Variations on the theme of higher-dimensional designs*

Vedran Krčadinac

PMF-MO

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* This work was fully supported by the Croatian Science Foundation under the project 9752.

Outline

Higher-dimensional variations of. . .

1. Hadamard matrices
2. Symmetric block designs
3. Mosaics of symmetric designs

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Brief survey of known results, alternative definitions, **open problems** and **research directions**...

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$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes H = \begin{bmatrix} H & H \\ H & -H \end{bmatrix}$$

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Proposition.

Hadamard matrices exist for all orders of the form $v = 2^m$

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I. S. Kotsireas, 130+ *years of the Hadamard conjecture*, Combinatorial Designs and Codes (CODESCO'24), July 8-12, 2024, Sevilla, Spain.

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It is Hadamard if all $(n-1)$-dimensional parallel slices are orthogonal:

$$\sum_{1\le i_1,\ldots,\widehat{i_j},\ldots,i_n\le v} H(i_1,\ldots,a,\ldots,i_n)H(i_1,\ldots,b,\ldots,i_n) = v^{n-1}\delta_{ab}$$

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Parallel slices of dimension k are obtained by varying some k variables and fixing the remaining $n-k$ variables so that they agree in all but one of the fixed variables.

Higher-dimensional Hadamard matrices

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Proposition.

For any dimension $n \geq 2$, there exist proper n-dimensional Hadamard matrices of orders $v = 2^m$.

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Theorem (Product construction)

If $h : \{1, \ldots, v\}^2 \to \{-1, 1\}$ is a 2-dimensional Hadamard matrix of order v, then

$$H(i_1, \ldots, i_n) = \prod_{1 \le j < k \le n} h(i_j, i_k)$$

is a proper n-dimensional Hadamard matrix of order v, for all $n \ge 3$.

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Question: Are there examples with inequivalent 2-dimensional slices?

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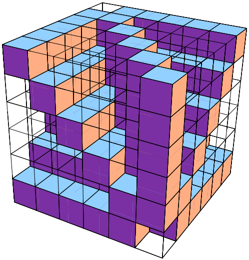

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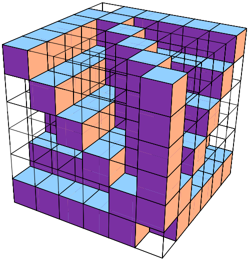

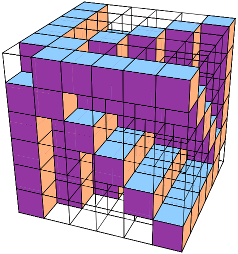

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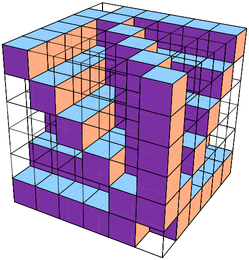

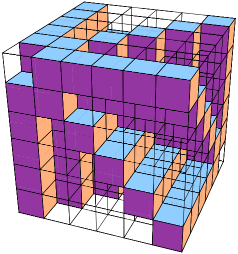

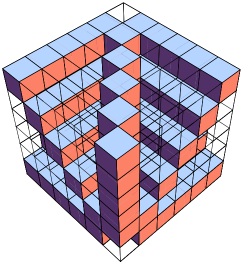

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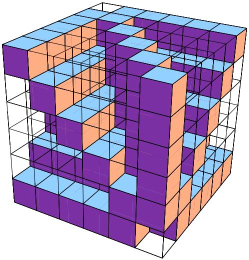
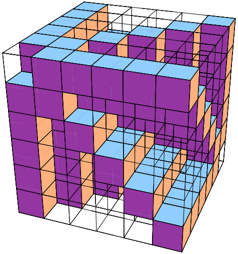
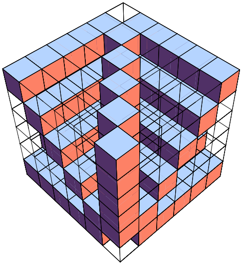

Y. X. Yang, X. X. Niu, C. Q. Xu, *Theory and applications of higher-dimensional Hadamard matrices, Second edition*, Chapman and Hall / CRC Press, 2010.

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Theorem (Dimension++)

If h is an n-dimensional Hadamard matrix of order v, then

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Theorem (Digit construction)

If h is a 2-dimensional Hadamard matrix of order $v = (2t)^s$, $s > 1$, then

$$H(i_0, \ldots, i_{s-1}, j_0, \ldots, j_{s-1}) =$$
$$= h(i_0 + (2t)i_1 + \ldots + (2t)^{s-1} i_{s-1},\ j_0 + (2t)j_1 + \ldots + (2t)^{s-1} j_{s-1})$$

is a $(2s)$-dimensional Hadamard matrix of order $2t$.

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Corollary.

If the Hadamard conjecture is true, then n-dimensional Hadamard matrices exist for all even orders v and all dimensions $n \geq 4$.

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Questions: (from Y. X. Yang's book)

1. Prove or disprove the existence of 3-dimensional Hadamard matrices of orders $4k + 2 \neq 2 \cdot 3^m$.
2. Construct more 3-dimensional Hadamard matrices of orders $4k + 2$.

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P. J. Shlichta:

VI. FUTURE RESEARCH AND APPLICATIONS

The present exposition suggests a number of unsolved problems and unproven conjectures. Some examples follow.

a) The algebraic approach to the derivation of two-dimensional Hadamard matrices [2]–[7] suggests that a similar procedure may be feasible for three- or higher dimensional matrices.

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V. Krčadinac, M. O. Pavčević, K. Tabak, *Three-dimensional Hadamard matrices of Paley type*, Finite Fields Appl. **92** (2023), 102306.

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$H : (\mathbb{F}_q \cup \{\infty\})^3 \to \{1, -1\}$, q odd prime power

$$H(x,y,z) = \begin{cases} -1, & \text{if } x = y = z, \\ 1, & \text{if } x = y \neq z \\ & \quad \text{or } x = z \neq y \\ & \quad \text{or } y = z \neq x, \\ \chi(z-y), & \text{if } x = \infty, \\ \chi(x-z), & \text{if } y = \infty, \\ \chi(y-x), & \text{if } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{otherwise.} \end{cases}$$

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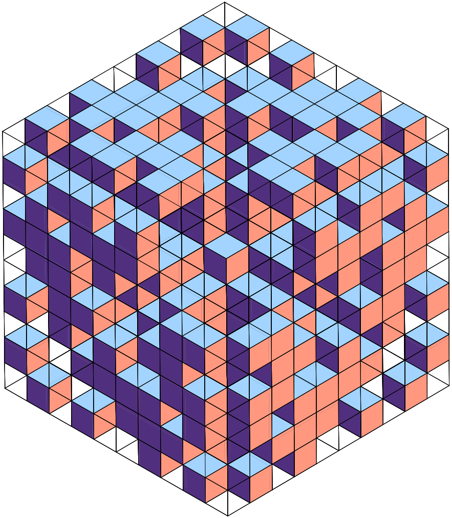

