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On Polynomial Spline Collocation Methods for Neutral Volterra Integro–Differential Equations with Delay Arguments*

Vilmoš Horvat†

Abstract. In this paper we construct and give an analysis of the global convergence and local superconvergence properties of polynomial collocation solution $u \in S_{m+d}^{(d)}(Z_N)$ of neutral Volterra integro–differential equations with constant delay, thus extending the existing theory for $d = 0$ to the general case.

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1. Introduction

We consider the numerical discretization of neutral Volterra integro–differential equations (VIDE) with (constant) delay $\tau > 0$,

$$y'(t) = f(t, y(t)) + \int_0^t k_1\big(t, s, y(s), y'(s)\big)\, ds + \int_0^{t-\tau} k_2\big(t, s, y(s), y'(s)\big)\, ds, \qquad (1)$$

for $t \in I := [0, T]$, whose solution $y(t)$ is to agree with a given initial C^1 function ϕ

$$y(t) = \phi(t), \quad t \in [-\tau, 0], \qquad (2)$$

by collocation methods in certain polynomial spline spaces.

It will be assumed that functions $\phi : [-\tau, 0] \to \mathbb{R}$, $f : I \times \mathbb{R} \to \mathbb{R}$, $k_1 : S \times \mathbb{R}^2 \to \mathbb{R}$ ($S := \{(t, s) \mid 0 \le s \le t \le T\}$), and $k_2 : S_\tau \times \mathbb{R}^2 \to \mathbb{R}$ ($S_\tau := I \times [-\tau, T - \tau]$) are sufficiently smooth on their domains for equation (1) to have a unique solution $y \in C^l(I)$, with $l \in \mathbb{N}$. We will not discuss the "classical" neutral VIDE ($k_2 \equiv 0$) and we assume that $k_2(t, s, y, y')$ does not vanish identically on its domain. Existence and uniqueness results for (1) can be found, for example, in [6, 7].

As the delay VIDEs became important in the mathematical modelling of biological and physical phenomena, for example [13], there has been a growing interest in the

*Supported by Grant 037011 of the Ministry of Science and Technology, Republic of Croatia.
†Department of Mathematics, University of Zagreb, Bijenička cesta 30, 10000 Zagreb, Croatia, e–mail: horvat@math.hr

numerical solution of VIDEs. One-step and linear multistep methods were discussed for neutral Volterra functional differential equations and VIDEs in [10, 12] (one-step methods) and in [11] (multistep methods). Collocation methods for ordinary VIDEs were studied in polynomial spline spaces $S_m^{(0)}(Z_N)$ and $S_m^{(1)}(Z_N)$ in [1, 5]. Details of the notation are given in Section 2. The solution of (1) is constructed in piecewise polynomial space $S_m^{(0)}(Z_N)$ in [4], and in [2, 3] (ordinary delay VIDE with unbounded delay).

The aim of this paper is to construct a new polynomial collocation solution of (1) in the polynomial space $S_{m+d}^{(d)}(Z_N)$, with $m-2 \geq d \geq 0$. An approximate solution of VIDE with constant delay in the polynomial space $S_{m+d}^{(d)}(Z_N)$, with $m-2 \geq d \geq 0$, is studied in [9]. Compare also [8] for an analysis of collocation methods for delay Volterra integral equations in the polynomial space $S_{m+d}^{(d)}(Z_N)$, with $d \geq -1$. The approximation $u \in S_{m+d}^{(d)}(Z_N)$ will be determined by collocation. The attainable order of global and local convergence of these methods, both in exact and discretized case, is analysed in detail.

2. The collocation method

Let $t_n = nh$ $(n = 0, \dots, N-1,\ t_N = T)$ define a uniform partition for $I = [0, T]$, and let $\Pi_N := \{t_0, \dots, t_N\}$, $\sigma_n := [t_n, t_{n+1}]$, for $n = 0, \dots, N-1$. The mesh Π_N is constrained in the following sense:

$$h = \frac{\tau}{r}, \quad \text{for some} \quad r \in \mathbb{N}. \tag{3}$$

With a given mesh Π_N we associate the set $Z_N := \{t_n \mid n = 1, \dots, N-1\}$ of its interior points. For a fixed $N \geq 1$, and for given integers $d \geq -1$ and $m \geq 1$, the piecewise polynomial space $S_{m+d}^{(d)}(Z_N)$ is defined by

$$S_{m+d}^{(d)}(Z_N) := \big\{u : I \to \mathbb{R} \bigm| u|_{\sigma_n} =: u_n \in \pi_{m+d},\, u_{n-1}^{(\nu)}(t_n) = u_n^{(\nu)}(t_n),\, \nu = 0, \dots, d\big\},$$

where π_{m+d} denotes the space of (real) polynomials of a degree not exceeding $m+d$. The dimension of $S_{m+d}^{(d)}(Z_N)$ is $Nm + d + 1$. In this paper, we will assume that $m-2 \geq d \geq 0$. Let $u \in S_{m+d}^{(d)}(Z_N)$, $u_n = u|_{\sigma_n}$. For all $t \in \sigma_n$ we have

$$u_n(t) = \sum_{l=0}^{d} \frac{u_{n-1}^{(l)}(t_n)}{l!}(t-t_n)^l + \sum_{l=1}^{m} a_{n,l}(t-t_n)^{d+l}, \quad n = 0, \dots, N-1, \tag{4}$$

where

$$u_{-1}^{(l)}(0) = y^{(l)}(0), \quad l = 0, \dots, d. \tag{5}$$

From (4) we see that an element $u \in S_{m+d}^{(d)}(Z_N)$ is well defined if we know the coefficients $\{a_{n,l}\}$ for all $n = 0, \dots, N-1$. In order to compute these coefficients, we consider the set of collocation parameters $\{c_j\}$, where $0 \le c_1 < \cdots < c_m \le 1$, and define the set $X_N := \{t_{n,j}\}_{j=1,\,n=0}^{m,\;N-1}$ of collocation points by

$$t_{n,j} := t_n + c_j h, \quad j = 1, \dots, m, \quad n = 0, \dots, N-1. \tag{6}$$

The collocation solution $u \in S_{m+d}^{(d)}(Z_N)$ will be determined by imposing the condition that u satisfies the neutral integro–differential equation (1) on the finite set X_N

$$u'(t) = f(t, u(t)) + \int_0^t k_1\big(t, s, u(s), u'(s)\big)\, ds + \int_0^{t-\tau} k_2\big(t, s, u(s), u'(s)\big)\, ds, \tag{7}$$

with

$$u(t) = \phi(t) \quad \text{on} \quad [-\tau, 0]. \tag{8}$$

If $t = t_{n,j} < \tau$, or $t_{n-r,j} := t_{n,j} - \tau < 0$ (from (3), $\tau = rh = t_r$), then (1) becomes

$$u'(t) = f(t, u(t)) + \int_0^t k_1\big(t, s, u(s), u'(s)\big)\, ds - \Phi(t), \quad t = t_{n,j}, \tag{9}$$

for $j = 1, \dots, m$, $n = 0, \dots, r-1$, where

$$\Phi(t) := \int_{t-\tau}^0 k_2\big(t, s, \phi(s), \phi'(s)\big)\, ds. \tag{10}$$

In contrast to the classical neutral VIDE corresponding to $k_2 = 0$, the occurrence of the term $\Phi(t)$ in the collocation equation (9) reveals that, for $t = t_{n,j} < \tau$, we have to evaluate (or approximate) a *functional* containing the given initial function $\phi(t)$.

