Projektivni pravci i trodimenzionalne Hadamardove matrice*

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Prirodoslovno-matematički fakultet

Sveučilište u Zagrebu

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4. Permutacije koje čuvaju dvoomjer: $(ABCD) = \frac{(a-c)(b-d)}{(a-d)(b-c)}$.

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$$f(x) = \frac{ax + b}{cx + d}, \quad ad - bc \neq 0.$$

Ako je nazivnik nula uzimamo $f(x) = \infty$. Stavljamo $f(\infty) = \frac{a}{c}$.

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$PSL(2, \mathbb{F})$ djeluje 2-tranzitivno na $PG(1, \mathbb{F})$.

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Ako postoji Hadamardova matrica reda v, onda je $v = 1$, $v = 2$ ili $v = 4u$.

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Teorem.

Neka je q potencija prostog broja. Ako je $q \equiv 3 \pmod 4$, onda postoji Hadamardova matrica reda $v = q + 1$. Ako je $q \equiv 1 \pmod 4$, onda postoji Hadamardova matrica reda $v = 2(q + 1)$.

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Neka je q neparna prim potencija i $\mathbb{F}_q$ konačno polje reda q.

$$\chi : \mathbb{F}_q^* \to \{1, -1\}, \quad \chi(a) = \begin{cases} 1, & \text{ako je } a \text{ kvadrat u } \mathbb{F}_q^* \\ -1, & \text{ako je } a \text{ nekvadrat u } \mathbb{F}_q^* \end{cases}$$

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Funkcija χ je homomorfizam iz multiplikativne grupe $\mathbb{F}_q^*$ u $\{1, -1\}$:

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Paleyevu matricu tipa I, tj. za $q \equiv 3 \pmod 4$ možemo indeksirati točkama projektivnog pravca $PG(1, \mathbb{F}_q)$:

$$H(x, y) = \begin{cases} -1, & \text{ako je } x = y = \infty \\ 1, & \text{ako je } x = y \neq \infty \text{ ili } x = \infty \neq y \text{ ili } y = \infty \neq x \\ \chi(y - x), & \text{inače} \end{cases}$$

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Najmanji red za koji je egzistencija nepoznata: $v = 668 = 4 \cdot 167$.

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Hadamardova matrica dimenzije n i reda v je $H : \{1, \ldots, v\}^n \to \{-1, 1\}$ takva da su paralelni $(n-1)$-dimenzionalni slojevi međusobno ortogonalni:

$$\sum_{1 \le i_1, \ldots, \widehat{i_j}, \ldots, i_n \le v} H(i_1, \ldots, a, \ldots, i_n) H(i_1, \ldots, b, \ldots, i_n) = v^{n-1} \delta_{ab}$$

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Višedimenzionalne Hadamardove matrice

Propozicija.

- Ako postoji prava Hadamardova matrica reda $v > 2$ (bilo koje dimenzije), onda je v djeljiv s 4.
- Ako postoji Hadamardova matrica dimenzije $n \geq 3$ i reda $v > 1$, onda je v paran.

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Teorem [Produktna konstrukcija].