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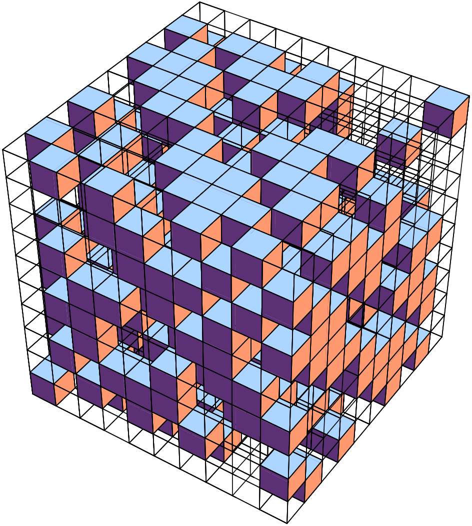

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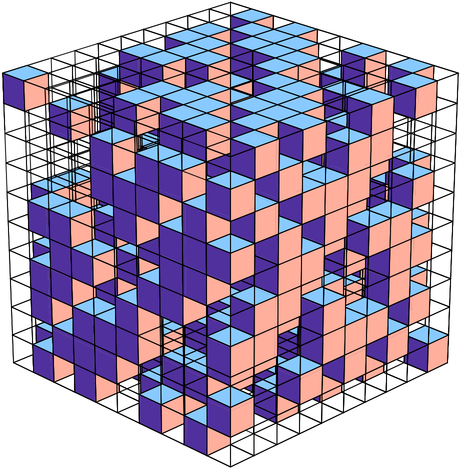

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1. What about dimension $n = 3$ and orders $v = 22$, 34, 46, $58, \ldots$?

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Hadamard matrices of dimension $n = 3$ and order $v = q + 1$ exist for all odd prime powers q. The Paley-type construction gives proper Hadamard matrices if $q \equiv 3 \pmod 4$, and ordinary H. matrices if $q \equiv 1 \pmod 4$.

Existence: $v = 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \ldots$

Questions:

1. What about dimension $n = 3$ and orders $v = 22$, 34, 46, $58, \ldots$?
2. Can other known construction techniques for 2-dimensional Hadamard matrices be generalized to higher dimensions?

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1. What about dimension $n = 3$ and orders $v = 22,\ 34,\ 46,\ 58, \ldots$?
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At the CODESCO conference. . .

Combinatorial Designs and Codes, July 8-12, 2024, Sevilla, Spain

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A variation on n-dimensional Hadamard matrices

Edinah K. Gnang, A. Elgammal, V. Retakh, *A spectral theory for tensors*, Ann. Fac. Sci. Toulouse Math. (6) **20** (2011), no. 4, 801–841.

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The Bhattacharya-Mesner product of hypermatrices...

D. M. Mesner, P. Bhattacharya, *Association schemes on triples and a ternary algebra*, J. Combin. Theory Ser. A **55** (1990), no. 2, 204–234.

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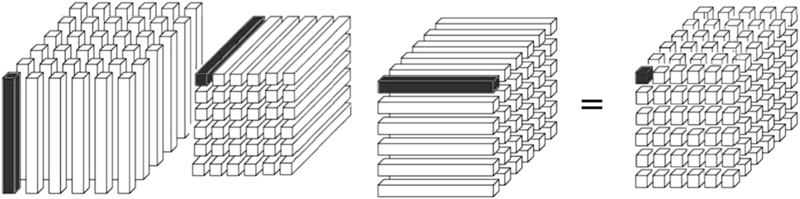

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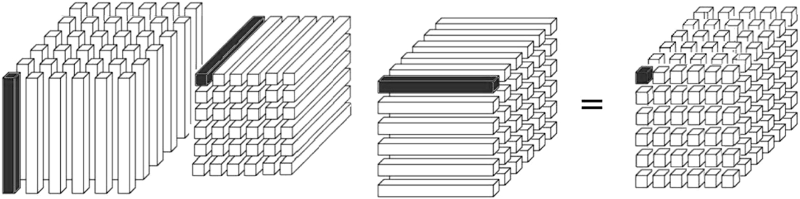

Let $A^{(1)}$, $A^{(2)}$, $A^{(3)}$, $A^{(4)}$ be 4-dimensional matrices of order v

$$\left[\mathrm{Prod}(A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)})\right]_{i_1,i_2,i_3,i_4} = \sum_{j=1}^{v} a^{(1)}_{i_1,j,i_3,i_4} a^{(2)}_{i_1,i_2,j.i_4} a^{(3)}_{i_1,i_2,i_3,j} a^{(4)}_{j,i_2,i_3,i_4}$$

A variation on n-dimensional Hadamard matrices

More operations for n-dimensional matrices of order v:

$$\left[\, A^{\tau} \,\right]_{i_1,i_2,\ldots,i_{n-1},i_n} = \left[\, A \,\right]_{i_n,i_1,\ldots,i_{n-2},i_{n-1}}$$

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$$n = 3: \qquad I = \mathrm{Prod}(J, J, \Delta), \qquad \mathrm{Prod}\left(I, A, I^{\tau^2}\right) = A, \ \forall A$$

A variation on n-dimensional Hadamard matrices

An n-dimensional matrix $H : \{1, \ldots, v\}^n \to \{-1, 1\}$ is Hadamard if

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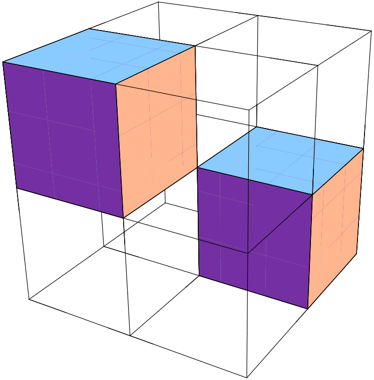

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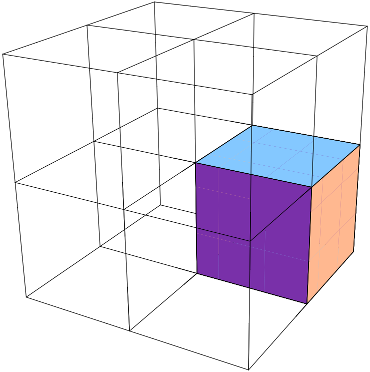

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$$H : \begin{array}{cccccccccc}
(1,1,1) & \mapsto & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
(1,1,2) & \mapsto & 1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\
(1,2,1) & \mapsto & 1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\
(1,2,2) & \mapsto & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 \\
(2,1,1) & \mapsto & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\
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3. Apart from the Kronecker product construction, can other known constructions for $n = 2$ be generalized to odd dimensions?

