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Reduction of Dimension for Parabolic Equations via Two–Scale Convergence

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Abstract. Inspired by the similar ideas from homogenization theory, in [9] we introduced the notion of two-scale convergence for thin domains that allow lower-dimensional approximations. We generalize that idea to evolutional spaces appearing in study of parabolic equations. We prove the compactness theorem, analogous to the one in stationary case. We apply our method to quasi-static lubrication problem.

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1. Introduction

The notion of two scale convergence was introduced for periodic homogenization by Nguetseng [11] and fully developed by Allaire [1]. It is a powerful tool that avoids formal two-scale expansions and generalizing the idea of the energy method enables an easy proof of convergence of homogenization process.

Two-scale asymptotic expansions are also the most common tools for study of processes in thin domains and derivation of lower-dimensional models for their description (see e.g., [5, 6] (elasticity), [2, 3, 4, 7, 10] (fluid mechanics)). The method used in those papers can be roughly described as follows:

1. The problem, originally posed in a thin domain, is rewritten on a domain with unit thickness by introducing a new, dilated (or fast) variable. To do so, the differential operator has to be replaced with a new one, containing the derivatives with respect to the dilated variable. As a consequence, the negative powers of the domain thickness appear, singularly perturbing the operator.

2. On such rescaled domain, independent of the small parameter one can derive the *a priori* estimates and pass to the limit in the perturbed equation in order to get the lower-dimensional approximation. Very often passage to the limit is not straightforward, but we need to have a good candidate for the limit in order to

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choose an appropriate test function. Such a candidate has to be computed by an asymptotic expansion (see e.g., [2]). Furthermore, the obtained convergence is the convergence of the rescaled sequence of the solutions, and not of the original solutions.

3. If the problem is strongly nonlinear and non-monotone the weak convergence from step 2 is insufficient to pass to the limit and the correctors giving the strong convergence have to be found (see e.g., [3, 4]). If one wants to find the correctors and get the strong convergence, the asymptotic expansion of the solution has to be computed even for linear problems (see, for instance, [10, 6]).

In analogy with the homogenization theory in [9] we have developed the notion of two-scale convergence as a tool for deriving lower-dimensional approximations for stationary problems in thin domains. We generalize that idea to evolutional spaces $L^q(0,T;W^{s,r})$ typical for the study of parabolic PDEs. It enables us to avoid the three steps described above. To apply the two-scale convergence, we only have to find sharp *a priori* estimates. Directly from those estimates we can deduce the form of our limit and choose the test functions of the same form. We do not need to change the domain and write the problem in new variables, but can pass to the limit in the original equation. In addition, our method gives the convergence of the traces.

Finally, in this paper, as an illustration of our method, we study the viscous fluid flow between two rigid, rough surfaces in relative motion (the lubrication problem).

2. Definition of the two-scale convergence for thin domains

Definition 1. *Let $\omega \subset \mathbb{R}^m$ be a bounded $C^{0,1}$ domain, and let $\{S(x^1)\}_{x^1\in\omega}$ be a family of bounded $C^{0,1}$ domains $S(x^1) \subset \mathbb{R}^\ell$. We define a thin domain $\Omega_\varepsilon \subset \mathbb{R}^{m+\ell}$*

$$\begin{aligned}
\Omega_\varepsilon &= \{x = (x^1, x^2) \in \mathbb{R}^{m+\ell} \mid x^1 \in \omega, x^2 \in \varepsilon S(x^1)\},\\
\Gamma_\varepsilon &= \{x = (x^1, x^2) \in \mathbb{R}^{m+\ell} \mid x^1 \in \omega, x^2 \in \varepsilon \partial S(x^1)\},\\
\Sigma_\varepsilon &= \partial\Omega_\varepsilon \setminus \Gamma_\varepsilon.
\end{aligned}$$

We put $\Omega = \Omega_1$, $\Gamma = \Gamma_1$, $\Sigma = \Sigma_1$. Let $T > 0$. We say that a sequence $\{v^\varepsilon\}_{\varepsilon>0}$, such that $v^\varepsilon \in L^q(0,T;L^r(\Omega_\varepsilon))$, $L^q(0,T;L^r)$-two-scale converges to a function $V \in L^q(0,T;L^r(\Omega))$ (notation: $(L^q(0,T;L^r)$–$2s)$) if