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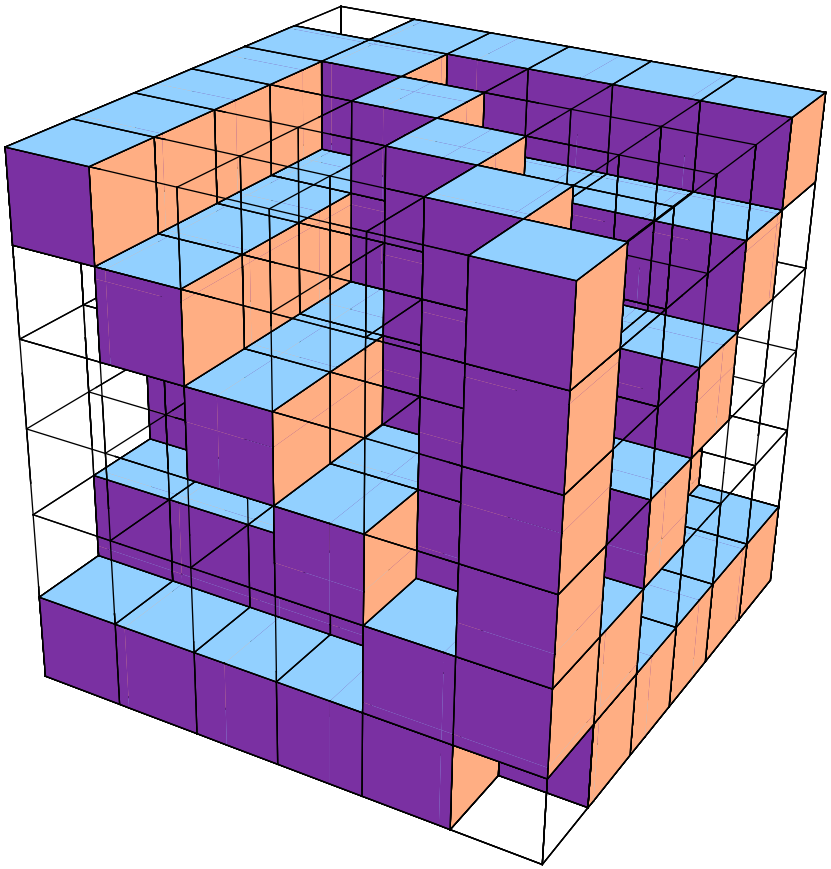

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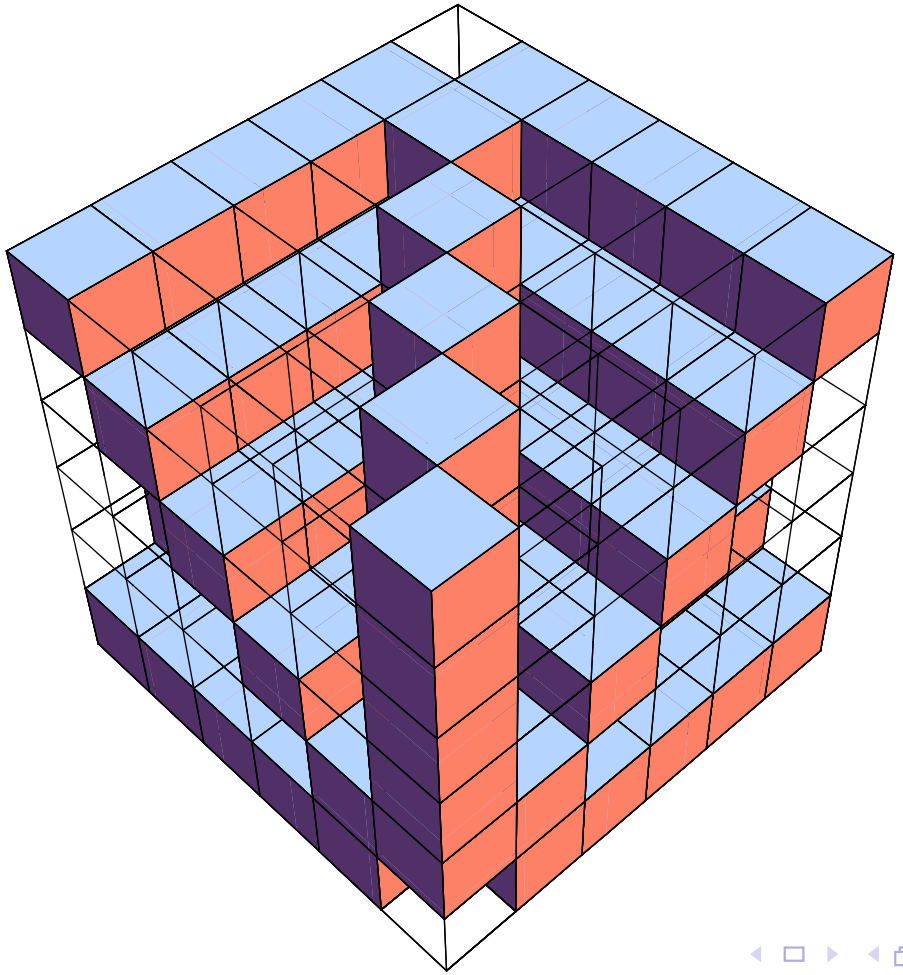