A conference on Hadamard matrices

8th Workshop on Design Theory, Hadamard Matrices and Applications (Hadamard 2025)

26-30 May, 2025, Sevilla

The purpose of the workshop is to bring together researchers and students interested in design theory, especially as it relates to Hadamard matrices and their applications, as well as in related areas in coding theory, association schemes, sequences, finite geometry, difference sets, quantum information theory, theoretical physics and computer security. The audiences would learn about the latest developments in these areas, discuss the latest findings, take stock of what remains to be done on classical problems and explore different visions for setting the direction for future work.

https://gestioneventos.us.es/hadamard2025

Another conference

5TH PYTHAGOREAN CONFERENCE

KALAMATA, GREECE, JUNE 1-6, 2025
AN ADVANCED RESEARCH WORKSHOP IN FINITE GEOMETRY, COMBINATORIAL DESIGNS, ALGEBRAIC COMBINATORICS, CODING THEORY, CRYPTOGRAPHY & CRYPTOLOGY

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- Marco Buratti, Sapienza Università di Roma, Italy
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- Spyros S. Magliveras, Florida Atlantic University, Boca Raton, FL, USA
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https://cargo.wlu.ca/5thPythagorean/

Symmetric designs

A symmetric (v, k, λ) design is a $v \times v$ matrix with $\{0, 1\}$-entries such that $A \cdot A^\tau = (k - \lambda)\, I + \lambda\, J$ holds. The order of the design is $m = k - \lambda$.

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Example:

$(7, 3, 1)$

$m = 2$

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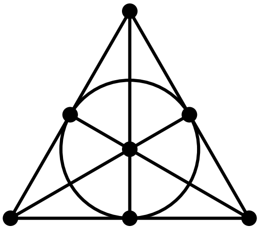

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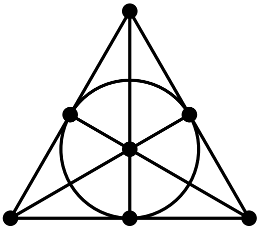

Proposition.

If a symmetric (v, k, λ) design exists, then $\lambda(v - 1) = k(k - 1)$.

Symmetric designs

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A Hadamard matrix of order $v = 4m$ exists if and only if there exists a symmetric $(4m-1, 2m-1, m-1)$ design.

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Theorem.

If D is a (v, k, λ) difference set in $G = \{g_1, \ldots, g_v\}$, then the $v \times v$ matrix A with entries

$$a_{ij} = [\, g_i \cdot g_j \in D\,] = \begin{cases} 1, & \text{if } g_i \cdot g_j \in D, \\ 0, & \text{otherwise} \end{cases}$$

is a symmetric (v, k, λ) design with G as a regular automorphism group.

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Symmetric $(25, 9, 3)$ designs exist, but there are no $(25, 9, 3)$ difference sets in any group of order 25.

Cubes of symmetric designs

V. Krčadinac, M. O. Pavčević, K. Tabak, *Cubes of symmetric designs*, Ars Math. Contemp. (to appear). https://arxiv.org/abs/2304.05446

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An *n*-dimensional cube of symmetric (v, k, λ) designs is a function

$$A : \{1, \ldots, v\}^n \to \{0, 1\}$$

such that all 2-dimensional slices are symmetric (v, k, λ) designs.

Cubes of symmetric designs

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An *n*-dimensional cube of symmetric (v, k, λ) designs is a function

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Theorem (Difference cubes)

If D is a (v, k, λ) difference set in $G = \{g_1, \ldots, g_v\}$, then

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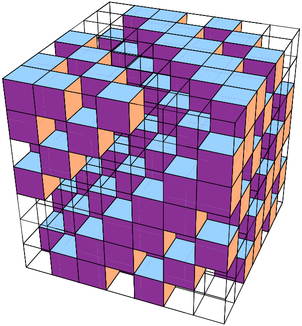
A 3-cube of symmetric $(7,3,1)$ designs

$\rightsquigarrow$ “Fano cube”



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Cubes of symmetric designs

Theorem (Group cubes)

If $\{D_1, \ldots, D_v\}$ is a family of (v, k, λ) difference sets in $G = \{g_1, \ldots, g_v\}$ that are blocks of a symmetric (v, k, λ) design, then

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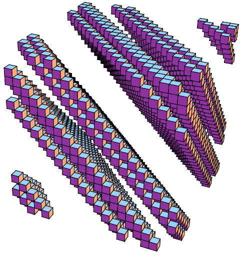
A 3-cube of $(21, 5, 1)$ designs

(projective planes of order 4)



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$G = \langle a, b \mid a^3 = b^7 = 1,\ ba = ab^2 \rangle$

$$D_1 = \{1, a, b, b^3, a^2 b^2\}$$
$$D_2 = \{a^2 b^6, b^6, a^2 b^3, a^2 b^4, a\}$$
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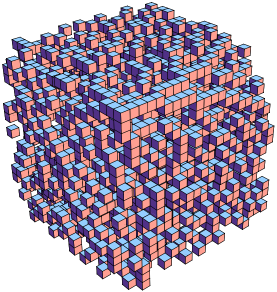

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Theorem.

For every $m \geq 2$ and $n \geq 3$, there are n-cubes of symmetric

$$(4^m,\ 2^{m-1}(2^m-1),\ 2^{m-1}(2^{m-1}-1))$$

designs that are group cubes, but not difference cubes.

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Red design, Green design, Blue design

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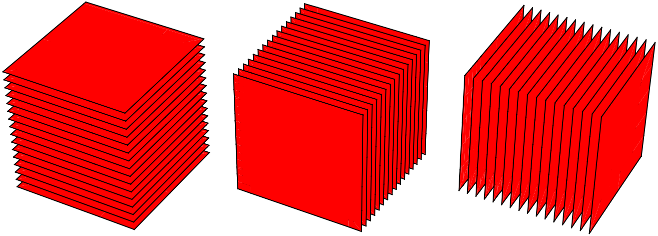

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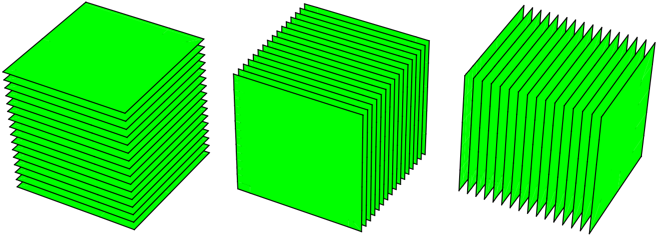