In order to put (7) into a form amenable to numerical computation, let $t \in \sigma_n$, and define

$$F_n(t) := \int_0^{t_n} k_1\big(t, s, u(s), u'(s)\big)\, ds.$$

Further, let

$$D(t) := \int_0^{t-\tau} k_2\big(t, s, u(s), u'(s)\big)\, ds, \quad \text{with} \quad D(t) := -\Phi(t), \quad \text{if} \quad t < \tau.$$

By using (6), equation (7) can be written as

$$\begin{aligned} u'(t_{n,j}) &= f(t_{n,j}, u(t_{n,j})) + F_n(t_{n,j}) + D(t_{n,j}) \\ &\quad + h \int_0^{c_j} k_1\big(t_{n,j}, t_n + vh, u(t_n + vh), u'(t_n + vh)\big)\, dv, \end{aligned} \tag{11}$$

for $j = 1, \dots, m$.

Generally, the integrals on the right-hand side in (11), including those in $F_n(t_{n,j})$ and $D(t_{n,j})$, cannot be evaluated analytically, but have to be approximated by suitable quadrature formulæ.

Let μ_0 and μ_1 be given positive integers. Suppose that the quadrature parameters $\{d_l\}$ and $\{d_{j,l}\}$ satisfy $0 \leq d_1 < \cdots < d_{\mu_1} \leq 1$ and $0 \leq d_{j,1} < \cdots < d_{j,\mu_0} \leq c_j$, $j = 1, \ldots, m$, respectively, and w_l, $w_{j,l}$ denote the corresponding quadrature weights. The fully discretized collocation equation corresponding to (11) is thus given by

$$\begin{aligned} \hat{u}'(t_{n,j}) &= f(t_{n,j}, \hat{u}(t_{n,j})) + \widehat{F}_n(t_{n,j}) + \widehat{D}(t_{n,j}) \\ &\quad + h \sum_{l=1}^{\mu_0} w_{j,l}\, k_1\big(t_{n,j}, t_n + d_{j,l}h, \hat{u}(t_n + d_{j,l}h), \hat{u}'(t_n + d_{j,l}h)\big), \end{aligned} \tag{12}$$

for $j = 1, \ldots, m$, with

$$\widehat{F}_n(t_{n,j}) := h \sum_{i=0}^{n-1} \sum_{l=1}^{\mu_1} w_l\, k_1\big(t_{n,j}, t_i + d_l h, \hat{u}(t_i + d_l h), \hat{u}'(t_i + d_l h)\big), \tag{13}$$

and, if $n - r \geq 0$, then

$$\begin{aligned} \widehat{D}_n(t_{n,j}) &:= h \sum_{i=0}^{n-r-1} \sum_{l=1}^{\mu_1} w_l\, k_2\big(t_{n,j}, t_i + d_l h, \hat{u}(t_i + d_l h), \hat{u}'(t_i + d_l h)\big) \\ &\quad + h \sum_{l=1}^{\mu_0} w_{j,l}\, k_2\big(t_{n,j}, t_{n-r} + d_{j,l}h, \hat{u}(t_{n-r} + d_{j,l}h), \hat{u}'(t_{n-r} + d_{j,l}h)\big). \end{aligned} \tag{14}$$

If $n - r < 0$, then $\widehat{D}_n(t_{n,j})$ is given either by the exact value of $-\Phi(t_{n,j})$ (recall (10)),

$$\begin{aligned} \widehat{D}_n(t_{n,j}) = D_n(t_{n,j}) &= -h \int_{c_j}^{1} k_2\big(t_{n,j}, t_{n-r} + vh, \phi(t_{n-r} + vh), \phi'(t_{n-r} + vh)\big)\, dv \\ &\quad - h \sum_{i=n-r+1}^{-1} \int_0^1 k_2\big(t_{n,j}, t_i + vh, \phi(t_i + vh), \phi'(t_i + vh)\big)\, dv, \end{aligned} \tag{15}$$

or by a suitable quadrature approximation to $-\Phi(t_{n,j})$,

$$\begin{aligned} \widehat{D}_n(t_{n,j}) &= -h \sum_{l=1}^{\mu_1} \widetilde{w}_{j,l}\, k_2\big(t_{n,j}, t_{n-r} + \xi_{j,l}h, \phi(t_{n-r} + \xi_{j,l}h), \phi'(t_{n-r} + \xi_{j,l}h)\big) \\ &\quad - h \sum_{i=n-r+1}^{-1} \sum_{l=1}^{\mu_1} w_l\, k_2\big(t_{n,j}, t_i + d_l h, \phi(t_i + d_l h), \phi'(t_i + d_l h)\big), \end{aligned} \tag{16}$$

where $\xi_{j,l} := c_j + (1 - c_j)d_l$, $\widetilde{w}_{j,l} := (1 - c_j)w_l$, for j, $l = 1, \ldots, m$.

Since the quadrature error terms will be disregarded, we generate an approximation $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ of the following form, for $n = 0, \ldots, N-1$:

$$\begin{aligned} \hat{u}(t) = \hat{u}_n(t) &= \sum_{l=0}^{d} \frac{\hat{u}_{n-1}^{(l)}(t_n)}{l!}(t - t_n)^l + \sum_{l=1}^{m} \hat{a}_{n,l}(t - t_n)^{d+l}, \quad t \in \sigma_n, \\ \hat{u}_{-1}^{(l)}(0) &= y^{(l)}(0), \quad l = 0, \ldots, d. \end{aligned} \tag{17}$$

Equations (11) and (12) represent, for each $n = 0, \ldots, N-1$, a recursive system of m nonlinear algebraic equations with the unknowns $\{a_{n,r}\}$, and $\{\hat{a}_{n,r}\}$, respectively. Since the solutions of the systems have been found, the values of u and $\hat{u}$, and their derivatives on σ_n are determined by (4) and (17), respectively.

3. Global convergence

Let $u \in S_{m+d}^{(d)}(Z_N)$ be the (exact) collocation solution of (1) defined by (7)–(9). For simplicity, we will focus on the linear version of (1),

$$y'(t) = p(t)y(t) + q(t) + (V_1 y)(t), \quad t \in I, \tag{18}$$

with

$$(V_1 y)(t) := \int_0^t H^{[1]}\big(t, s, y(s), y'(s)\big)\, ds + \int_0^{t-\tau} H^{[2]}\big(t, s, y(s), y'(s)\big)\, ds$$

and

$$H^{[i]}\big(t, s, y(s), y'(s)\big) := \sum_{\nu=0}^{1} K_\nu^{[i]}(t,s)\, y^{(\nu)}(s), \quad i = 1, 2,$$

where $K_\nu^{[1]} \in C(S)$, $K_\nu^{[2]} \in C(S_\tau)$, $\nu = 0, 1$. A comment on the extension of the convergence results for the nonlinear equation (1) can be found at the end of this section.