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\ell} \int_0^T\int_{\Omega_\varepsilon} v^\varepsilon(x,t)\,\phi\left(x^1, \frac{x^2}{\varepsilon}, t\right) dx\,dt = \int_0^T\int_\Omega V(x^1,y,t)\,\phi(x^1,y,t)\,dx^1\,dy\,dt,$$
$$\forall \phi \in L^{q'}(0,T;L^{r'}(\Omega)), \quad 1/r + 1/r' = 1, \quad 1/q + 1/q' = 1.$$

We say that a sequence $\{v^\varepsilon\}_{\varepsilon>0}$ strongly $L^q(0,T;L^r)$-two-scale converges to $V \in L^q(0,T;L^r(\Omega))$ (notation: $(s$–$L^q(0,T;L^r)$–$2s)$) if

$$\lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|^{1/r}} \left| v^\varepsilon(x,t) - V\left(x^1, \frac{x^2}{\varepsilon}, t\right)\right|_{L^q(0,T;L^r(\Omega_\varepsilon))} = 0.$$

Obviously, if $\{v^\varepsilon\}$ $(L^q(0,T;L^r)$–$2s)$ converges to $V^0 \in L^q(0,T;L^r(\Omega))$, $1 < q, r < \infty$, and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\ell/r}} |v^\varepsilon|_{L^q(0,T;L^r(\Omega_\varepsilon))} = |V^0|_{L^q(0,T;L^r(\Omega))},$$

then $v^\varepsilon \to V^0$ $(s$–$L^q(0,T;L^r)$–$2s)$. Furthermore, if $v^\varepsilon \to V^0$ $(s$–$L^q(0,T;L^r)$–$2s)$ and $w^\varepsilon \to W^0$ $(L^\alpha(0,T;L^\gamma)$–$2s)$ where $1/r + 1/\gamma = 1/\beta$, $1/q + 1/\alpha = 1/\sigma$, $\sigma, \beta \geq 1$, then

$$v^\varepsilon w^\varepsilon \to V^0 W^0 \quad (L^\sigma(0,T;L^\beta)\text{–}2s).$$

As an easy consequence of Definition 1, we also get that $v^\varepsilon \to V$ $(L^q(0,T;L^r)$–$2s)$ implies

$$\frac{1}{\varepsilon^\ell} \int_{\varepsilon S(\cdot)} v^\varepsilon(\cdot, x^2)\, dx^2 \rightharpoonup \int_{S(\cdot)} V(\cdot, y)\, dy \quad \text{weakly in } L^q(0,T;L^r(\omega)).$$

3. Two-scale compactness

The main goal of this paper is to give the compactness theorem for such convergence. We use the obvious notations for the formal partial differential operators

$$\begin{aligned}
\nabla_{x^1}\phi &= \frac{\partial\phi}{\partial x_1}e_1 + \cdots + \frac{\partial\phi}{\partial x_m}e_m, & \operatorname{div}_{x^1}\phi &= \frac{\partial\phi_1}{\partial x_1} + \cdots + \frac{\partial\phi_m}{\partial x_m},\\
\nabla_{x^2}\phi &= \frac{\partial\phi}{\partial x_{m+1}}e_{m+1} + \cdots + \frac{\partial\phi}{\partial x_n}e_n, & \operatorname{div}_{x^2}\phi &= \frac{\partial\phi_{m+1}}{\partial x_{m+1}} + \cdots + \frac{\partial\phi_n}{\partial x_n},\\
\nabla_{y}\phi &= \frac{\partial\phi}{\partial y_{m+1}}e_{m+1} + \cdots + \frac{\partial\phi}{\partial y_n}e_n, & \operatorname{div}_{y}\phi &= \frac{\partial\phi_{m+1}}{\partial y_{m+1}} + \cdots + \frac{\partial\phi_n}{\partial y_n}.
\end{aligned}$$

Our next result is the main compactness theorem for the two-scale convergence. This is, in fact, a simple generalization of the analogous result from [9] to evolutional spaces.

Theorem 1. *Let $\{v^\varepsilon\}_{\varepsilon>0}$, be a sequence of functions such that $v^\varepsilon \in L^q(0,T;L^r(\Omega_\varepsilon))$, $1 < q, r \leq \infty$ and*

$$\frac{1}{|\Omega_\varepsilon|^{1/r}} |v^\varepsilon|_{L^q(0,T;L^r(\Omega_\varepsilon))} \leq C.$$

(i) Then there exists a subsequence $\{v^{\varepsilon'}\}_{\varepsilon'>0}$ and a function $V^0 \in L^q(0,T;L^r(\Omega))$ such that $v^{\varepsilon'} \to V^0$ $(L^q(0,T;L^r)$–$2s)$.