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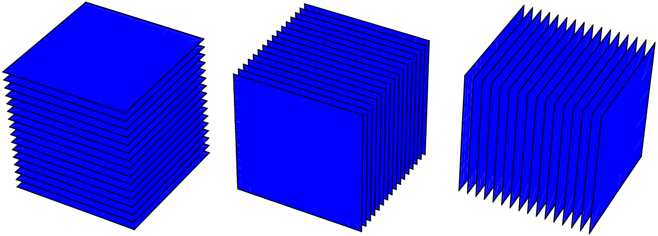

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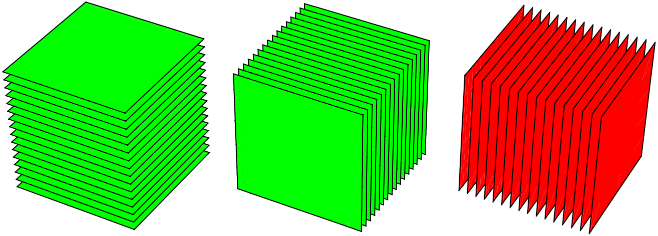

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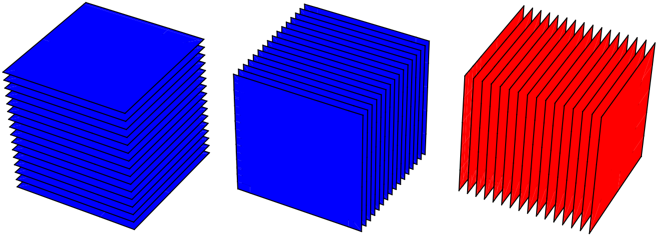

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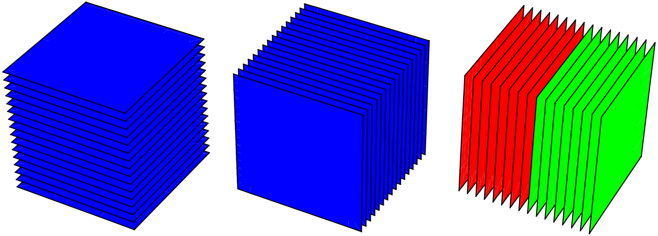

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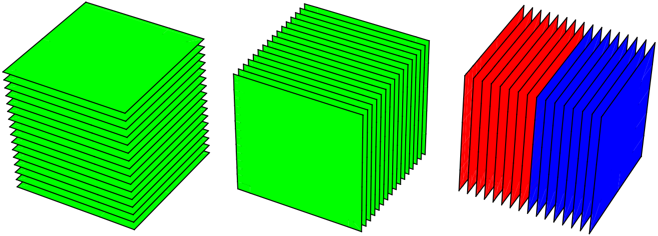

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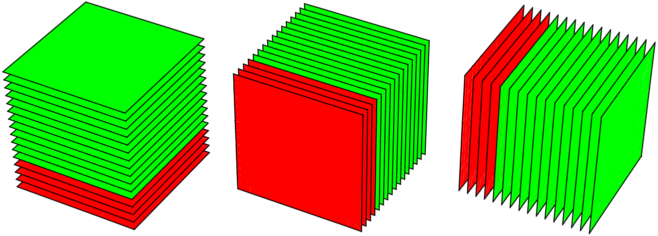

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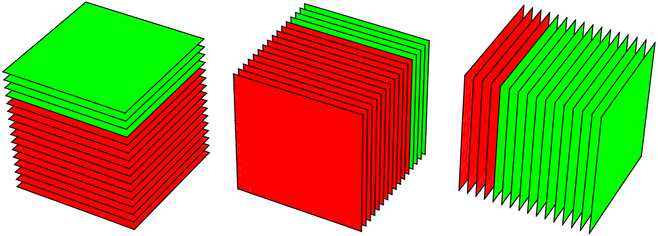

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Proposition.

Up to equivalence, the set $\mathcal{C}^3(16,6,2)$ contains exactly 27 difference cubes and 946 non-difference group cubes. Furthermore, it contains at least 1423 inequivalent non-group cubes.

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The parameters are of Menon type: $(4u^2, 2u^2 - u, u^2 - u)$. By exchanging $0 \to -1$, the cubes are transformed to n-dimensional Hadamard matrices with inequivalent slices!

Cubes of symmetric designs

Questions:

1. There are exactly 78 symmetric $(25, 9, 3)$ designs, but no difference sets. Are there cubes of $(25, 9, 3)$ designs of dimension $n \geq 3$?

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3. **Is there a product construction for cubes of symmetric designs?**

4. Hadamard matrices coming from Menon designs are of square orders. Are there n-dimensional Hadamard matrices with inequivalent slices of non-square orders?

Room squares

Room squares

T. G. Room, *A new type of magic square*, Math. Gaz. **39** (1955), 307.

Thomas Gerald Room

Article Talk

From Wikipedia, the free encyclopedia

Thomas Gerald Room FRS FAA (10 November 1902 – 2 April 1986) was an Australian mathematician who is best known for Room squares. He was a Foundation Fellow of the Australian Academy of Science.[1][2]

Room squares

T. G. Room, *A new type of magic square*, Math. Gaz. **39** (1955), 307.

Let S be a set of $v+1$ elements, say $S = \{\infty, 1, 2, \ldots, v\}$.

A Room square of order v is a $v \times v$ matrix M such that:

- the entries of M are empty or 2-element subsets of S
- each 2-subset of S appears once in M
- elements of S appear once in every row and column of M

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Example.

$v = 7$

$\infty 1$			26		57	34
45	$\infty 2$			37		16
27	56	$\infty 3$			14	
	13	67	$\infty 4$			25
36		24	17	$\infty 5$		
	47		35	12	$\infty 6$	
		15		46	23	$\infty 7$

Room squares

Equivalent objects:

Theorem.

A Room square of order v is equivalent to a pair of orthogonal 1-factorizations of the complete graph K_{v+1}.

Room squares

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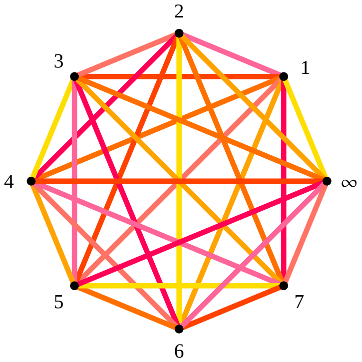

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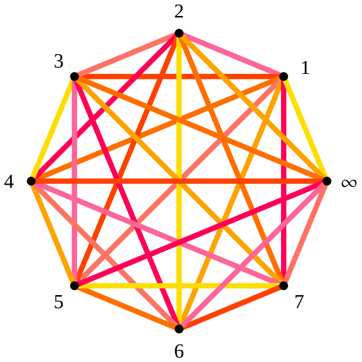

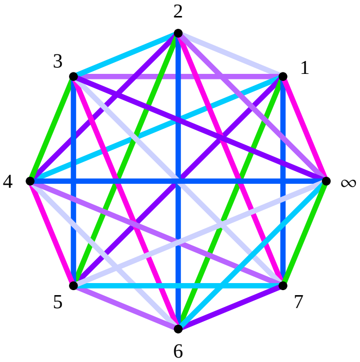

Equivalent objects:

Theorem.