Theorem 1. *Assume that the given functions in* (18) *and* (2) *satisfy* $p, q \in C^{m+d}(I)$, $K_\nu^{[1]} \in C^{m+d}(S)$, $K_\nu^{[2]} \in C^{m+d}(S_\tau)$, $\nu = 0, 1$, $\phi \in C^{m+d}([-\tau, 0])$, *and that for* $t \in [0, \tau]$ *the integral* (10)

$$\Phi(t) := \int_{t-\tau}^{0} \sum_{\nu=0}^{1} K_\nu^{[2]}(t,s)\, \phi^{(\nu)}(s)\, ds$$

is exactly known. Then for all sufficiently small $h = \tau/r$, $r \in \mathbb{N}$, *the collocation solution* $u \in S_{m+d}^{(d)}(Z_N)$, $m - 2 \geq d \geq 0$, *of* (18) *satisfies*

$$\|y - u\|_\infty \leq C_0 h^{m+d}, \quad \|y^{(k)} - u^{(k)}\|_\infty \leq C_k h^{m+d+1-k}, \tag{19}$$

for all $k = 1, \ldots, m+d$, *where* C_k *are positive constants not depending on* h. *This estimate holds for all collocation parameters* $\{c_j\}$, *with* $0 \leq c_1 < \cdots < c_m \leq 1$.

Proof. Assume without any loss of generality, that $T = M\tau$ for some $M \in \mathbb{N}$. In each interval $J_\mu := (\mu\tau, (\mu+1)\tau)$, the exact solution y of (18) is $m+d+1$ times continuously differentiable. This follows from the smoothness hypotheses we have imposed on ϕ, p, q, $K_\nu^{[i]}$, $\nu = 0, 1$, $i = 1, 2$, and from the expressions for $y^{(\nu)}(t)$ obtained by successively differentiating (18) with respect to t. From this it is obvious that left and right limits of $y^{(\nu)}(t)$, as t tends to $\mu\tau$, exist and are finite for $\nu = 0, \ldots, m+d+1$.

We will prove the estimates (19) by induction, using a similar technique as in [9]. Let $e := y - u$ be the collocation error, and let $e_n := e|_{\sigma_n} = y - u_n$ on σ_n. For $n = 0, \ldots, N-1$, and all $t = t_n + vh \in \sigma_n$, the exact solution y can be expanded in Taylor series:

$$y(t_n + vh) = \sum_{l=0}^{m+d} \frac{y^{(l)}(t_n)}{l!}\, v^l h^l + R_n(v) h^{m+d+1}, \tag{20}$$

with $R_n(v)$ denoting the remainder term in Taylor's formula. So, by (4) and (20) we have

$$e(t_n + vh) = h^p \Big(\bar{\beta}_{n,0} + \sum_{l=1}^{d} \bar{\beta}_{n,l} v^l + \sum_{l=1}^{m} \beta_{n,l} v^{d+l} + h^{m+d+1-p} R_n(v) \Big), \tag{21}$$

where

$$h^p \bar{\beta}_{n,l} = \frac{e_{n-1}^{(l)}(t_n)}{l!}\, h^l, \quad l = 0, \ldots, d, \tag{22}$$

$$h^p \beta_{n,l} = \Big(\frac{y^{(d+l)}(t_n)}{(d+l)!} - a_{n,l} \Big) h^{d+l}, \quad l = 1, \ldots, m. \tag{23}$$

The exponent p in h^p will be suitably chosen later. From (21) it follows that

$$e'(t_n + vh) = h^{p-1} \Big(\sum_{l=1}^{d} l \bar{\beta}_{n,l} v^{l-1} + \sum_{l=1}^{m} (d+l) \beta_{n,l} v^{d+l-1} + h^{m+d+1-p} R_n'(v) \Big). \tag{24}$$

As y is the solution of (18), and $u \in S_{m+d}^{(d)}(Z_N)$ satisfies the exact collocation equation (7)–(9), the collocation error e satisfies

$$e'(t_{n,j}) = p_{n,j} e(t_{n,j}) + (V_1 e)(t_{n,j}), \quad j = 1, \ldots, m, \quad n = 0, \ldots, N-1, \tag{25}$$

where $p_{n,j} := p(t_{n,j})$. The abbreviations $K_{n,j}^{[\nu,i]}(\cdot) := K_\nu^{[i]}(t_{n,j}, \cdot)$, $\nu = 0, 1$, $i = 1, 2$, and $(H_{n,j}^{[i]} y)(s) := H^{[i]}\big(t_{n,j}, s, y(s), y'(s)\big)$, $i = 1, 2$, will be used.

If $t_n < \tau = t_r$, then $t_{n-r,j} = t_n + c_j h - \tau \le 0$. Since $u(t) = \phi(t)$ on $[-\tau, 0]$, the equation (25) is reduced, and can be written as

$$e'(t_{n,j}) = p_{n,j} e(t_{n,j}) + h \int_0^{c_j} (H_{n,j}^{[1]} e)(t_n + vh)\, dv + h \sum_{i=0}^{n-1} \int_0^1 (H_{n,j}^{[1]} e)(t_i + vh)\, dv. \tag{26}$$

If $t_n \ge t_r$, the equation (25) can be written as

$$\begin{aligned} e'(t_{n,j}) = p_{n,j} e(t_{n,j}) &+ h \int_0^{c_j} (H_{n,j}^{[1]} e)(t_n + vh)\, dv + h \sum_{i=0}^{n-1} \int_0^1 (H_{n,j}^{[1]} e)(t_i + vh)\, dv \\ &+ h \int_0^{c_j} (H_{n,j}^{[2]} e)(t_{n-r} + vh)\, dv + h \sum_{i=0}^{n-r-1} \int_0^1 (H_{n,j}^{[2]} e)(t_i + vh)\, dv. \end{aligned} \tag{27}$$