(ii) Let $X^{q,r}$ be a space of measurable functions ϕ on $]0,T[\, \times \omega$ such that $|S|^{1/r}\phi \in L^q(0,T;L^r(\omega))$. If

$$\frac{1}{|\Omega_\varepsilon|^{1/r}} |v^\varepsilon|_{L^q(0,T;W^{1,r}(\Omega_\varepsilon))} \leq C \tag{1}$$

then there exist a subsequence $\{v^{\varepsilon'}\}_{\varepsilon'>0}$,

$$V^0 = V^0(x^1, t) \in Z^{q,r} = \{\phi \in X^{q,r} \mid \nabla_{x^1}\phi \in (X^{q,r})^m\},$$

and

$$W^0 \in Y^{q,r} = \{\phi \in L^q(0,T;L^r(\Omega)) \mid \nabla_y \phi \in L^q(0,T;L^r(\Omega))^\ell\}$$

such that

$$v^{\varepsilon'} \to V^0 \quad (L^q(0,T;L^r)\text{–}2s)$$
$$\nabla v^{\varepsilon'} \to \nabla_{x^1} V^0 + \nabla_y W^0 \quad (L^q(0,T;L^r)\text{–}2s).$$

(*iii*) *If*

$$\frac{\varepsilon}{|\Omega_\varepsilon|^{1/r}} |\nabla v^\varepsilon|_{L^q(0,T;L^r(\Omega_\varepsilon))} \le C \tag{2}$$

then there exist a subsequence $\{v^{\varepsilon'}\}_{\varepsilon'>0}$ *and* $W^0 \in Y^{q,r}$ *such that*

$$v^{\varepsilon'} \to W^0 \quad (L^q(0,T;L^r)\text{–}2s)$$
$$\varepsilon \nabla v^{\varepsilon'} \to \nabla_y W^0 \quad (L^q(0,T;L^r)\text{–}2s).$$

(*iv*) *If*

$$\lim_{\varepsilon\to 0} \frac{1}{|\Omega_\varepsilon|^{1/r}} |\nabla_{x^2} v^\varepsilon|_{L^q(0,T;W^{-1,r}(\Omega_\varepsilon))} = 0$$

then the limit V^0 *satisfies* $V^0 = V^0(x^1) \in X^{q,r}$.

Proof. Since time t acts as a parameter, the above theorem is an easy generalization of the stationary version of the compactness theorem from [9]. We give a sketch of the proof and refer to [9] for details.

(*i*) We define the rescaled function $V^\varepsilon(x^1,y,t) = v^\varepsilon(x^1,\varepsilon y,t)$ for which

$$|V^\varepsilon|_{L^q(0,T;L^r(\Omega))} \le C.$$

We can extract a subsequence $\{V^{\varepsilon'}\}$ converging to some $V^0 \in L^q(0,T;L^r(\Omega))$ weakly in $L^q(0,T;L^r(\Omega))$. That is equivalent to the $(L^q(0,T;L^r)\text{–}2s)$ convergence of $v^{\varepsilon'}$.

(*ii*) If, in addition, (1) holds, then

$$|\nabla_{x^1} V^\varepsilon|_{L^q(0,T;L^r(\Omega))} \le C, \quad |\nabla_y V^\varepsilon|_{L^q(0,T;L^r(\Omega))} \le C\varepsilon.$$

We first get that $\nabla_y V^0 = 0$ implying $V^0 = V^0(x^1,t)$. We have, in addition, that $\varepsilon^{-1}|\nabla_y V^\varepsilon|_{L^q(0,T;L^r(\Omega))} \le C$. Therefore, there exists $W^0 \in Y^{q,r}$ such that

$$\varepsilon^{-1}\nabla_y V^\varepsilon \rightharpoonup \nabla_y W^0 \quad \text{weakly in } L^q(0,T;L^r(\Omega)).$$

The proof of (*iii*) is the same as the proof of existence of W^0 in (*ii*). The proof of (*iv*) is analogous to the proof that $V^0 = V^0(x^1,t)$ in (*ii*). ■

We also have the same convergence results for traces as in the stationary case.

Proposition 1.