A Room square of order v is equivalent to a pair of orthogonal 1-factorizations of the complete graph K_{v+1}.

Theorem.

A Room square of order v is equivalent to a pair of orthogonal-symmetric latin squares of order v.

Room squares

	1	2	3	4	5	6	7
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2	45	∞2			37		16
3	27	56	∞3			14	
4		13	67	∞4			25
5	36		24	17	∞5		
6		47		35	12	∞6	
7			15		46	23	∞7

1	6	4	3	7	2	5
6	2	7	5	4	1	3
4	7	3	1	6	5	2
3	5	1	4	2	7	6
7	4	6	2	5	3	1
2	1	5	7	3	6	4
5	3	2	6	1	4	7

1	5	2	6	3	7	4
5	2	6	3	7	4	1
2	6	3	7	4	1	5
6	3	7	4	1	5	2
3	7	4	1	5	2	6
7	4	1	5	2	6	3
4	1	5	2	6	3	7

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Existence:

Theorem.

A Room square of order v exists if and only if v is odd and $v \neq 3, 5$.

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Proof: 1955–1973.

Room cubes

A Room cube is an n-dimensional matrix of order v with entries that are empty or 2-subsets of $S = \{\infty, 1, 2, \ldots, v\}$ such that every 2-dimensional projection is a Room square.

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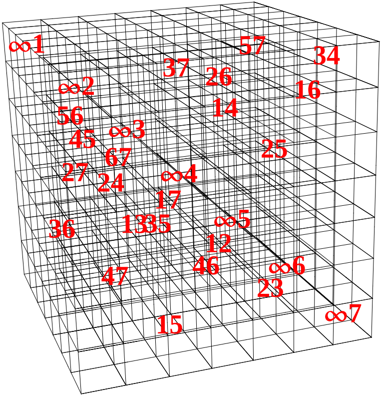
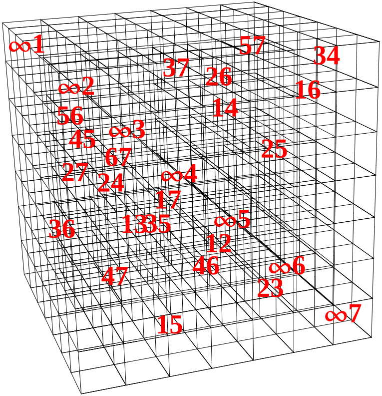


Room cubes

A Room cube is an n-dimensional matrix of order v with entries that are empty or 2-subsets of $S = \{\infty, 1, 2, \ldots, v\}$ such that every 2-dimensional projection is a Room square.



Front view:

∞1	56	24		37		
	∞2	67	35		14	
		∞3	17	46		25
36			∞4	12	57	
	47			∞5	23	16
27		15			∞6	34
45	13		26			∞7

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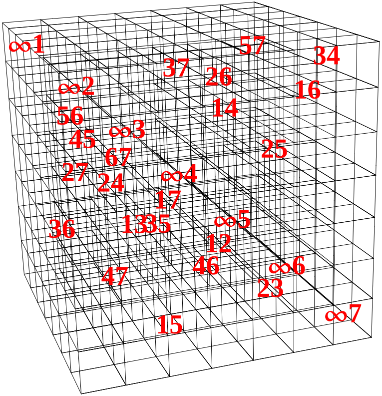

Top view:

∞1			36		27	45
56	∞2			47		13
24	67	∞3			15	
	35	17	∞4			26
37		46	12	∞5		
	14		57	23	∞6	
		25		16	34	∞7

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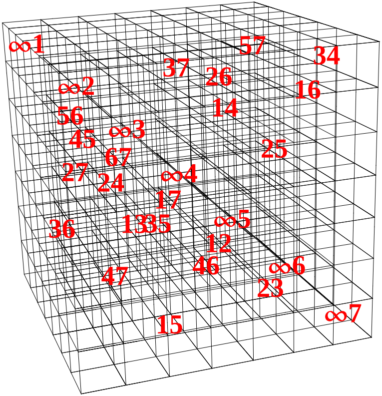

Side view:

26	34		57			∞1
45		16			∞2	37
	27			∞3	14	56
13			∞4	25	67	
		∞5	36	17		24
	∞6	47	12		35	
∞7	15	23		46		

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An n-dimensional Room cube of order v is equivalent to:

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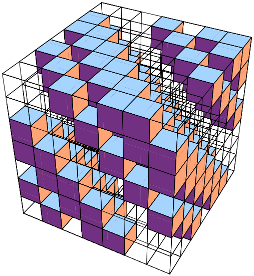
Conjecture (W. D. Wallis): $\mu(v) \leq \frac{1}{2}(v-1)$

A variation on cubes of symmetric designs



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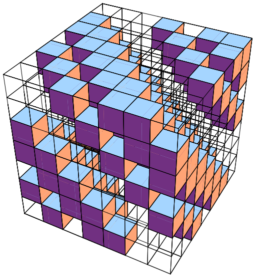
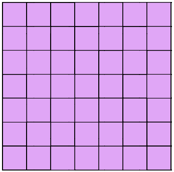


Front view:



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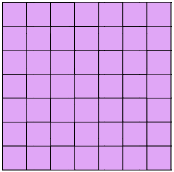

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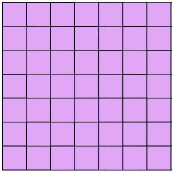

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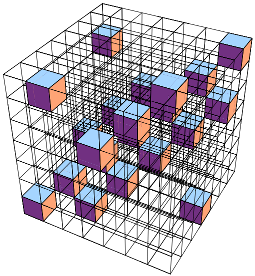

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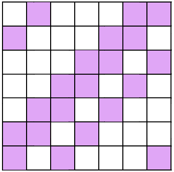

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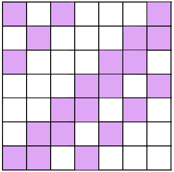

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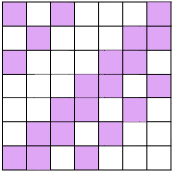

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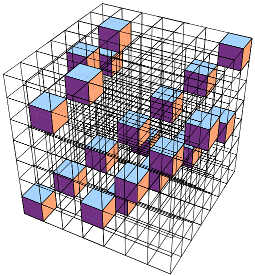