In order to make the following analysis more transparent, we introduce the vectors

$$\beta_n := (\beta_{n,1}, \ldots, \beta_{n,m})^T, \quad \bar{\beta}_n := (\bar{\beta}_{n,1}, \ldots, \bar{\beta}_{n,d})^T, \quad n = 0, \ldots, N-1,$$
$$q_n^{(\eta)} := (q_{n,1}^{(\eta)}, \ldots, q_{n,m}^{(\eta)})^T, \quad n = (\eta-1)r, \ldots, (\eta-1)(N-r)+r-1, \quad \eta = 1, 2,$$

with

$$q_{n,j}^{(1)} := -R_n'(c_j) + hp_{n,j}R_n(c_j) + h\int_0^{c_j} \sum_{\nu=0}^{1} h^{1-\nu} K_{n,j}^{[\nu,1]}(t_n + vh)\, R_n^{(\nu)}(v)\, dv$$
$$+ h \sum_{i=0}^{n-1} \int_0^1 \sum_{\nu=0}^{1} h^{1-\nu} K_{n,j}^{[\nu,1]}(t_i + vh)\, R_i^{(\nu)}(v)\, dv, \quad j = 1, \ldots, m,$$

and

$$q_{n,j}^{(2)} := q_{n,j}^{(1)} + h\int_0^{c_j} \sum_{\nu=0}^{1} h^{1-\nu} K_{n,j}^{[\nu,2]}(t_{n-r} + vh)\, R_{n-r}^{(\nu)}(v)\, dv$$
$$+ h \sum_{i=0}^{n-r-1} \int_0^1 \sum_{\nu=0}^{1} h^{1-\nu} K_{n,j}^{[\nu,2]}(t_i + vh)\, R_i^{(\nu)}(v)\, dv, \quad j = 1, \ldots, m.$$

By using (22)–(23), and $p = m+d$ in the equations (26)–(27), we obtain the recurrence relation for the vectors β_n of the form

$$\big(V - hP_n - hQ_{n,n}^{(1)}\big)\beta_n = \begin{cases} h\displaystyle\sum_{i=0}^{n-1} Q_{n,i}^{(1)}\beta_i + r_n^{(1)} + \bar{r}_n^{(1)} + hq_n^{(1)}, \\ \qquad\qquad\qquad\qquad \text{for } n = 0, \ldots, r-1, \\ h\displaystyle\sum_{i=0}^{n-1} Q_{n,i}^{(1)}\beta_i + h\sum_{i=0}^{n-r} Q_{n,i}^{(2)}\beta_i + r_n^{(2)} + \bar{r}_n^{(2)} + hq_n^{(2)}, \\ \qquad\qquad\qquad\qquad \text{for } n = r, \ldots, N-1, \end{cases} \tag{28}$$

where the matrices V, P_n, $Q_{n,i}^{(1)}$, $Q_{n,i}^{(2)}$, W, W', $F_{n,i}^{(1)}$, $F_{n,i}^{(2)}$, and the vectors $r_n^{(1)}$, $\bar{r}_n^{(1)}$, $r_n^{(2)}$, $\bar{r}_n^{(2)}$, are given by

$$V := ((d+l)\, c_j^{d+l-1}), \quad P_n := (c_j^{d+l} p_{n,j}), \quad j, l = 1, \ldots, m,$$
$$W := (c_j^l p_{n,j}), \quad W' := (lc_j^{l-1}), \quad j = 1, \ldots, m, \quad l = 1, \ldots, d,$$

$$Q_{n,i}^{(\eta)} := \begin{cases} \displaystyle\int_0^1 \sum_{\nu=0}^{1} h^{1-\nu}\, (d+l)^\nu K_{n,j}^{[\nu,\eta]}(t_i + vh)\, v^{d+l-\nu}\, dv, & \text{if } 0 \le i \le n-1-r\delta_{2,\eta}, \\ \displaystyle\int_0^{c_j} \sum_{\nu=0}^{1} h^{1-\nu}\, (d+l)^\nu K_{n,j}^{[\nu,\eta]}(t_i + vh)\, v^{d+l-\nu}\, dv, & \text{if } i = n - r\delta_{2,\eta}, \end{cases}$$

for $j, l = 1, \ldots, m$, and $\eta = 1, 2$,

$$F_{n,i}^{(\eta)} := \begin{cases} \displaystyle\int_0^1 \sum_{\nu=0}^{1} h^{1-\nu}\, l^\nu K_{n,j}^{[\nu,\eta]}(t_i+vh)\, v^{l-\nu}\, dv, & \text{if } 1 \le i \le n-1-r\delta_{2,\eta}, \\ \displaystyle 4\int_0^{c_j} \sum_{\nu=0}^{1} h^{1-\nu}\, l^\nu K_{n,j}^{[\nu,\eta]}(t_i+vh)\, v^{l-\nu}\, dv, & \text{if } i = n-r\delta_{2,\eta}, \end{cases}$$

for $j = 1, \dots, m$, $l = 1, \dots, d$, and $\eta = 1, 2$, where $\delta_{2,\eta}$ is the Kronecker symbol,

$$r_n^{(1)} := \big(-W' + hW + hF_{n,n}^{(1)}\big)\bar\beta_n + h\sum_{i=1}^{n-1} F_{n,i}^{(1)}\,\bar\beta_i,$$

$$r_n^{(2)} := r_n^{(1)} + h\sum_{i=1}^{n-r} F_{n,i}^{(2)}\,\bar\beta_i,$$

$$\bar r_{n,j}^{(1)} := \left(hp_{n,j} + h^2\int_0^{c_j} K_{n,j}^{[0,1]}(t_n+vh)\,dv\right)\bar\beta_{n,0} + h^2\sum_{i=0}^{n-1}\bar\beta_{i,0}\int_0^1 K_{n,j}^{[0,1]}(t_i+vh)\,dv,$$

$$\bar r_{n,j}^{(2)} := \bar r_{n,j}^{(1)} + h^2\bar\beta_{n-r,0}\int_0^{c_j} K_{n,j}^{[0,2]}(t_n+vh)\,dv + h^2\sum_{i=1}^{n-r-1}\bar\beta_{i,0}\int_0^1 K_{n,j}^{[0,2]}(t_i+vh)\,dv,$$

for $j = 1, \dots, m$.

The continuity conditions of the approximating polynomial spline at the knots in Z_N yield an additional relationship between $\bar\beta_{n,0}$ and the vectors $\bar\beta_i$ and β_i, $i < n$,

$$\bar\beta_{n,0} = \bar\beta_{n-1,0} + \sum_{l=1}^{d}\bar\beta_{n-1,l} + \sum_{l=1}^{m}\beta_{n-1,l} + hR_{n-1}(1).$$

By the initial conditions (5), it follows that

$$\bar\beta_{n,0} = \sum_{i=1}^{n-1}\sum_{l=1}^{d}\bar\beta_{i,l} + \sum_{i=0}^{n-1}\left(\sum_{l=1}^{m}\beta_{i,l} + hR_i(1)\right), \quad n = 1, \dots, N-1. \tag{29}$$

For all collocation parameters $\{c_j\}$, with $0 \le c_1 < \cdots < c_m \le 1$, V is the Vandermonde matrix and thus nonsingular. Let $\bar K_\nu^{[1]} := \max\{|K_\nu^{[1]}(t,s)| \mid (t,s) \in S\}$, $\bar K_\nu^{[2]} := \max\{|K_\nu^{[2]}(t,s)| \mid (t,s) \in S_\tau\}$, $\nu = 0, 1$, and $\bar P := \max\{|p(t)| \mid t \in I\}$. Since by the assumptions of Theorem 1, $K_\nu^{[1]} \in C(S)$, $K_\nu^{[2]} \in C(S_\tau)$, $\nu = 0, 1$, and $p \in C(I)$, it follows that $\bar K_\nu^{[i]}$, $\nu = 0, 1$, $i = 1, 2$, and $\bar P$ are finite.