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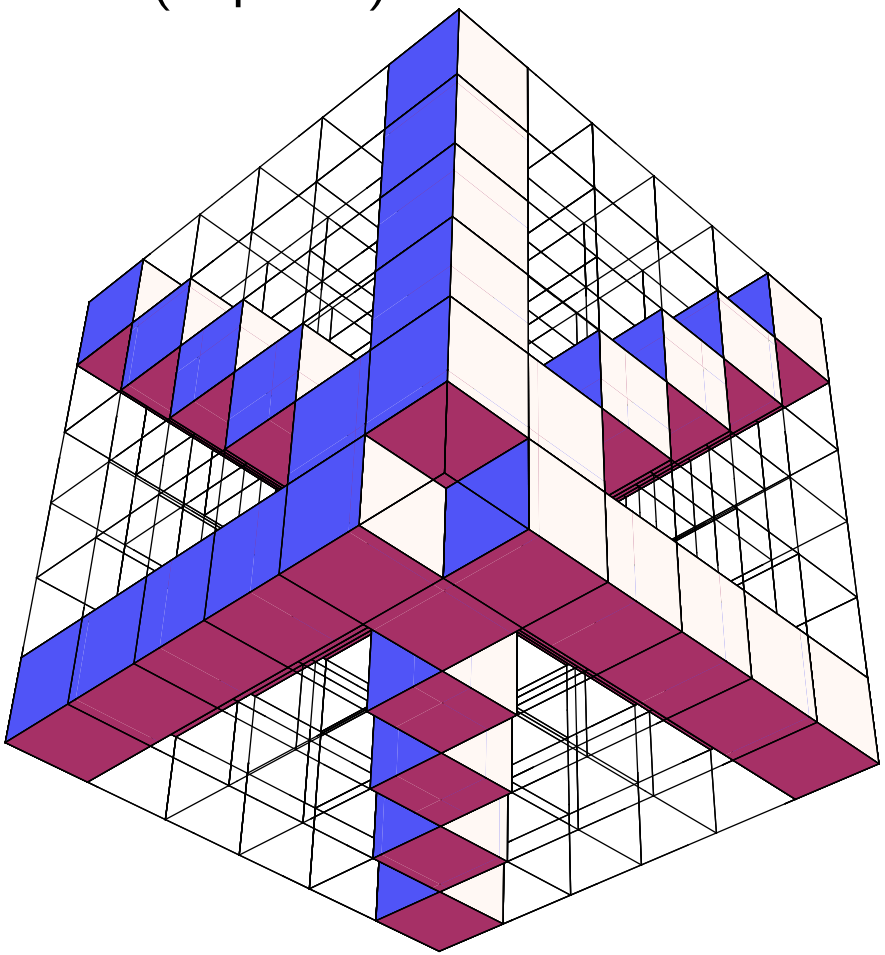

Yangovi rezultati

Teorem [Dimenzija++].

Ako je h Hadamardova matrica dimenzije n, onda je

$$H(i_1, \ldots, i_n, i_{n+1}) = h(i_1, \ldots, i_{n-1},\ i_n + i_{n+1} \bmod v)$$

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Hadamardova matrica dimenzije $n+1$ istog reda v.

Teorem [2-dim. reda $(2t)^s \rightsquigarrow 2s$-dim. reda $2t$].

