' \\ &= -P(y,t)r_1 + (-1)^i\big[-Q(y,t)(y-\psi) + F(y,t) \\ &\qquad\qquad - P(y,t)\varphi - Q(y,t)\psi - \varphi'\big] - r_1', \\ \pi_i^2(x,y,t) &= (-1)^i x + (-1)^{i-1}\psi' - r_2' = (-1)^i(x-\varphi) - r_2'.\end{aligned}$$

(*i*) According to the conditions (6)–(8), (9) and (10), the following estimates for π_i^1 on X_i and π_i^2 on Y_i are valid, respectively:

$$\begin{aligned}\pi_i^1(x,y,t) &\le -P(y,t)r_1 + |Q(y,t)|r_2 \\ &\qquad + |F(y,t) - P(y,t)\varphi - Q(y,t)\psi - \varphi'| - r_1' \\ &\le -P(y,t)r_1 + |Q(y,t)|r_2 + |F(y,t) - F(\psi,t)| \\ &\qquad + |P(\psi,t) - P(y,t)|\,|\varphi| + |Q(\psi,t) - Q(y,t)|\,|\psi| - r_1' \\ &\le -P(y,t)r_1 + |Q(y,t)|r_2 + \big(L_3 + L_1|\varphi| + L_2|\psi|\big)r_2 - r_1' < 0, \\ \pi_i^2(x,y,t) &\le r_1 - r_2' < 0.\end{aligned}$$

Accordingly, set $\partial\sigma = X_1 \cup X_2 \cup Y_1 \cup Y_2$ is a set of points of strict entrance of integral curves of the system (5) with respect to the sets σ and Ω. Hence, all solutions of the system (5) which satisfy the conditions

$$|x_0 - \varphi_0| \le r_1(t_0), \quad |y_0 - \psi_0| \le r_2(t_0), \quad (x_0 = x(t_0)),$$

also satisfy conditions

$$|x(t) - \varphi(t)| < r_1(t), \quad |y(t) - \psi(t)| < r_2(t), \quad \text{for} \quad t > t_0.$$

Since, in view of (4),

$$x_0 - \varphi_0 = y_0' - \psi_0',$$

all solutions of the problem (1), (2) satisfy the conditions (11).

(ii) According to the conditions (6)–(8), (12) and (13), the following estimates for π_i^1 on X_i and π_i^2 on Y_i are valid, respectively:

$$\begin{aligned}\pi_i^1(x,y,t) &\geq -P(y,t)r_1 + |Q(y,t)|\,(-r_2)\\ &\qquad - |F(y,t) - P(y,t)\varphi - Q(y,t)\psi - \varphi'| - r_1'\\ &\geq -P(y,t)r_1 - |Q(y,t)|r_2 - |F(y,t) - F(\psi,t)|\\ &\qquad - |P(\psi,t) - P(y,t)|\,|\varphi| - |Q(\psi,t) - Q(y,t)|\,|\psi| - r_1'\\ &\geq -P(y,t)r_1 - |Q(y,t)|r_2 - \big(L_3 + L_1|\varphi| + L_2|\psi|\big)r_2 - r_1' > 0,\\ \pi_i^2(x,y,t) &\geq -r_1 - r_2' > 0.\end{aligned}$$

Accordingly, set $\partial\sigma$ is a set of points of strict exit of integral curves of the system (5) with respect to sets σ and Ω. Hence, according to T .Ważewski's retraction method [6], the system (5) has at least one solution belonging to the set σ for all $t \in I$. Consequently, the problem (1), (2) has at least one solution satisfying the conditions (11).

(iii) In this case $X_1 \cup X_2$ is a set of points of strict exit, and $Y_1 \cup Y_2$ is a set of points of strict entrance (or reversely) of integral curves of the system (5) with respect to the sets σ and Ω. According to the retraction method, the system (5) has one-parameter class of solutions belonging to the set σ for all $t \in I$. Hence, the problem (1), (2) also has one-parameter class of solutions satisfying the conditions (11). ■

Let us consider now the solutions $y(t)$ of equation (1) which satisfy the condition (3).