For $n = 0$, by (4) and (28), we obtain:

$$\big(V - hP_0 - hQ_{0,0}^{(1)}\big)\beta_0 = hq_0^{(1)}. \tag{30}$$

By the standard Neumann series argument it follows that there exists $\bar h > 0$ such that the matrix $V - hP_0 - hQ_{0,0}^{(1)}$ has a uniformly bounded inverse for all $h \in (0, \bar h)$. It follows from (30) that

$$\|\beta_0\|_1 \le hC_0. \tag{31}$$

Thus, (31) together with (21) proves that

$$|e(t_0 + vh)| \le C_0 h^{m+d+1}, \quad \text{for all } v \in [0,1].$$

We take the derivative of the relation (21) k times, and use (31) to obtain

$$|e^{(k)}(t_0 + vh)| \le C_{0,k} h^{m+d+1-k}, \quad \text{for all } v \in [0,1], \quad k = 1, \dots, m+d.$$

Suppose that for $j = 0, \dots, n-1$

$$|e(t_j + vh)| \le C_{j,0} h^{m+d}, \quad \text{for all } v \in [0,1], \tag{32}$$

$$|e^{(k)}(t_j + vh)| \le C_{j,k} h^{m+d+1-k}, \quad \text{for all } v \in [0,1], \quad k = 1, \dots, m+d. \tag{33}$$

We will prove that (32)–(33) hold for $j = n$. By (22) and the assumptions (32)–(33), it follows that

$$\|\bar{\beta}_i\|_1 \le h\widehat{B}, \quad i = 1, \dots, n. \tag{34}$$

By the assumptions and the arguments above, it follows that there exists $\bar{h} > 0$ such that the matrix $V - hP_n - hQ^{(1)}_{n,n}$ has a uniformly bounded inverse for all $h \in (0, \bar{h})$. Also, by the assumptions of the Theorem 1, all other matrices in (28) are uniformly bounded. Since $|R_n(v)| \le M_n$, for all $v \in [0,1]$, where $M_n > 0$ is a finite constant, for $q_n := q_n^{(1)} + q_n^{(2)}$ we obtain

$$\|q_n\|_1 \le \begin{cases} \Delta := m\big(M_n' + hM_n\big(\bar{P} + nh\bar{K}_0^{[1]}\big) + nh\bar{K}_1^{[1]} M_n'\big), & 0 < n < r, \\ \Delta + m(n-r)\big(h\bar{K}_0^{[2]} M_n + \bar{K}_1^{[2]} M_n'\big), & r \le n \le N-1. \end{cases} \tag{35}$$

By using these estimates and (29), it follows from (28) that

$$\|\beta_n\|_1 \le \Delta_1 := h(D_1 + D_0) \sum_{i=0}^{n-1} \|\beta_i\|_1 + h^2 D_2 \sum_{i=0}^{n-1} \sum_{s=0}^{i-1} \|\beta_s\|_1 + h(B + R), \tag{36}$$

if $n < r$, and

$$\|\beta_n\|_1 \le \Delta_1 + h^2 \bar{D}_0 \sum_{i=0}^{n-r-1} \|\beta_i\|_1 + h^2 \bar{D}_2 \sum_{i=0}^{n-r-1} \sum_{s=0}^{i-1} \|\beta_s\|_1 + h(\bar{B} + \bar{R}), \tag{37}$$

if $n \ge r$, where B, $\bar{B}$ are given by an expression involving (34), and R, $\bar{R}$ by upper bounds similar to (35).

After the simplification, (36) and (37) yield the discrete Gronwall inequality

$$\|\beta_n\|_1 \le \begin{cases} h\widehat{D}_0 \displaystyle\sum_{i=0}^{n-1} \|\beta_i\|_1 + hR_1, & 0 < n < r, \\ h\widehat{D}_0 \displaystyle\sum_{i=0}^{n-1} \|\beta_i\|_1 + h\widehat{D}_1 \sum_{i=0}^{n-r} \|\beta_i\|_1 + hR_2, & r \le n \le N-1. \end{cases}$$

A well-known result on discrete Gronwall inequalities [5, p.41] leads to

$$\|\beta_n\|_1 \le hB_1, \quad 0 \le n \le N-1, \tag{38}$$

uniformly as $h \to 0$, with $Nh = T$. By using (34) and (38), from (29) it follows that

$$|\beta_{n,0}| \le hN(B_1 + \widehat{B} + M_n), \quad n = 1, \dots, N-1. \tag{39}$$

The above estimate (39), (34) and (38), by (21), imply that $|e(t_n + vh)| \le C_{n,0}h^{m+d}$, for all $v \in [0,1]$. By using (34), (38), and taking the derivative of the relation (24) $k-1$ times, we obtain $|e^{(k)}(t_n + vh)| \le C_{n,k}h^{m+d+1-k}$, for all $v \in [0,1]$, $k = 1, \dots, m+d$. Hence, the Theorem 1 holds. ∎

If $\Phi(t)$ in (10) cannot be found analytically, it has to be approximated by a suitable numerical quadrature.

Theorem 2. *Let the assumptions of Theorem* 1 *hold, except that the integrals*

$$\Phi(t) = \int_{t-\tau}^{0} \sum_{\nu=0}^{1} K_\nu^{[2]}(t,s)\,\phi^{(\nu)}(s)\,ds, \quad t = t_{n,j}, \quad n = 0, \dots, r-1,$$

are now approximated by $\widehat{\Phi}(t)$*, with quadrature errors such that, for some* $s > 0$*,*

$$|\Phi(t) - \widehat{\Phi}(t)| \le h^s, \quad t = t_{n,j}, \quad 0 \le n < r.$$

Then the collocation solution $u \in S_{m+d}^{(d)}(Z_N)$ *satisfies, for all sufficiently small* $h > 0$*,*

$$\|e\|_\infty \le C_0 h^p, \quad \|e^{(k)}\|_\infty \le C_k h^{p+1-k}, \quad k = 1, \dots, p,$$

with $p := \min\{m+d, s\}$*, where* C_k *are finite constants not depending on* h*.*

The global convergence of the fully discretized collocation method is described in the following theorem.

Theorem 3. *Let the assumptions of Theorem* 2 *hold, and assume that the approximations* $\widehat{\Phi}(t_{n,j})$*,* $0 \le n < r$*, are given by the interpolatory quadrature formulæ* (16)*. Suppose that the quadrature formulæ used in* (12)–(14) *satisfy:*

$$\int_0^1 \rho(t_i + vh)\,dv - \sum_{l=1}^{\mu_1} w_l\,\rho(t_i + d_l h) = O(h^{\mu_1}),$$

and, for $j = 1, \dots, m$

$$\int_0^{c_j} \rho(t_n + vh)\,dv - \sum_{l=1}^{\mu_0} w_{j,l}\,\rho(t_n + d_{j,l} h) = O(h^{\mu_0}).$$

Then the error $\hat{e} := y - \hat{u}$ *associated with collocation solution* $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ *defined by* (12)–(14)*,* (16) *and* (17)*, satisfies, for all sufficiently small* $h = \tau/r$*,*

$$\|\hat{e}\| \le C_0 h^p, \quad \|\hat{e}^{(k)}\|_\infty \le C_k h^{p+1-k}, \quad k = 1, \dots, p,$$

with $p := \min\{m+d, \mu_0 + 1, \mu_1\}$*, where* C_k *are finite constants not depending on* h*.*

Theorems 2 and 3 can be proved by the techniques used in [8, 9].