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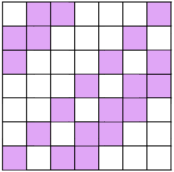

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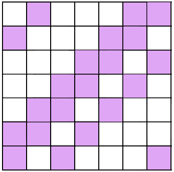

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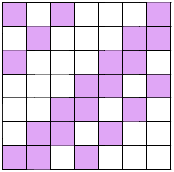

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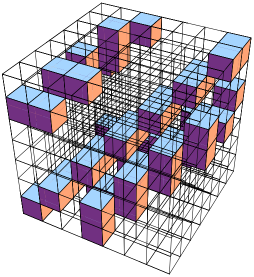

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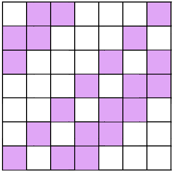

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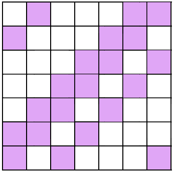

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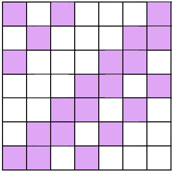

A variation on cubes of symmetric designs

An n-dimensional **projection cube** of (v, k, λ) designs is a function

$$A : \{1, \ldots, v\}^n \to \{0, 1\}$$

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More incidences can appear if we take sums in the binary semifield $\mathbb{B}_2$. In this case we can make 2^{21} examples out of the second $(7, 3, 1)$ projection cube.

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0	0	1
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If we work in the binary field $\mathbb{F}_2$, XORing with any cube with an even number of 1's in every direction does not affect the sums. This would produce many more examples.

$+$	0	1
0	0	1
1	1	0

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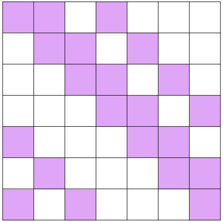

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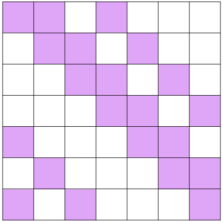

$(1,1)$
$(1,2)$
$(1,4)$
$(2,2)$
$(2,3)$
$(2,5)$
$(3,3)$
$(3,4)$
$(3,6)$
$(4,4)$
$(4,5)$

$(4,7)$
$(5,5)$
$(5,6)$
$(5,1)$
$(6,6)$
$(6,7)$
$(6,2)$
$(7,7)$
$(7,1)$
$(7,3)$

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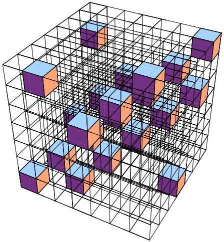

$(1,1,1)$	$(4,7,3)$
$(1,2,3)$	$(5,5,5)$
$(1,4,7)$	$(5,6,7)$
$(2,2,2)$	$(5,1,4)$
$(2,3,4)$	$(6,6,6)$
$(2,5,1)$	$(6,7,1)$
$(3,3,3)$	$(6,2,5)$
$(3,4,5)$	$(7,7,7)$
$(3,6,2)$	$(7,1,2)$
$(4,4,4)$	$(7,3,6)$
$(4,5,6)$	

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$(1,1,1,1)$	$(4,7,3,6)$
$(1,2,3,4)$	$(5,5,5,5)$
$(1,4,7,3)$	$(5,6,7,1)$
$(2,2,2,2)$	$(5,1,4,7)$
$(2,3,4,5)$	$(6,6,6,6)$
$(2,5,1,4)$	$(6,7,1,2)$
$(3,3,3,3)$	$(6,2,5,1)$
$(3,4,5,6)$	$(7,7,7,7)$
$(3,6,2,5)$	$(7,1,2,3)$
$(4,4,4,4)$	$(7,3,6,2)$
$(4,5,6,7)$	

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Question: Are there projection cubes of dimension $n > 3$?

$$(1,1,1,1,4) \quad (4,7,3,6,5)$$
$$(1,2,3,4,1) \quad (5,5,5,5,1)$$
$$(1,4,7,3,2) \quad (5,6,7,1,5)$$
$$(2,2,2,2,5) \quad (5,1,4,7,6)$$
$$(2,3,4,5,2) \quad (6,6,6,6,2)$$
$$(2,5,1,4,3) \quad (6,7,1,2,6)$$
$$(3,3,3,3,6) \quad (6,2,5,1,7)$$
$$(3,4,5,6,3) \quad (7,7,7,7,3)$$
$$(3,6,2,5,4) \quad (7,1,2,3,7)$$
$$(4,4,4,4,7) \quad (7,3,6,2,1)$$
$$(4,5,6,7,4)$$

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$(1,1,1,1,4,1)$	$(4,7,3,6,5,5)$
$(1,2,3,4,1,6)$	$(5,5,5,5,1,5)$
$(1,4,7,3,2,2)$	$(5,6,7,1,5,3)$
$(2,2,2,2,5,2)$	$(5,1,4,7,6,6)$
$(2,3,4,5,2,7)$	$(6,6,6,6,2,6)$
$(2,5,1,4,3,3)$	$(6,7,1,2,6,4)$
$(3,3,3,3,6,3)$	$(6,2,5,1,7,7)$
$(3,4,5,6,3,1)$	$(7,7,7,7,3,7)$
$(3,6,2,5,4,4)$	$(7,1,2,3,7,5)$
$(4,4,4,4,7,4)$	$(7,3,6,2,1,1)$
$(4,5,6,7,4,2)$	

A variation on cubes of symmetric designs

Question: Are there projection cubes of dimension $n > 3$?

$$
\begin{array}{ll}
(1,1,1,1,4,1,2) & (4,7,3,6,5,5,2) \\
(1,2,3,4,1,6,1) & (5,5,5,5,1,5,6) \\
(1,4,7,3,2,2,6) & (5,6,7,1,5,3,5) \\
(2,2,2,2,5,2,3) & (5,1,4,7,6,6,3) \\
(2,3,4,5,2,7,2) & (6,6,6,6,2,6,7) \\
(2,5,1,4,3,3,7) & (6,7,1,2,6,4,6) \\
(3,3,3,3,6,3,4) & (6,2,5,1,7,7,4) \\
(3,4,5,6,3,1,3) & (7,7,7,7,3,7,1) \\
(3,6,2,5,4,4,1) & (7,1,2,3,7,5,7) \\
(4,4,4,4,7,4,5) & (7,3,6,2,1,1,5) \\
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\end{array}
$$

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Question: Are there projection cubes of dimension $n > 3$?