Theorem 2. *Let $P(y,t)$, $Q(y,t)$, $F(y,t) \in C(\mathbb{R} \times I)$, and let the conditions* (6), (7) *and* (8) *be satisfied. Let r_1, $r_2 \in C^1(I)$, $r_1(t) > 0$, $r_2(t) > 0$, and*

$$\Big(\big(L_1|\varphi| + L_2|\psi| + L_3\big)r_2^2 + \big|r_1^2 - Q(y,t)r_2^2\big|\Big)^2 < 4r_1r_2\big(P(y,t)r_1 + r_1'\big)r_2'. \qquad (14)$$

Then:

(i) *If*

$$r_2' > 0, \qquad (15)$$

then all solutions $y(t)$ of the problem (1), (3) *satisfy the condition*

$$\frac{(y(t) - \psi(t))^2}{r_2^2(t)} + \frac{(y'(t) - \psi'(t))^2}{r_1^2(t)} < 1, \quad \text{for} \quad t > t_0. \qquad (16)$$

(ii) *If*

$$r_2' < 0, \qquad (17)$$

then at least one solution of the problem (1), (3) *satisfies the condition* (16).

Proof. We shall consider the equation (1) through the equivalent system (5). Let us consider the integral curves of the system (5) with respect to the set ω. For the scalar product $\pi(x,y,t) = (\frac{1}{2}\nabla H, T)$ on the surface W, we have:

$$\begin{aligned}\pi(x,y,t) = \big[-P(y,t)x - Q(y,t)y + F(y,t)\big]\,\frac{x-\varphi}{r_1^2} + x\,\frac{y-\psi}{r_2^2}\\ - \frac{(x-\varphi)^2 r_1'}{r_1^3} - \frac{(y-\psi)^2 r_2'}{r_2^3} - \frac{(x-\varphi)\varphi'}{r_1^2} - \frac{(y-\psi)\psi'}{r_2^2}.\end{aligned}$$

If we introduce the notation

$$X = \frac{x-\varphi}{r_1}, \quad Y = \frac{y-\psi}{r_2},$$

we have:

$$\begin{aligned}\pi(x,y,t) &= \left[-P(y,t) - \frac{r_1'}{r_1}\right] X^2 + \left[-Q(y,t)\,\frac{r_2}{r_1} + \frac{r_1}{r_2}\right] X\,Y - \frac{r_2'}{r_2}\,Y^2\\ &\quad + \big[-P(y,t)\varphi - Q(y,t)\psi + F(y,t) - \varphi'\big]\,\frac{X}{r_1}\\ &= \left[-P(y,t) - \frac{r_1'}{r_1}\right] X^2 + \left[-Q(y,t)\,\frac{r_2}{r_1} + \frac{r_1}{r_2}\right] X\,Y - \frac{r_2'}{r_2}\,Y^2\\ &\quad + \Big[\big(P(\psi,t) - P(y,t)\big)\varphi + \big(Q(\psi,t) - Q(y,t)\big)\psi + F(y,t) - F(\psi,t)\Big]\,\frac{X}{r_1}.\end{aligned}$$

In view of (6)–(8), the following estimates for $\pi(x,y,t)$ on W are valid:

$$\begin{aligned}\pi(x,y,t) &\le \left[-P(y,t) - \frac{r_1'}{r_1}\right] X^2 + \left|-Q(y,t)\,\frac{r_2}{r_1} + \frac{r_1}{r_2}\right| |X|\,|Y| + \left[-\frac{r_2'}{r_2}\right] Y^2\\ &\quad + \big(L_1|\varphi| + L_2|\psi| + L_3\big)\,\frac{r_2}{r_1}\,|X|\,|Y|\\ &= \left[-P(y,t) - \frac{r_1'}{r_1}\right] X^2 + \left[\big(L_1|\varphi| + L_2|\psi| + L_3\big)\,\frac{r_2}{r_1}\right.\\ &\quad \left. + \left|\frac{r_1}{r_2} - Q(y,t)\,\frac{r_2}{r_1}\right|\right] |X|\,|Y| + \left[-\frac{r_2'}{r_2}\right] Y^2,\end{aligned}$$