(i) *Suppose that* $\{v^\varepsilon\}_{\varepsilon>0}$ *is a sequence of functions*

$$v^\varepsilon \in L^q(0,T;W^{1,r}(\Omega_\varepsilon)), \quad 1 < q,r < \infty$$

such that (1) *holds. Then, in addition to* (ii) *from Theorem* 1, *we have*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\ell} \int_0^T\int_{\Sigma_\varepsilon} \left[v^\varepsilon(x,t) - V^0\left(x^1, \frac{x^2}{\varepsilon}, t\right)\right] \psi\left(x^1, \frac{x^2}{\varepsilon}, t\right) d\Sigma_\varepsilon\, dt = 0,$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\ell-1}} \int_0^T\int_{\Gamma_\varepsilon} \left[v^\varepsilon(x,t) - V^0\left(x^1, \frac{x^2}{\varepsilon}, t\right)\right] \phi\left(x^1, \frac{x^2}{\varepsilon}, t\right) d\Gamma_\varepsilon\, dt = 0,$$

for any $\psi \in L^{q'}(0,T;L^{r'}(\Sigma))$ *and* $\phi \in L^{q'}(0,T;L^{r'}(\Gamma))$.

(ii) *Suppose that* $|S(x^1)| \geq c_0 > 0$ *(for simplicity) and that each* $S(x_1)$ *is star-shaped with respect to* $(x^1,0)$. *If* (2) *holds, then, in addition to Theorem* 1 (iii), *we get*

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{\ell-1}} \int_0^T\int_{\Gamma_\varepsilon} \left(v^\varepsilon(x,t) - V^0\left(x^1, \frac{x^2}{\varepsilon}, t\right)\right) \phi\left(x^1, \frac{x^2}{\varepsilon}, t\right) d\Gamma_\varepsilon\, dt = 0,$$

for any $\phi \in L^{q'}(0,T;L^{r'}(\Gamma))$.

Proof.

(i) As in the proof of Theorem 1, we get the uniform estimates for rescaled sequence V^ε in $L^q(0,T;W^{1,r}(\Omega))$, and the result follows from the boundedness of the trace operator $\gamma_0 : L^q(0,T;W^{1,r}(\Omega)) \to L^q(0,T;L^r(\partial\Omega))$.

(ii) The rescaled sequence $V^\varepsilon(x^1,y) = v^\varepsilon(x^1,\varepsilon y)$ satisfies the uniform estimate $|V^\varepsilon|_{Y^{q,r}} \leq C$ implying that $|\gamma_0(V^\varepsilon)|_{L^q(0,T;L^r(\Gamma))} \leq C$, and we see that $V^{\varepsilon'} \rightharpoonup V^0$ weakly in $L^q(0,T;L^r(\Gamma))$. ∎

Remark 1. *The estimate* (1) *is typical for the case when the boundary condition on* Γ_ε *is dynamic, like Neumann's or Robin's. Consequently, the convergence of traces on* Γ_ε *is not of particular interest as is the convergence on* Σ_ε, *where Dirichlet's condition can be imposed. On the other hand, the estimate* (2) *is typical for Dirichlet's problems, and the above result is important for trace on* Γ. *However, we have no information about the value of* V^0 *on* Σ. *This effect, typical for Dirichlet's problems in thin domains is called the boundary layer phenomenon.*

However, the boundary layer on the lateral boundary Σ for incompressible fluids appears only in direction tangential to the boundary, while the normal traces converge.

Proposition 2. *Let* $\{v^\varepsilon\}$ *be a sequence such that* $v^\varepsilon \in L^q(0,T;L^r(\Omega_\varepsilon))$, $1 < q,r < \infty$, $\operatorname{div} v^\varepsilon = 0$, $v^\varepsilon \cdot \mathbf{n} = 0$ *on* $\Gamma_\varepsilon^T = [0,T[\times\Gamma_\varepsilon$ *and* (2) *holds. Then there exist a subsequence* $\{v^{\varepsilon'}\}$ *that* $(L^q(0,T;L^r)$*–*$2s)$ *converges to some* $V^0 \in L^q(0,T;L^r(\Omega))$, $\operatorname{div}_y V^0 = 0$ *and* $\operatorname{div}_{x^1}\left(\int_{S(x^1)} V^0\, dy\right) = 0$. *Furthermore* $\left(\frac{1}{(\varepsilon')^\ell}\int_{\varepsilon' S(\cdot)} v^{\varepsilon'}\, dx^2\right)\cdot\mathbf{n} \to \left(\int_{S(\cdot)} V^0\, dy\right)\cdot\mathbf{n}$ *weakly in* $L^{q'}(0,T;W^{-1/r,r}(\partial\omega)$.