Ako je h Hadamardova mat. dimenzije 2 i reda $v = (2t)^s$, $s > 1$, onda je

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$$= h(i_0 + (2t)i_1 + \ldots + (2t)^{s-1} i_{s-1},\ j_0 + (2t)j_1 + \ldots + (2t)^{s-1} j_{s-1})$$

Hadamardova matrica dimenzije $2s$ i reda $2t$.

Yangovi rezultati

Teorem [Dimenzija++].

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Teorem [DigitConstructionMat].

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Ako je istinita Hadamardova hipoteza, tj. postoje 2-dimenzionalne Hadamardove matrice svih redova djeljivih s 4, onda postoje Hadamardove matrice svih parnih redova za dimenzije $n \geq 4$.

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Što je s dimenzijom $n = 3$?

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Egzistencija: $v = 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \ldots$

Trodimenzionalne Hadamardove matrice

"Concluding questions"

5. Prove or disprove the existence of three-dimensional Hadamard matrices of orders $4k + 2 \neq 2 \cdot 3^m$.

6. Construct more three-dimensional Hadamard matrices of orders $4k + 2$.

Trodimenzionalne Hadamardove matrice

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Paul J. Shlichta, *Higher dimensional Hadamard matrices*, IEEE Trans. Inform. Theory **25** (1979), no. 5, 566–572.

VI. Future Research and Applications

The present exposition suggests a number of unsolved problems and unproven conjectures. Some examples follow.

a) The algebraic approach to the derivation of two-dimensional Hadamard matrices [2]–[7] suggests that a similar procedure may be feasible for three- or higher dimensional matrices.

Trodimenzionalne Hadamardove matrice

J. Hammer, J. R. Seberry, *Higher-dimensional orthogonal designs and applications*, IEEE Trans. Inform. Theory **27** (1981), no. 6, 772–779.

"Paleyeva kocka": $\chi(0) = -1$, reci i stupci $PG(1, q)$, $q \equiv 3 \pmod 4$,

$$H(x_1, \ldots, x_n) = \begin{cases} 1, & \text{ako je } x_i = \infty \text{ za bar jedan } i, \\ \chi(x_1 + \ldots + x_n), & \text{inače.} \end{cases}$$