Question:

Other combinatorial objects equivalent to projection cubes?

$$
\begin{array}{ll}
(1,1,1,1,4,1,2) & (4,7,3,6,5,5,2) \\
(1,2,3,4,1,6,1) & (5,5,5,5,1,5,6) \\
(1,4,7,3,2,2,6) & (5,6,7,1,5,3,5) \\
(2,2,2,2,5,2,3) & (5,1,4,7,6,6,3) \\
(2,3,4,5,2,7,2) & (6,6,6,6,2,6,7) \\
(2,5,1,4,3,3,7) & (6,7,1,2,6,4,6) \\
(3,3,3,3,6,3,4) & (6,2,5,1,7,7,4) \\
(3,4,5,6,3,1,3) & (7,7,7,7,3,7,1) \\
(3,6,2,5,4,4,1) & (7,1,2,3,7,5,7) \\
(4,4,4,4,7,4,5) & (7,3,6,2,1,1,5) \\
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\end{array}
$$

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Some computational results:

- $\mu(3, 2, 1) = 5$
- $\mu(7, 3, 1) \geq 7$
- $\mu(11, 5, 2) \geq 11$
- $\mu(13, 4, 1) \geq 13$
- $\mu(15, 7, 3) \geq 3$
- $\mu(16, 6, 2) \geq 4$
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1. Is there an upper bound on $\mu(v, k, \lambda)$?

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2. Difference sets for projection cubes?

A variation on cubes of symmetric designs

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3. ...

A variation on cubes of symmetric designs

Question: Are there projection cubes with inequivalent projections?

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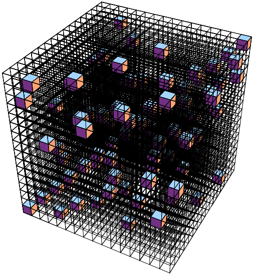
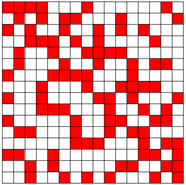
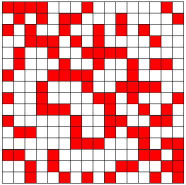
$(16, 6, 2)$



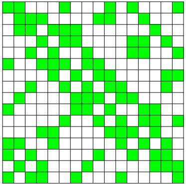
Front view:



Top view:



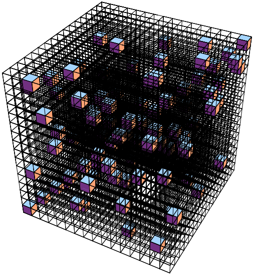
Side view:



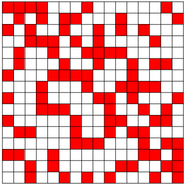
A variation on cubes of symmetric designs

Question: Are there projection cubes with inequivalent projections?

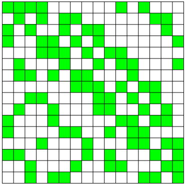
$(16, 6, 2)$



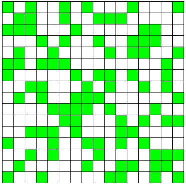
Front view:



Top view:



Side view:



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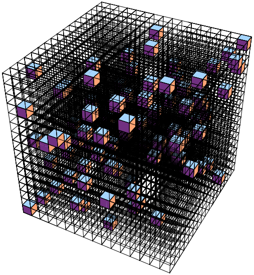
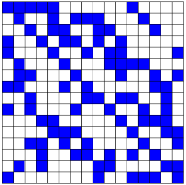
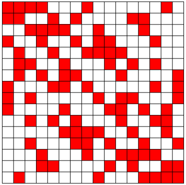


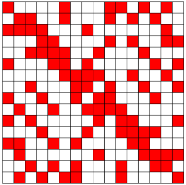
Front view:



Top view:



Side view:



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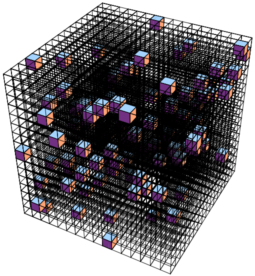

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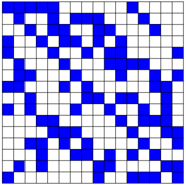

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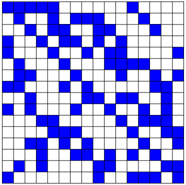

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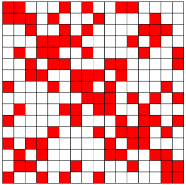

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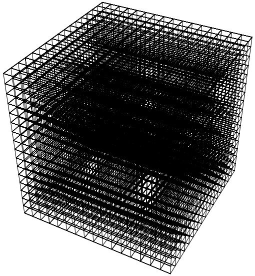

Question: Is there an example with all three colors?

Mosaics of combinatorial designs

O. W. Gnilke, M. Greferath, M. O. Pavčević, *Mosaics of combinatorial designs*, Des. Codes Cryptogr. **86** (2018), no. 1, 85–95.

Mosaics of combinatorial designs

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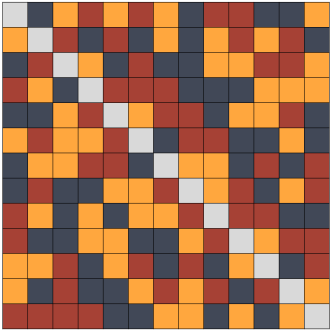
V. Krčadinac, *Small examples of mosaics of combinatorial designs*, to appear in Examples and Counterexamples. https://arxiv.org/abs/2405.12672



$$2\text{-}(13,4,1) \oplus 2\text{-}(13,4,1) \oplus 2\text{-}(13,4,1) \oplus 2\text{-}(13,1,0)$$

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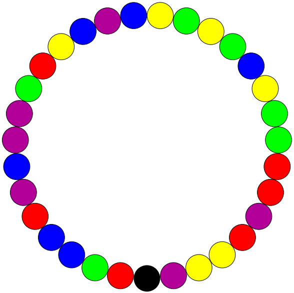

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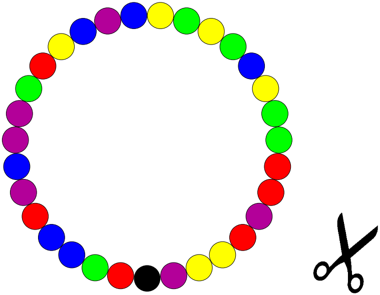
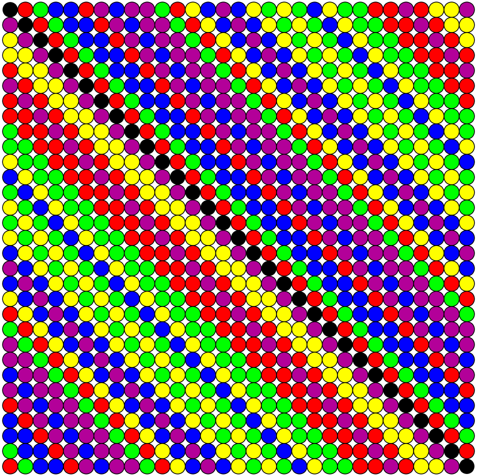
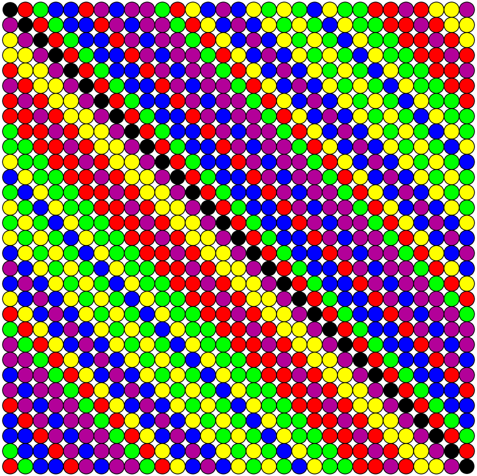