We conclude this section with a comment on the extension of the results of Theorems 1–3 to the nonlinear equation (1). Under the assumption of the existence of a (unique) solution $y(t)$ on I, the nonlinear analogue of the error equation (25) is

$$e'(t_{n,j}) = f(t_{n,j}, y(t_{n,j})) - f(t_{n,j}, u(t_{n,j})) + (V_1 y)(t_{n,j}) - (V_1 u)(t_{n,j}), \qquad (40)$$

for $j = 1, \ldots, m$, where the operator V_1 is now given by

$$(V_1 g)(t) := \int_0^t k_1\big(t, s, g(s), g'(s)\big)\, ds + \int_0^{t-\tau} k_2\big(t, s, g(s), g'(s)\big)\, ds. \qquad (41)$$

If the partial derivatives $\partial f(t,y)/\partial y$ and $\partial k_i(t,s,y,y')/\partial y^{(\nu)}$, $\nu = 0, 1$, $i = 1, 2$, are continuous and bounded on $I \times D$, $S \times D$, and $S_\tau \times D_\tau$, respectively, with

$$D := \big\{(y, y') \in \mathbb{R}^2 \;\big|\; |y - y(s)| < Y,\, |y' - y'(s)| < Y',\, s \in I\big\},$$

and

$$D_\tau := \big\{(y, y') \in \mathbb{R}^2 \;\big|\; |y - y(s)| < Y,\, |y' - y'(s)| < Y',\, s \in [-\tau, T - \tau]\big\},$$

for some $Y < \infty$, $Y' < \infty$, and if $h > 0$ is sufficiently small (assuring the existence of a unique collocation solution u), (40) may again be written in the form (25). The roles of p and $K_\nu^{[i]}$ are now assumed by

$$p^{(1)}(t) := \frac{\partial f(t, z_0(t))}{\partial y}, \quad k_\nu^{[i]}(t,s) := \frac{\partial k_i}{\partial y^{(\nu)}}\big(t, s, z_i(s), z_i'(s)\big), \quad \nu = 0, 1, \quad i = 1, 2,$$

where

$$z_0(t) := \theta_0\, y(t) + (1 - \theta_0)\, u(t), \quad 0 \le \theta_0 = \theta_0(t) \le 1,$$
$$z_i^{(\nu)}(s) := \theta_i\, y^{(\nu)}(s) + (1 - \theta_i)\, u^{(\nu)}(s), \quad 0 \le \theta_i = \theta_i(s) \le 1.$$

Hence, the above proofs are easily adapted to deal with the nonlinear case (1), and the convergence results of Theorems 1–3 remain valid for nonlinear delay integral equations.

4. Local superconvergence on $\bar{Z}_N$

The notion of local superconvergence is used when, on a set of interior points Z_N, or $\bar{Z}_N := Z_N \cup \{T\}$, the approximate solution has a convergence order greater than the global one. From Theorem 1 we notice that the only conditions imposed on the collocation parameters $\{c_j\}$ are that they must be distinct, and belong to $[0, 1]$. The local superconvergence on $\bar{Z}_N$ is closely connected with the choice of the collocation parameters, and with the relation between their number and the number of coefficients of the approximate solution determined from the smoothness conditions. (See [4] for delay, and [5] for "classical" VIDEs.) Without loss of generality, we assume that $T = t_N = M\tau$ for some $M \in \mathbb{N}$.

Theorem 4. *Assume that the functions given in* (18) *and* (2) *satisfy* $p, q \in C^{m+p}(I)$, $K_\nu^{[1]} \in C^{m+p}(S)$, $K_\nu^{[2]} \in C^{m+p}(S_\tau)$, $\nu = 0, 1$, $\phi \in C^{m+p}([-\tau, 0])$, *for some (given) integer* p, *with* $d + 1 < p \le m$. *Suppose that the delay integral* $\Phi(t)$ (10) *can be evaluated analytically. If* $m \ge d + 2$, *and the collocation parameters* $\{c_j\}$ *are determined from the orthogonality property*

$$J_k := \int_0^1 s^k \prod_{j=1}^{m} (s - c_j)\, ds = 0, \quad \textit{for} \quad k = 0, \dots, p-1, \quad J_p \neq 0, \tag{42}$$

with $0 \le c_1 < \cdots < c_m \le 1$, *then for all sufficiently small* $h = \tau/r$ *the collocation solution* $u \in S_{m+d}^{(d)}(Z_N)$ *defined by* (11), (15) *is locally superconvergent at the mesh points*

$$\max_{1 \le n \le N} |y(t_n) - u(t_n)| \le C h^{m+p}, \tag{43}$$

and, if $c_m = 1$, *then* u' *is locally superconvergent at the mesh points*

$$\max_{1 \le n \le N} |y'(t_n) - u'(t_n)| \le C h^{m+p}. \tag{44}$$

Proof. The theorem can be proved by the techniques used in [4, 9]. ■

The maximum value of p in the orthogonality condition (42) occurs iff the collocation parameters are the Gauss–Legendre points in $(0, 1)$. We are looking for a collocation approximation $u \in S_{m+d}^{(d)}(Z_N)$, which yields superconvergence both for u and its derivative u'. Hence, if we have fixed $c_m = 1$, the degree of precision in (43) and (44) cannot exceed $2m - 1$.

The proofs of Theorems 2, 3 and 4 suggest that the local superconvergence results are also true for discretized collocation solution $\hat{u} \in S_{m+d}^{(d)}(Z_N)$ defined by (12)–(14), (16) and (17), and characterized by $\mu_0 = \mu_1 = m$. The quadrature approximation to the delay integral

$$\Phi(t_n) = -\int_{t_{n-r}}^{0} k_2\big(t_n, s, \phi(s), \phi'(s)\big)\, ds$$

is given by

$$\widehat{\Phi}(t_n) = -h \sum_{i=n-r}^{-1} \sum_{l=0}^{m} w_l\, k_2\big(t_n, t_i + d_l h, \phi(t_i + d_l h), \phi'(t_i + d_l h)\big), \quad 0 \le n < r. \tag{45}$$

Theorem 5. *Under the conditions of Theorem* 4, *assume that the approximations to the delay integrals* $\Phi(t_{n,j})$, $\Phi(t_n)$ *are given by the quadrature formulæ* (16), (45), *respectively. If* $h = \tau/r$ *is sufficiently small, and the orthogonality conditions* (42) *hold, then the solution* $\hat{u}$ *given by* (12)–(14), (16), *and* (17), *has the property*

$$\max_{1 \le n \le N} |y(t_n) - \hat{u}(t_n)| \le C h^{m+p},$$

while $\hat{u}'$, *if* $c_m = 1$, *satisfies*

$$\max_{1 \le n \le N} |y'(t_n) - \hat{u}'(t_n)| \le C h^{m+p}.$$

The proof uses the same technique as the proof of Theorem 4.