Proof. Follows from Theorem 1 and a simple partial integration. ∎

4. An application

4.1. Description of the problem

We consider the flow of a viscous fluid in a thin domain between two plates in relative motion. This is a classical problem of a lubricant injected in a slipper bearing (see [2, 7] for a stationary case). We suppose that two surfaces $\Gamma^{\pm}$ are rough (so that the bearing needs to be lubricated), and that they are moving with time dependent velocities $\mathbf{s}_{\pm}(t)$, respectively. The microscopic flow of lubricant, which is supposed to be a viscous, Newtonian fluid, is governed by the Navier–Stokes system. We are interested in a global (macroscopic) behaviour of the flow of lubricant. In fact, by asymptotic analysis of the Navier–Stokes system in a thin domain (with thickness that tends to 0), we want to find a 2–D model for description of such a thin film flow. In the stationary case the macroscopic law is the Reynolds law. Here, we get the quasi-stationary Reynolds law, where time appears only as a parameter.

We suppose that $\omega \subset \mathbb{R}^2$ is a bounded $C^{0,1}$ domain, and that the shape functions $h^{\pm} \in C^2(\overline{\omega})$ satisfy $h^+ > 0$, $h^- < 0$. We finally define

$$\Omega_\varepsilon = \{x = (x^1, x_3) \in \mathbb{R}^3 \mid x^1 = (x_1, x_2) \in \omega, \varepsilon h^-(x^1) < x_3 < \varepsilon h^+(x^1)\},$$
$$\Gamma_\varepsilon^{\pm} = \{x_3 = \varepsilon h^{\pm}\}.$$

For time interval $[0, T]$, $T > 0$, the flow is assumed to be governed by the Navier–Stokes system

$$\frac{\partial u^\varepsilon}{\partial t} - \mu \Delta u^\varepsilon + (u^\varepsilon \nabla) u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in } \Omega_\varepsilon^T = \Omega_\varepsilon \times]0, T[, \tag{3}$$

$$\operatorname{div} u^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon^T, \tag{4}$$

$$u^\varepsilon = \varepsilon^2 \mathbf{s}_{\pm} = \varepsilon^2 (s_1^{\pm}, s_2^{\pm}, 0) \quad \text{on } \Gamma_\varepsilon^{T\pm} = \Gamma_\varepsilon^{\pm} \times]0, T[, \tag{5}$$

$$u^\varepsilon = \varepsilon^2 g\left(x^1, \frac{x_3}{\varepsilon}\right) \quad \text{on } \Sigma_\varepsilon^T = \Sigma_\varepsilon \times]0, T[, \tag{6}$$

$$u^\varepsilon(\cdot, 0) = \varepsilon u_0 \quad \text{in } \Omega_\varepsilon, \tag{7}$$

where $s_1^+(t)$, $s_2^+(t)$, $s_1^-(t)$, $s_2^-(t) \in H^1(0, T)$, $f \in C^1(\overline{\omega} \times [0, T])^3$, $u_0 \in H^2(\omega)^3$, $g = (g_1, g_2, 0) \in H^1(0, T; H^{3/2}(\Sigma))^3$, $\int_\Sigma g(\cdot, t) \cdot \mathbf{n} = 0$ and $g(x^1, h^{\pm}, t) = \mathbf{s}_{\pm}(t)$ (a.e.) for $t \in]0, T[$.

Under the above conditions, due to the fact that we are treating the case of a small Reynolds number

$$Re^\varepsilon = \frac{U_\varepsilon \, L_\varepsilon}{\mu} = \frac{O(\varepsilon^2) \cdot O(\varepsilon)}{O(1)} = O(\varepsilon^3),$$

we can apply Theorem 3.7 from [12] and conclude that the problem (3)–(7) has a unique solution

$$u^\varepsilon \in H^1(0, T; H^1(\Omega_\varepsilon))^3 \cap W^{1,\infty}(0, T; L^2(\Omega_\varepsilon))^3$$

and $p^\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon)/\mathbb{R})$.

4.2. A priori estimates

The only part in applying our method that demands an effort is the derivation of sharp *a priori* estimates.