Mosaics of combinatorial designs

Mosaics of combinatorial designs

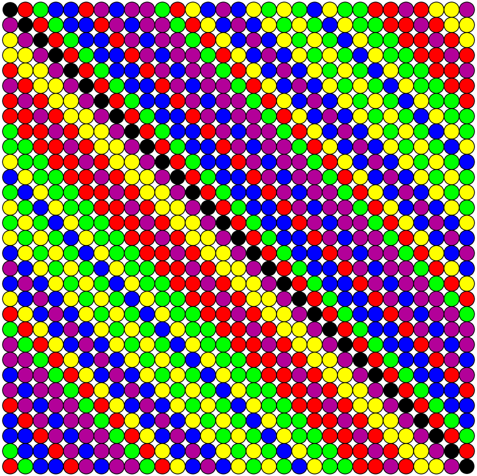


Mosaics of combinatorial designs



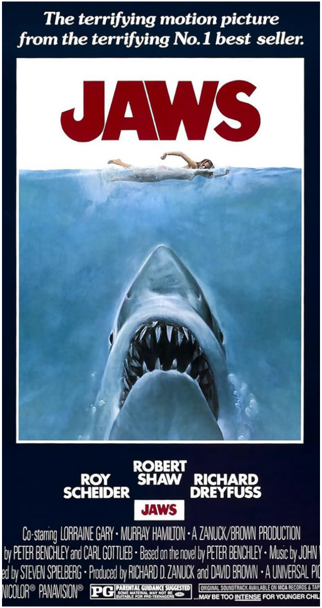
$2\text{-}(31,6,1) \oplus 2\text{-}(31,6,1) \oplus 2\text{-}(31,6,1) \oplus 2\text{-}(31,6,1) \oplus 2\text{-}(31,6,1) \oplus 2\text{-}(31,1,0)$

Mosaics of combinatorial designs

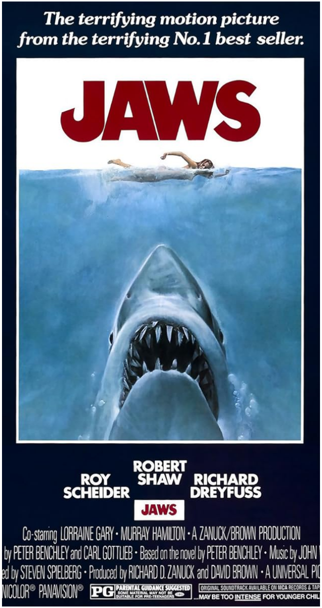


Question: Is there a higher-dimensional variation of mosaics?

Two classic films



Two classic films



What could go wrong if we take both ideas. . .

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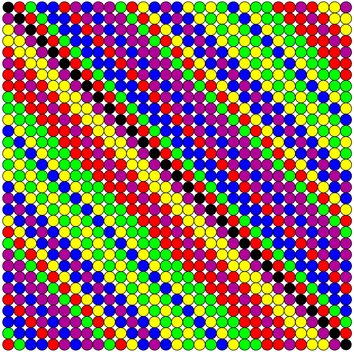
Two ideas to combine designs



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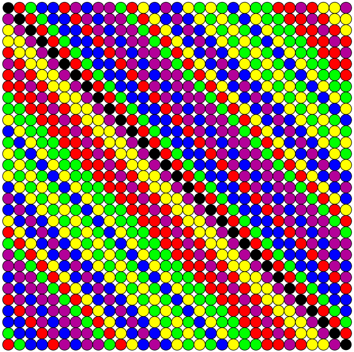
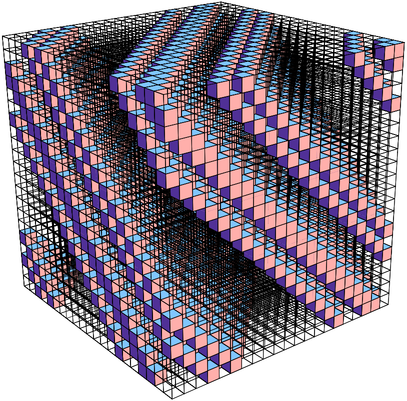
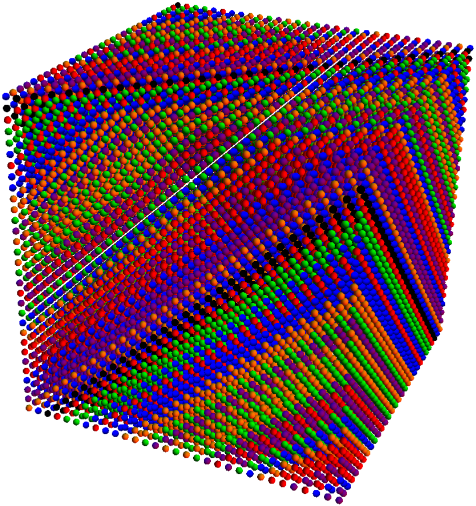
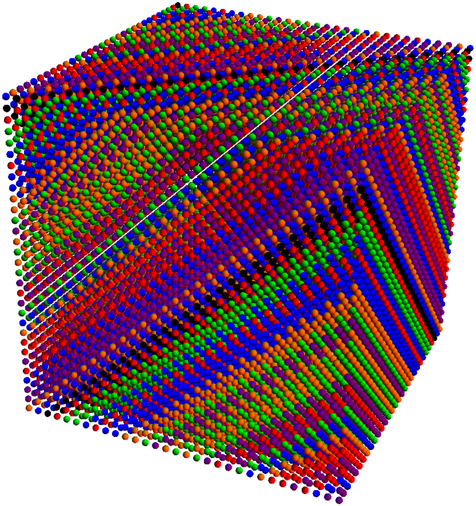




Cubes of mosaics of designs?



Sharknado designs!



The End

Thanks for your attention!