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Paleyeva matrica tipa I

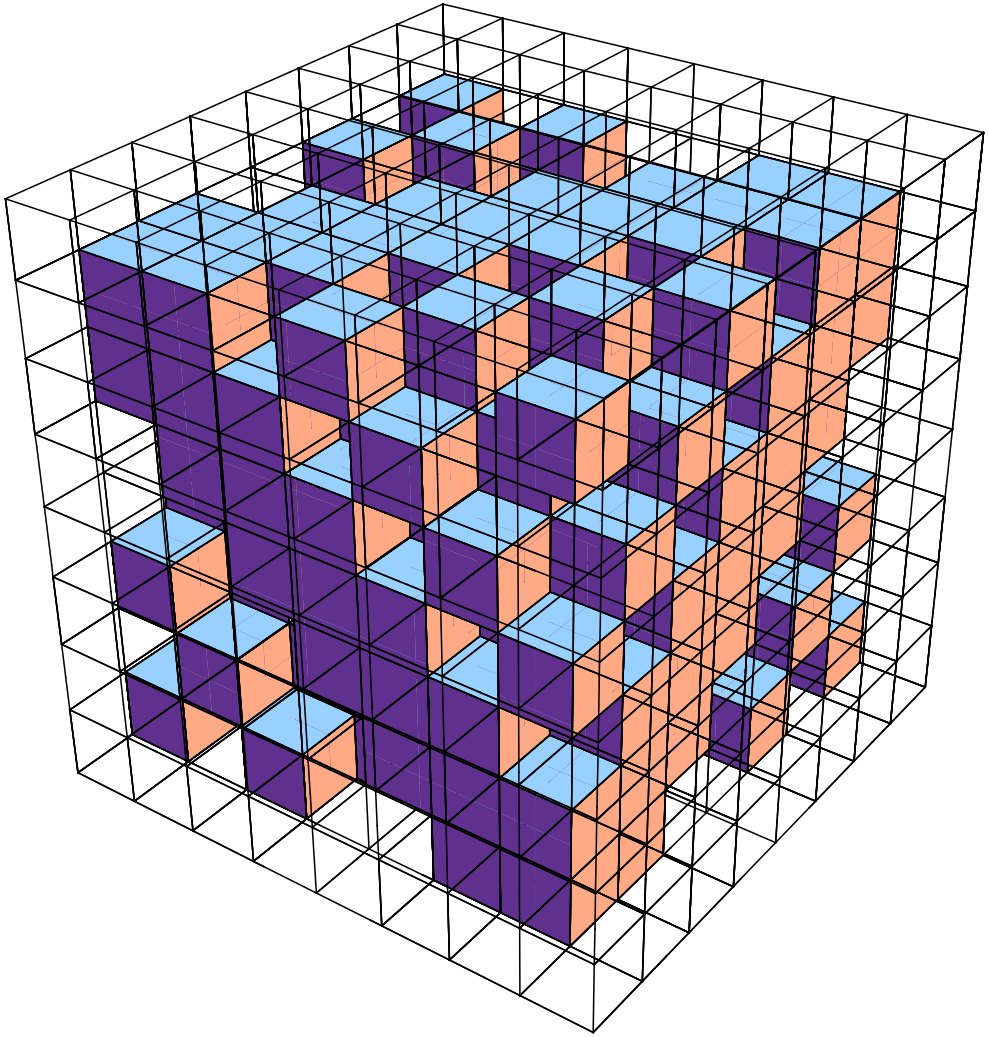
$$\begin{bmatrix} - & + & + & + & + & + & + & + \\ + & + & - & - & + & - & + & + \\ + & + & + & - & - & + & - & + \\ + & + & + & + & - & - & + & - \\ + & - & + & + & + & - & - & + \\ + & + & - & + & + & + & - & - \\ + & - & + & - & + & + & + & - \\ + & - & - & + & - & + & + & + \end{bmatrix}$$

Paleyeva kocka dim. $n = 2$

$$\begin{bmatrix} + & + & + & + & + & + & + & + \\ + & - & + & + & - & + & - & - \\ + & + & + & - & + & - & - & - \\ + & + & - & + & - & - & - & + \\ + & - & + & - & - & - & + & + \\ + & + & - & - & - & + & + & - \\ + & - & - & - & + & + & - & + \\ + & - & - & + & + & - & + & - \end{bmatrix}$$

Trodimenzionalne Hadamardove matrice

Paleyeva kocka za $n = 3$ i $q = 7$:



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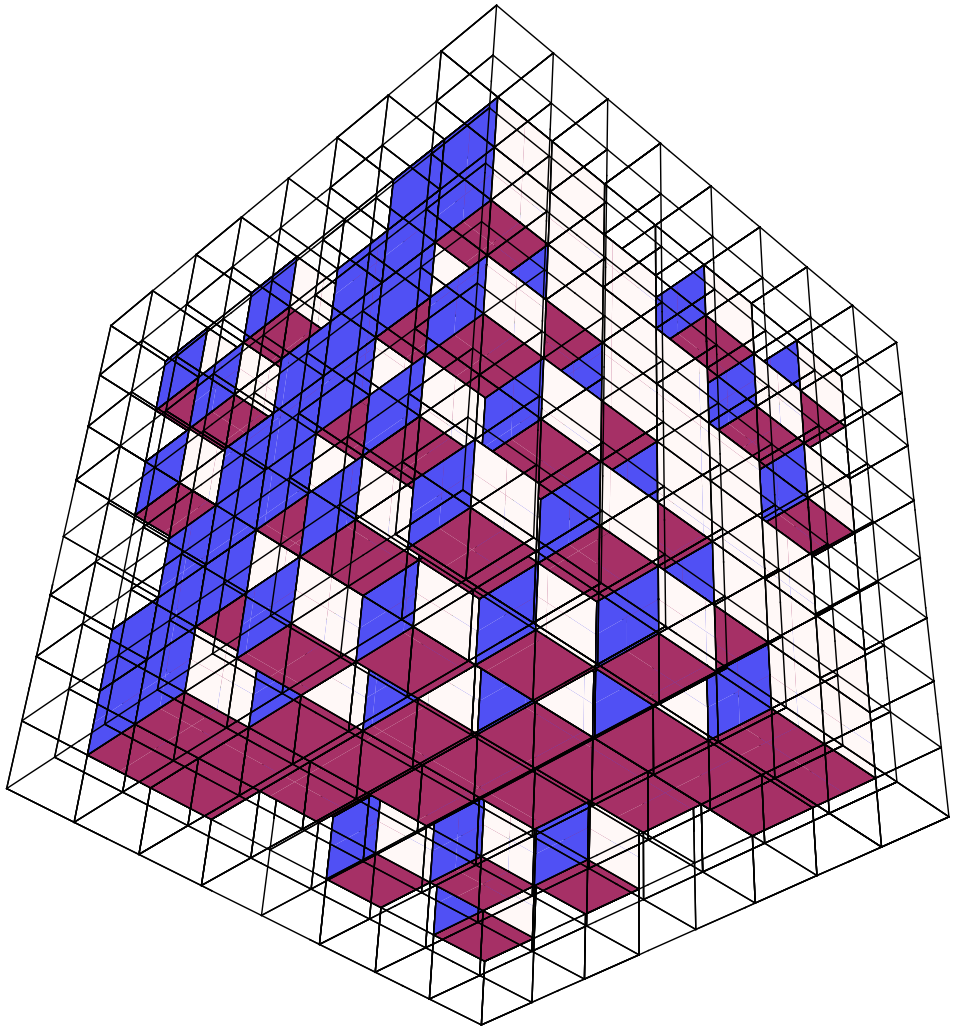

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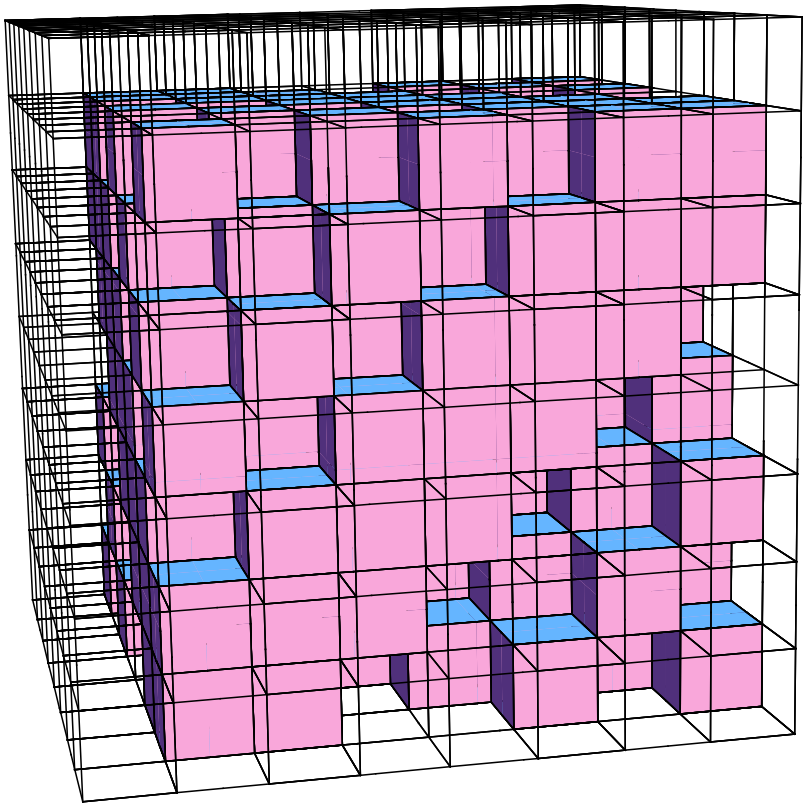