Finally, we comment on the extension of the results in Theorems 4 and 5 to the nonlinear case. The equation for the collocation error e now has the form

$$e'(t) = f(t, y(t)) - f(t, u(t)) + \delta(t) + (V_1 y)(t) - (V_1 u)(t),$$

where the operator V_1 is given by (41). Under appropriate differentiability and boundedness conditions for f and k_i, $i = 1, 2$, we can write

$$\begin{aligned} f(t, y(t)) - f(t, u(t)) &= \frac{\partial f(t, y(t))}{\partial y} \cdot e(t) + \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(t, z_0(t)) \cdot e^2(t), \\ k_i\big(t, s, y(s), y'(s)\big) - k_i\big(t, s, u(s), u'(s)\big) &= \frac{\partial k_i}{\partial y}\big(t, s, y(s), y'(s)\big) \cdot e(s) \\ &\quad + \frac{\partial k_i}{\partial y'}\big(t, s, y(s), y'(s)\big) \cdot e'(s) + O(h^{2(m+d)}). \end{aligned}$$

The global convergence of u and $\hat{u}$ has already been established (see Section 3), and we know that $\|e^2\|_\infty = O(h^{2(m+d)})$ for any $\{c_j\}$. The remaining part of the proof (both for u and $\hat{u}$) uses the same techniques as before.

5. Numerical examples

We have applied the collocation method to a model neutral Volterra integro–differential equation with constant delay.

Example 1. *The model equation is*

$$y'(t) = (\lambda - \mu)e^{\tau - t} - (1 + \lambda - \mu)y(t) - \lambda \int_{t-\tau}^{t} y(s)\, ds - \mu \int_{t-\tau}^{t} y'(s)\, ds,$$

with the initial function $\phi(t) = e^{-t}$, $t \in [-\tau, 0]$. *The exact solution is* $y(t) = e^{-t}$, *for all* $t \geq 0$.

The exact solution of the equation is approximated by the exact collocation method, the integrals occurring in the collocation equation (11) being evaluated analytically. We choose different approximating spline spaces $S^{(d)}_{m+d}(Z_N)$, with $m = 3, 4$, and $d = 0, 1$.

For the collocation parameters we choose: shifted Gauss points, shifted Radau II points, shifted Lobatto points, shifted $m - 1$ Gauss points with $c_m = 1$ (denoted by Gauss 1), and points named as Other. The collocation parameters named as Other are $c_1 = 1/3$, $c_2 = 1/2$, $c_3 = 1$, for $m = 3$, and $c_1 = 1/4$, $c_2 = 1/2$, $c_3 = 2/3$, $c_4 = 1$, for $m = 4$, respectively.

The errors and the effective order of convergence have been calculated on $[0, T]$ and on $\bar{Z}_N$, both for the approximation and its derivative.

The theoretical results presented in the preceding sections agree with the numerical results given in the Tables 1–4. The results for $S_m^{(0)}(Z_N)$ given in the Tables 1 and 3 agree with the theoretical results given in [2, 4].

The stability properties of the collocation methods with $d \geq 1$ will be studied elsewhere.

	h	$\|e_N\|_\infty$	p^*	$\|e\|_\infty$	p	$\|e'_N\|_\infty$	$\bar{p}^*$	$\|e'\|_\infty$	$\bar{p}$
	0.2	$8.33e{-}09$	6.00	$2.10e{-}07$	4.22	$6.53e{-}05$	2.99	$2.51e{-}05$	2.89
Gauss	0.1	$1.30e{-}10$	6.00	$1.13e{-}08$	4.16	$8.24e{-}06$	2.99	$3.40e{-}06$	2.94
	0.05	$2.04e{-}12$	5.99	$6.32e{-}10$	4.10	$1.04e{-}06$	3.00	$4.41e{-}07$	2.97
	0.2	$1.43e{-}07$	4.93	$5.36e{-}07$	3.55	$4.64e{-}07$	4.95	$4.22e{-}05$	2.92
Radau II	0.1	$4.67e{-}09$	4.97	$4.58e{-}08$	3.79	$1.51e{-}08$	4.97	$5.57e{-}06$	2.97
	0.05	$1.49e{-}10$	4.98	$3.32e{-}09$	3.90	$4.81e{-}10$	4.99	$7.14e{-}07$	2.98
	0.2	$1.81e{-}06$	4.00	$2.32e{-}06$	3.83	$6.63e{-}06$	4.00	$4.03e{-}05$	2.77
Lobatto	0.1	$1.13e{-}07$	4.00	$1.63e{-}07$	3.91	$4.14e{-}07$	4.00	$5.92e{-}06$	2.88
	0.05	$7.06e{-}09$	4.00	$1.08e{-}08$	3.96	$2.58e{-}08$	4.00	$8.04e{-}07$	2.94
	0.2	$9.10e{-}07$	3.79	$2.33e{-}06$	3.69	$3.42e{-}06$	3.81	$4.35e{-}05$	3.02
Gauss 1	0.1	$6.57e{-}08$	3.90	$1.81e{-}07$	3.84	$2.43e{-}07$	3.91	$5.35e{-}06$	3.03
	0.05	$4.40e{-}09$	3.95	$1.26e{-}08$	3.92	$1.62e{-}08$	3.95	$6.54e{-}07$	3.02
	0.2	$4.25e{-}06$	2.86	$5.70e{-}06$	3.03	$2.17e{-}05$	2.86	$2.41e{-}05$	3.04
Other	0.1	$5.85e{-}07$	2.91	$6.98e{-}07$	3.03	$2.99e{-}06$	2.93	$2.93e{-}06$	3.02
	0.05	$7.76e{-}08$	2.95	$8.52e{-}08$	3.02	$3.92e{-}07$	2.96	$3.61e{-}07$	3.01

Table 1. Example 1 approximated in $S_3^{(0)}$, with $\lambda = 3$, $\mu = 1$, $\tau = 1$, and $T = 2$.