Proposition 3. *Under the above conditions there exists a constant $C > 0$, independent of ε, such that*

$$\frac{1}{\sqrt{|\Omega_\varepsilon|}} \left|\varepsilon^{-2} u^\varepsilon\right|_{L^2(\Omega_\varepsilon^T)} \leq C \tag{8}$$

$$\frac{\varepsilon}{\sqrt{|\Omega_\varepsilon|}} \left|\nabla\left(\varepsilon^{-2} u^\varepsilon\right)\right|_{L^2(\Omega_\varepsilon^T)} \leq C \tag{9}$$

$$\frac{1}{\sqrt{|\Omega_\varepsilon|}} \left|\frac{\partial u^\varepsilon}{\partial t}\right|_{L^2(\Omega_\varepsilon^T)} \leq C \tag{10}$$

$$\frac{1}{\sqrt{|\Omega_\varepsilon|}} \left|\nabla p^\varepsilon\right|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} \leq C\varepsilon \tag{11}$$

$$\frac{1}{\sqrt{|\Omega_\varepsilon|}} \left|p^\varepsilon\right|_{L^2(\Omega_\varepsilon^T)} \leq C. \tag{12}$$

Before starting the proof, we recall the result from [8] (see also [9]).

Lemma 1. *For any $r \in [1, 3[$ and $q \in [1, \frac{3r}{3-r}[$ there exists a constant $C(r,q)$, independent of ε, such that*

$$|\phi|_{L^q(\Omega_\varepsilon)} \leq C(r,q)\, \varepsilon^{1+(q^{-1}-r^{-1})}\, |\nabla\phi|_{L^r(\Omega_\varepsilon)},$$

for any $\phi \in W^{1,r}(\Omega_\varepsilon)$ such that $\phi = 0$ on some part of $\Gamma_\varepsilon^\pm$ with positive measure.

To begin the proof of Proposition 3, we define the function $h \in H^1(0,T;H^1(\Omega))^3$ such that, for any $t \in [0,T]$

$$\begin{cases} \operatorname{div} h(\cdot,t) = 0 \quad \text{in } \Omega, \\ h(\cdot,t) = \mathbf{s}_\pm(t) \quad \text{on } \Gamma^\pm, \\ h(x^1,y,t) = g(x^1,y,t) \quad \text{on } \Sigma, \end{cases}$$

and $|h|_{H^1(0,T;H^1(\Omega))} \leq C$. We now define

$$H^\varepsilon(x) = \Big(h_1\Big(x^1, \frac{x_3}{\varepsilon}\Big), h_2\Big(x^1, \frac{x_3}{\varepsilon}\Big), \varepsilon h_3\Big(x^1, \frac{x_3}{\varepsilon}\Big)\Big)$$

so that the function $v^\varepsilon = u^\varepsilon - \varepsilon^2 H^\varepsilon$ is divergence free and has a zero trace on $\partial\Omega_\varepsilon$. We can now prove (9).

Lemma 2.

$$|\nabla u^\varepsilon|_{L^2(0,T;L^2(\Omega_\varepsilon))} \leq C\varepsilon^{3/2}.$$

Proof. Multiplying (3) by v^ε and integrating over Ω_ε^T, we obtain

$$|\nabla u^\varepsilon|^2_{L^2(\Omega_\varepsilon^T)} = \frac{\varepsilon^3}{2}\,|u_0|^2_{L^2(\omega)} + \mu\varepsilon^2 \int_{\Omega_\varepsilon^T} \nabla u^\varepsilon \nabla H^\varepsilon + \varepsilon^2 \int_{\Omega_\varepsilon^T} (u^\varepsilon\nabla)H^\varepsilon v^\varepsilon + \int_{\Omega_\varepsilon^T} f v^\varepsilon.$$

Since

$$\int_{\Omega_\varepsilon^T} (u^\varepsilon\nabla)H^\varepsilon u^\varepsilon \le C\varepsilon^{3/2}|\nabla u^\varepsilon|^2_{L^2(\Omega_\varepsilon^T)}|\nabla H^\varepsilon|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \le C\varepsilon|\nabla u^\varepsilon|^2_{L^2(\Omega_\varepsilon^T)},$$

the claim is proved. ■

Lemma 3.

$$\left|\frac{\partial u^\varepsilon}{\partial t}\right|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} \le C\sqrt{\varepsilon}.$$