Novi rezultat

V. Krčadinac, M. O. Pavčević, K. Tabak, *Three-dimensional Hadamard matrices of Paley type*, 2023. https://arxiv.org/abs/2305.12415

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$H : PG(1,q)^3 \to \{1,-1\}$, $q \equiv 1$ ili $3 \pmod 4$,

$$H(x,y,z) = \begin{cases} -1, & \text{ako je } x = y = z, \\ 1, & \text{ako je } x = y \neq z \\ & \text{ili } x = z \neq y \\ & \text{ili } y = z \neq x, \\ \chi(z-y), & \text{ako je } x = \infty, \\ \chi(x-z), & \text{ako je } y = \infty, \\ \chi(y-x), & \text{ako je } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{inače.} \end{cases}$$

Novi rezultat

Lema 1.

Matrica H je invarijanta na cikličke permutacije koordinata, tj. $H(x, y, z) = H(y, z, x) = H(z, x, y)$.

Novi rezultat

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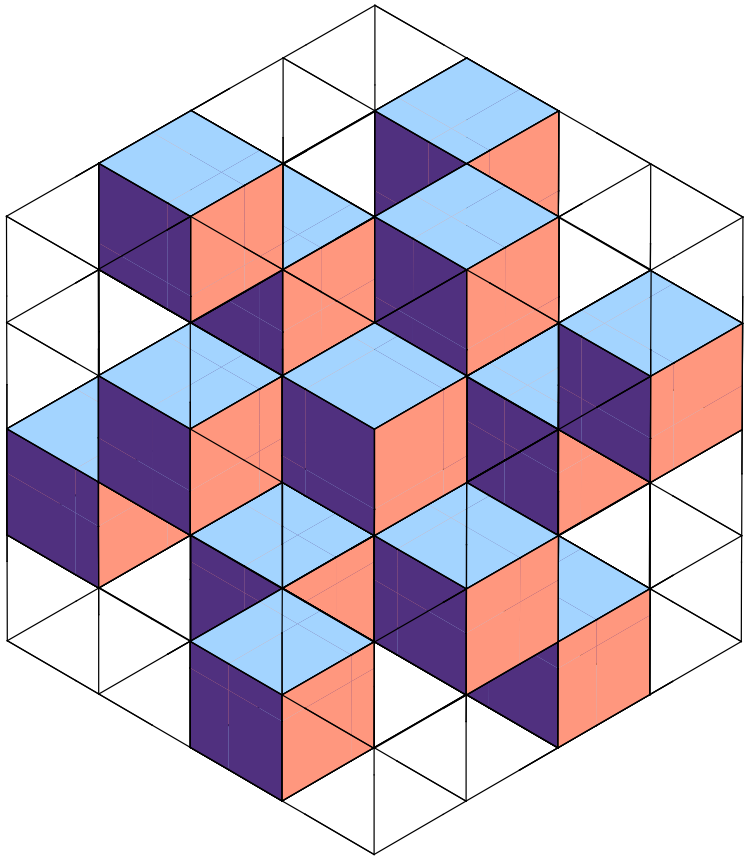

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Matrica H je invarijanta na cikličke permutacije koordinata, tj. $H(x, y, z) = H(y, z, x) = H(z, x, y)$.