	h	$\|e_N\|_\infty$	p^*	$\|e\|_\infty$	p	$\|e'_N\|_\infty$	$\bar{p}^*$	$\|e'\|_\infty$	$\bar{p}$
	0.2	$1.76e{-}09$	6.12	$4.10e{-}08$	6.67	$1.29e{-}05$	5.44	$4.87e{-}06$	5.31
Gauss	0.1	$2.53e{-}11$	6.04	$9.07e{-}10$	6.25	$6.81e{-}07$	5.14	$2.87e{-}07$	5.18
	0.05	$3.86e{-}13$	6.03	$2.57e{-}11$	6.16	$4.27e{-}08$	5.12	$1.87e{-}08$	5.21
	0.2	$1.98e{-}08$	5.08	$1.27e{-}08$	4.92	$7.14e{-}08$	5.08	$6.10e{-}07$	3.90
Radau II	0.1	$5.89e{-}10$	5.05	$4.19e{-}10$	4.96	$2.11e{-}09$	5.04	$4.08e{-}08$	3.95
	0.05	$1.78e{-}11$	5.02	$1.34e{-}11$	4.98	$6.41e{-}11$	5.02	$2.64e{-}09$	3.97
	0.2	$5.17e{-}08$	3.98	$3.46e{-}08$	3.68	$2.55e{-}07$	3.96	$5.60e{-}07$	3.94
Gauss 1	0.1	$3.29e{-}09$	3.99	$2.70e{-}09$	3.85	$1.64e{-}08$	4.00	$3.65e{-}08$	3.97
	0.05	$2.06e{-}10$	4.00	$1.87e{-}10$	3.93	$1.03e{-}09$	4.00	$2.33e{-}09$	3.99
	0.2	$9.84e{-}08$	4.16	$8.56e{-}08$	4.05	$4.58e{-}07$	4.10	$3.66e{-}07$	4.04
Other	0.1	$5.50e{-}09$	4.08	$5.18e{-}09$	4.02	$2.67e{-}08$	4.06	$2.23e{-}08$	4.04
	0.05	$3.26e{-}10$	4.03	$3.18e{-}10$	4.02	$1.60e{-}09$	4.03	$1.36e{-}09$	4.02

Table 2. Example 1 approximated in $S_4^{(1)}$, with $\lambda = 3$, $\mu = 1$, $\tau = 1$, and $T = 2$.

	h	$\|e_N\|_\infty$	p^*	$\|e\|_\infty$	p	$\|e'_N\|_\infty$	$\bar{p}^*$	$\|e'\|_\infty$	$\bar{p}$
Gauss	0.2	$4.18e{-12}$	8.01	$6.67e{-09}$	4.91	$9.17e{-07}$	3.97	$6.87e{-08}$	3.66
	0.1	$1.62e{-14}$	6.60	$2.22e{-10}$	4.96	$5.84e{-08}$	3.99	$5.44e{-09}$	3.84
	0.05	$1.67e{-16}$	-0.42	$7.13e{-12}$	4.98	$3.68e{-09}$	3.99	$3.79e{-10}$	3.92
Radau II	0.2	$1.03e{-10}$	6.96	$9.43e{-09}$	4.98	$3.16e{-10}$	6.96	$4.20e{-07}$	3.83
	0.1	$8.26e{-13}$	7.00	$2.98e{-10}$	5.00	$2.54e{-12}$	6.99	$2.94e{-08}$	3.92
	0.05	$6.44e{-15}$	4.86	$9.33e{-12}$	5.00	$2.00e{-14}$	3.79	$1.95e{-09}$	3.96
Lobatto	0.2	$2.08e{-09}$	5.99	$1.78e{-08}$	4.80	$6.70e{-09}$	6.00	$1.97e{-07}$	4.19
	0.1	$3.27e{-11}$	6.00	$6.38e{-10}$	4.90	$1.05e{-10}$	6.00	$1.08e{-08}$	4.13
	0.05	$5.10e{-13}$	5.98	$2.14e{-11}$	4.95	$1.64e{-12}$	5.98	$6.17e{-10}$	4.08
Gauss 1	0.2	$1.33e{-09}$	5.87	$3.45e{-09}$	6.14	$4.30e{-09}$	5.89	$8.64e{-07}$	3.87
	0.1	$2.26e{-11}$	5.94	$4.87e{-11}$	7.03	$7.27e{-11}$	5.94	$5.90e{-08}$	3.94
	0.05	$3.68e{-13}$	5.97	$4.67e{-13}$	5.37	$1.18e{-12}$	5.95	$3.85e{-09}$	3.97
Other	0.2	$2.00e{-08}$	3.18	$6.68e{-08}$	4.17	$1.38e{-07}$	3.48	$4.26e{-07}$	4.10
	0.1	$2.21e{-09}$	3.68	$3.70e{-09}$	4.08	$1.23e{-08}$	3.78	$2.48e{-08}$	4.05
	0.05	$1.72e{-10}$	3.86	$2.19e{-10}$	4.04	$8.95e{-10}$	3.90	$1.50e{-09}$	4.02

Table 3. Example 1 approximated in $S_4^{(0)}$, with $\lambda = 3$, $\mu = 1$, $\tau = 1$, and $T = 2$.

	h	$\|e_N\|_\infty$	p^*	$\|e\|_\infty$	p	$\|e'_N\|_\infty$	$\bar{p}^*$	$\|e'\|_\infty$	$\bar{p}$
Gauss	0.2	$2.64e{-12}$	8.43	$3.42e{-09}$	6.26	$4.82e{-07}$	5.30	$3.65e{-08}$	5.06
	0.1	$1.05e{-14}$	6.56	$1.13e{-10}$	6.28	$3.02e{-08}$	5.30	$2.83e{-09}$	5.19
	0.05	$1.11e{-16}$	-1.00	$3.64e{-12}$	6.29	$1.89e{-09}$	5.30	$1.95e{-10}$	5.25
Radau II	0.2	$1.84e{-11}$	7.05	$1.57e{-10}$	5.88	$5.89e{-11}$	7.05	$5.06e{-09}$	4.84
	0.1	$1.39e{-13}$	6.97	$2.66e{-12}$	5.94	$4.43e{-13}$	6.83	$1.77e{-10}$	4.92
	0.05	$1.11e{-15}$	3.32	$4.33e{-14}$	6.02	$3.89e{-15}$	1.67	$5.84e{-12}$	4.96
Gauss 1	0.2	$1.46e{-10}$	5.98	$2.03e{-10}$	5.89	$5.28e{-10}$	5.99	$1.02e{-08}$	4.86
	0.1	$2.32e{-12}$	6.00	$3.41e{-12}$	5.95	$8.31e{-12}$	6.00	$3.49e{-10}$	4.93
	0.05	$3.62e{-14}$	6.03	$5.54e{-14}$	5.96	$1.30e{-13}$	5.80	$1.15e{-11}$	4.97
Other	0.2	$5.02e{-10}$	5.40	$2.65e{-10}$	5.02	$2.15e{-09}$	5.31	$2.58e{-09}$	5.13
	0.1	$1.19e{-11}$	5.24	$8.15e{-12}$	5.00	$5.43e{-11}$	5.19	$7.40e{-11}$	5.09
	0.05	$3.14e{-13}$	5.16	$2.56e{-13}$	5.00	$1.49e{-12}$	5.10	$2.17e{-12}$	5.07

Table 4. Example 1 approximated in $S_5^{(1)}$, with $\lambda = 3$, $\mu = 1$, $\tau = 1$, and $T = 2$.

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