Proof. We follow the proof of Theorem 3.7 from [12]. Let $\{u_m^\varepsilon\}$ denote the sequence of Galerkin's approximations for u^ε. For $t=0$, by direct integration we obtain

$$\begin{aligned}\left|\frac{\partial u_m^\varepsilon}{\partial t}(\cdot,0)\right|^2_{L^2(\Omega_\varepsilon)} &\le \big(\varepsilon\mu|\Delta u_0|_{L^2(\Omega_\varepsilon)} + \varepsilon^2\sqrt{\varepsilon}\,|u_0|_{L^\infty(\omega)}|\nabla u_0|_{L^2(\omega)} + \sqrt{\varepsilon}\,|f(\cdot,0)|_{L^2(\omega)}\big)\\ &\quad\cdot\left(\left|\frac{\partial u_m^\varepsilon}{\partial t}(\cdot,0)\right|_{L^2(\Omega_\varepsilon)} + \varepsilon^2\left|\frac{\partial H^\varepsilon}{\partial t}(\cdot,0)\right|^2_{L^2(\Omega_\varepsilon)}\right)\end{aligned}$$

leading to

$$\left|\frac{\partial u_m^\varepsilon}{\partial t}(\cdot,0)\right|_{L^2(\Omega_\varepsilon)} \le C\sqrt{\varepsilon}.$$

As in the proof of Theorem 3.7 from [12], we conclude that $\mu - C\varepsilon^{3/2}|\nabla u_m^\varepsilon|_{L^2(\Omega_\varepsilon)} > 0$ for any $t\in[0,T]$. Now, deriving with respect to t, multiplying by $\frac{\partial u_m^\varepsilon}{\partial t}$ and integrating over Ω_ε^T, we obtain

$$\left|\frac{\partial u_m^\varepsilon}{\partial t}\right|^2_{L^2(\Omega_\varepsilon)} \le C\left(\left|\frac{\partial u_m^\varepsilon}{\partial t}(\cdot,0)\right|^2_{L^2(\Omega_\varepsilon)} + \left|\frac{\partial f}{\partial t}\right|^2_{L^2(\Omega_\varepsilon^T)} + \varepsilon^4\left|\nabla\left(\frac{\partial H^\varepsilon}{\partial t}\right)\right|^2_{L^2(\Omega_\varepsilon^T)}\right) \le C\sqrt{\varepsilon}.$$

■

Lemma 4.

$$|\nabla p^\varepsilon|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))} \le \varepsilon^{3/2}.$$

Proof. Let $\phi\in H_0^1(\Omega_\varepsilon^T)^3$. Then

$$\langle\nabla p^\varepsilon|\phi\rangle = \mu\int_{\Omega_\varepsilon^T}\big(\nabla u^\varepsilon\nabla\phi - (u^\varepsilon\nabla)u^\varepsilon\phi + f\phi\big) - \int_{\Omega_\varepsilon^T}\frac{\partial u_m^\varepsilon}{\partial t}\,\phi \le C\varepsilon^{3/2}|\nabla\phi|_{L^2(0,T;L^2(\Omega_\varepsilon))}.$$

■

Finally, we need a variant of Nečas inequality (see [9] for the proof).

Lemma 5. *There exists $C > 0$ such that, for any $\phi \in L^2(\Omega_\varepsilon^T)$,*

$$\left| \phi(x,t) - |\Omega_\varepsilon|^{-1} \int_{\Omega_\varepsilon} \phi(\cdot,t)\,dx \right|_{L^2(\Omega_\varepsilon^T)} \le C\varepsilon^{-1} |\nabla\phi|_{L^2(0,T;H^{-1}(\Omega_\varepsilon))}.$$

Now an application of Lemma 5 on (11) proves (12), ending the proof of Proposition 3.

Theorem 1 and Propositions 1 and 2 imply the existence of $U^0 \in Y^{2,2}$, $P^0 = P^0(x^1,t) \in L^2(\omega_T)$, $\omega_T = \omega \times]0,T[$, such that (up to a subsequence)

$$\begin{aligned}
&\varepsilon^{-2}u^\varepsilon \to U^0, \quad \varepsilon^{-1}\nabla u^\varepsilon \to \frac{\partial U^0}{\partial y}\,e_3, \quad p^\varepsilon \to P^0 \quad (L^2(0,T;L^r)\text{–}2s), \qquad (13)\\
&\frac{\partial U_3^0}{\partial y} = 0,\\
&\mathrm{div}_{x^1}\left(\int_{h^-}^{h^+} U^0\,dy\right) = 0 \quad \text{in } \omega_T,\\
&U^0 = \mathbf{s}_\pm \quad \text{on } \Gamma_T^\pm = \Gamma^\pm \times]0,T[,\\
&U_3^0 = 0 \quad \text{on } \Gamma_T^\pm \implies U_3^0 = 0,\\
&\left(\int_{h_-}^{h^+} U^0\,dy\right)\cdot\mathbf{n} = \left(\int_{h^-}^{h^+} g\,dy\right)\cdot\mathbf{n} \quad \text{on } \partial\omega.
\end{aligned}$$