$$\begin{aligned}\pi(x,y,t) &\ge \left[-P(y,t) - \frac{r_1'}{r_1}\right] X^2 - \left|-Q(y,t)\,\frac{r_2}{r_1} + \frac{r_1}{r_2}\right| |X|\,|Y| + \left[-\frac{r_2'}{r_2}\right] Y^2\\ &\quad - \big(L_1|\varphi| + L_2|\psi| + L_3\big)\,\frac{r_2}{r_1}\,|X|\,|Y|\\ &= \left[-P(y,t) - \frac{r_1'}{r_1}\right] X^2 - \left[\big(L_1|\varphi| + L_2|\psi| + L_3\big)\,\frac{r_2}{r_1}\right.\\ &\quad \left. + \left|\frac{r_1}{r_2} - Q(y,t)\,\frac{r_2}{r_1}\right|\right] |X|\,|Y| + \left[-\frac{r_2'}{r_2}\right] Y^2.\end{aligned}$$

The right-hand sides of the above inequalities are the quadratic symmetric forms

$$a_{11}X^2 \pm 2a_{12}|X|\,|Y| + a_{22}Y^2,$$

where corresponding coefficients a_{11}, a_{12}, a_{22} are introduced.

(i) Conditions (14) and (15) imply

$$a_{22} < 0, \quad a_{11}a_{22} - a_{12}^2 > 0,$$

which, according to Sylvester's criterion, means that $\pi(x, y, t) < 0$ on W. Consequently, set W is a set of points of strict entrance of integral curves of the system (5) with respect to the sets ω and Ω. Hence, all solutions of the system (5) which satisfy the condition

$$\frac{(x_0 - \varphi_0)^2}{r_1^2(t_0)} + \frac{(y_0 - \psi_0)^2}{r_2^2(t_0)} < 1, \tag{18}$$

satisfy the inequality

$$\frac{(x(t) - \varphi(t))^2}{r_1^2(t)} + \frac{(y(t) - \psi(t))^2}{r_2^2(t)} < 1, \quad \text{for} \quad t > t_0. \tag{19}$$

Since $x_0 - \varphi_0 = y_0' - \psi_0'$, then all solutions of the problem (1), (3) satisfy condition (16).

(ii) Conditions (14), (17) imply

$$a_{22} > 0, \quad a_{11}a_{22} - a_{12}^2 > 0,$$

which, according to Sylvester's criterion, means that $\pi(x, y, t) > 0$ on W. Consequently, W is a set of points of strict exit of integral curves of the system (5) with respect to the sets ω and Ω. Hence, according to retraction method, the problem (5), (18) has at least one solution satisfying condition (19). Consequently, the problem (1), (3) has at least one solution satisfying condition (16). ■

Now let us consider solutions of equation (1) which satisfy the condition

$$y_0^2 + (y_0')^2 \le r^2(t_0). \tag{20}$$

Theorem 3. *Let $r \in C^1(I)$, $r(t) > 0$. Then:*

(i) *If the conditions*

$$F^2(y, t) < 2P(y, t)\big(2rr' - \big|1 - Q(y, t)\big|r^2\big), \tag{21}$$

$$P(y, t) > 0, \tag{22}$$

are satisfied, then all solutions $y(t)$ of the problem (1), (20) satisfy the condition

$$y^2(t) + {y'}^2(t) < r^2(t), \quad \textit{for} \quad t > t_0. \tag{23}$$

(*ii*) *If the conditions*

$$F^2(y,t) < 2P(y,t)\big(2rr' + \big|1 - Q(y,t)\big|r^2\big), \tag{24}$$

$$P(y,t) < 0, \tag{25}$$

are satisfied, then at least one solution of the problem (1), (20) *satisfies the condition* (23).

Proof. We consider the system (5). Let $(\varphi(t), \psi(t), t)$, $t \in I$, where $\varphi \in C^1(I)$, be an integral curve of the system (5), and consider the set

$$\omega_0 = \big\{(x,y,t) \in \Omega \;\big|\; x^2 + y^2 \le r^2(t)\big\}.$$

The boundary surface of ω_0 is

$$W_0 = \big\{(x,y,t) \in \operatorname{Cl}\omega_0 \;\big|\; H_0(x,y,t) \equiv x^2 + y^2 - r^2(t) = 0\big\}\,.$$

Let $\nu(x,y,t) = \frac{1}{2}\nabla H_0(x,y,t)$ be a vector of the outer normal on the surface W_0. For the scalar product $\pi_0(x,y,t) = (\nu, T)$ on the surface W_0, we have:

$$\begin{aligned}\pi_0(x,y,t) &= \big[-P(y,t)x - Q(y,t)y + F(y,t)\big]x + xy - rr' \\ &= -P(y,t)x^2 + \big[1 - Q(y,t)\big]xy + F(y,t)x - rr'.\end{aligned}$$

According to the conditions (21), (22), and (24), (25), and by using the inequality $ab \le \frac{1}{2}(a^2 + b^2)$, for $a, b \in \mathbb{R}$, the following estimates for $\pi_0(x,y,t)$ on W_0 are valid in the cases (*i*) and (*ii*), respectively:

$$\begin{aligned}\pi_0(x,y,t) &\le -P(y,t)x^2 + F(y,t)x + \frac{1}{2}\big|1 - Q(y,t)\big|(x^2 + y^2) - rr' \\ &= -P(y,t)x^2 + F(y,t)x + \frac{1}{2}\big|1 - Q(y,t)\big|r^2 - rr' < 0,\\ \pi_0(x,y,t) &\ge -P(y,t)x^2 + F(y,t)x - \frac{1}{2}\big|1 - Q(y,t)\big|(x^2 + y^2) - rr' \\ &= -P(y,t)x^2 + F(y,t)x - \frac{1}{2}\big|1 - Q(y,t)\big|r^2 - rr' > 0.\end{aligned}$$

According to Lemma 1 and Lemma 2, the above estimates for π_0 imply the statement of the theorem. ∎

Example 1. *For the problem*

$$y'' + f(y,t)y' - f(y,t)\sin 2t - 2\cos 2t = 0, \tag{26}$$

with

$$\big|y_0 - \sin^2 t_0\big| \le \beta \exp(-st_0), \quad \big|y_0' - \sin 2t_0\big| \le \alpha \exp(-st_0), \tag{27}$$

where α, β, $s \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $s \geq 0$, $\alpha < \beta s$, *we have:*

If function $f(y,t)$ *satisfies the Lipschitz's condition with respect to the variable* y, *with Lipschitz's constant* L, *and the condition*

$$f(y,t) > s + \frac{2\beta}{\alpha} L \quad on \quad \mathbb{R} \times I,$$

for all y, *then the problem* (26), (27) *has one-parameter class of solutions satisfying the condition*

$$\left|y(t) - \sin^2 t\right| \leq \beta \exp(-st), \quad \left|y'(t) - \sin 2t\right| \leq \alpha \exp(-st), \quad for \quad t > t_0.$$

This result follows from Theorem 1, *with* $r_1(t) = \alpha \exp(-st)$, $r_2(t) = \beta \exp(-st)$.

Example 2. *For the Van der Pol equation*

$$y'' - \mu(1 - \Phi(y))y' + y = 0, \quad \mu > 0, \tag{28}$$

and the condition

$$y^2(t_0) + {y'}^2(t_0) \leq \ln^2 t_0, \tag{29}$$

we can prove the following:

If function $\Phi(y) > 1$, *then all solutions of the problem* (28), (29) *satisfy the condition*

$$y^2(t) + {y'}^2(t) \leq \ln^2 t, \quad for \quad t \in (t_0, \infty), \quad t_0 > 1.$$

This result follows from Theorem 3, *with* $r(t) = \ln t$.

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