Since $U_3^0 = 0$, we choose a test function $\phi \in Y^{2,2}$ such that $\phi = 0$ on $\partial\Omega_T$, $\phi_3 = 0$. Using the above convergence, we pass to the limit in the variational form of (3)

$$-\int_{\Omega_\varepsilon^T} u^\varepsilon \frac{\partial\phi^\varepsilon}{\partial t} + \mu\int_{\Omega_\varepsilon^T} \nabla u^\varepsilon \nabla\phi^\varepsilon + \int_{\Omega_\varepsilon^T} (u^\varepsilon\nabla)u^\varepsilon\phi^\varepsilon = \int_{\Omega_\varepsilon^T} f\phi^\varepsilon + \int_{\Omega_\varepsilon^T} p^\varepsilon\,\mathrm{div}_{x^1}\phi^\varepsilon,$$

where $\phi^\varepsilon(x,t) = \phi\left(x^1, \frac{x_3}{\varepsilon}, t\right)$, and we get

$$\mu\int_{\Omega_T} \frac{\partial U^0}{\partial y}\frac{\partial\phi}{\partial y} = \int_{\Omega_T} f\phi + \int_{\omega_T} P^0\,\mathrm{div}_{x^1}\left(\int_{h^-}^{h^+}\phi\,dy\right). \qquad (14)$$

This is a two-scale problem.

Proposition 4. *There exist a unique $U^0 = (U_1^0, U_2^0, 0) \in Y^{2,2}$ and a unique (up to a constant) $P^0 \in L^2(\omega_T)$ solution of (14). Furthermore, we have*

$$\begin{cases}
-\mu\dfrac{\partial^2 U^0}{\partial y^2} = f - \nabla_{x^1}P^0 \quad \text{in } \Omega_T = \Omega\times]0,T[,\\
\mathrm{div}_{x^1}\left(\displaystyle\int_{h^-}^{h^+} U^0\,dy\right) = 0 \quad \text{in } \omega_T,\\
\left(\displaystyle\int_{h^-}^{h^+} U^0\,dy\right)\cdot\mathbf{n} = \left(\displaystyle\int_{h^-}^{h^+} g\,dy\right)\cdot\mathbf{n} \quad \text{on } \partial\omega_T, \quad U^0(x^1,h^\pm,t) = \mathbf{s}_\pm.
\end{cases} \qquad (15)$$

The uniqueness implies that not only subsequences, but whole sequences in (13) are convergent. The above problem (15) can be decoupled and we obviously have (16)–(18). To summarize, we have the following result.

Theorem 2. *Let* $(u^\varepsilon, p^\varepsilon)$ *be the solution of the Navier–Stokes system* (3)–(7)*. Let* $P^0 \in L^2(0,T;H^1(\omega))$ *and* $U^0 \in Y^{2,2}$ *be the solutions of the quasi-stationary Reynolds system (where t is only a parameter)*

$$U^0 = \frac{1}{2\mu}\,(y-h^-)(h^+-y)(f-\nabla_{x_1}P^0) + \frac{h^+-y}{\theta}\,\mathbf{s}_+ + \frac{y-h^-}{\theta}\,\mathbf{s}_-, \tag{16}$$

$$-\operatorname{div}\left(\theta^3(f-\nabla_{x^1}P^0)\right) = 6\mu\nabla\theta\cdot(\mathbf{s}_+ + \mathbf{s}_-) \quad \text{in } \omega_T, \tag{17}$$

$$\theta^3(f-\nabla_{x^1}P^0)\cdot\mathbf{n} = 12\mu\left(\int_{h^-}^{h^+} g\,dy\right)\cdot\mathbf{n} - 6\mu\theta(\mathbf{s}_+ + \mathbf{s}_-)\cdot\mathbf{n} \quad \text{on } \partial\omega_T, \tag{18}$$

where $\theta = h^+ - h^-$*. Then*

$$\frac{u^\varepsilon}{\varepsilon^2} \to U^0, \quad p^\varepsilon \to P^0 \quad (L^2(0,T;L^2)\text{-}2s).$$

■

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