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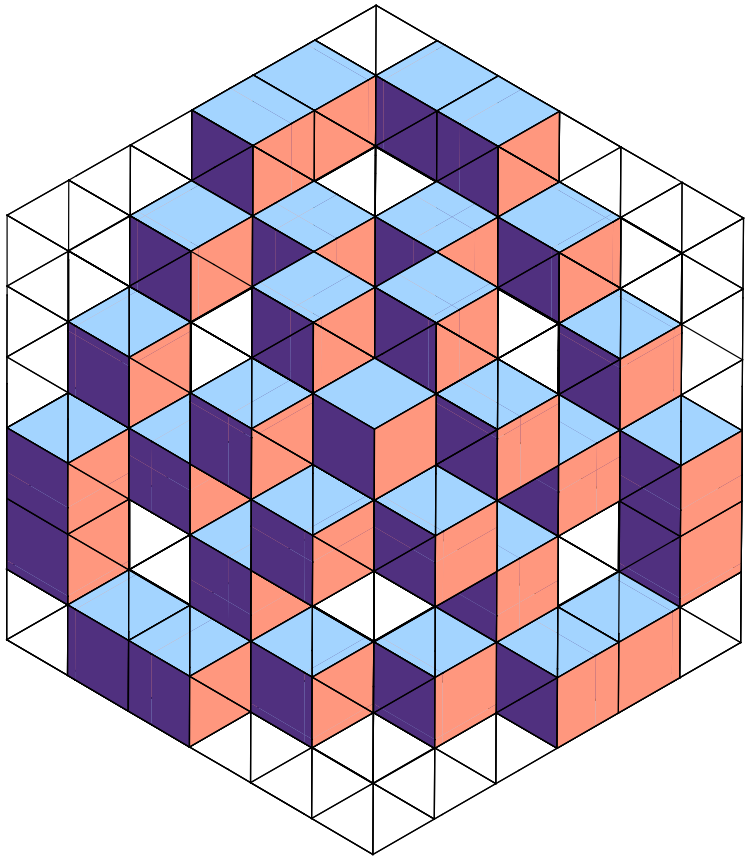

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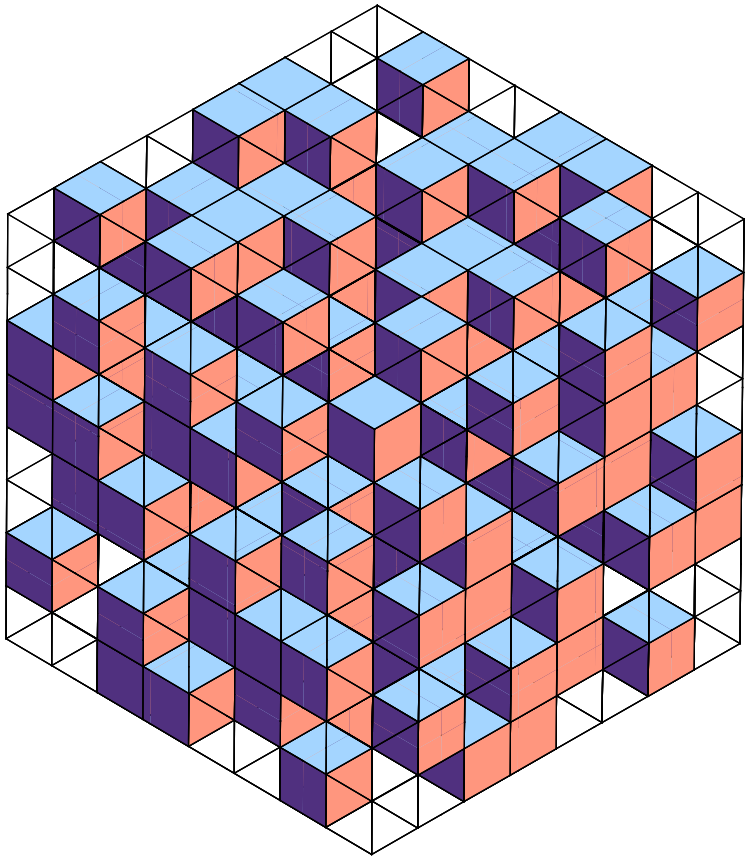

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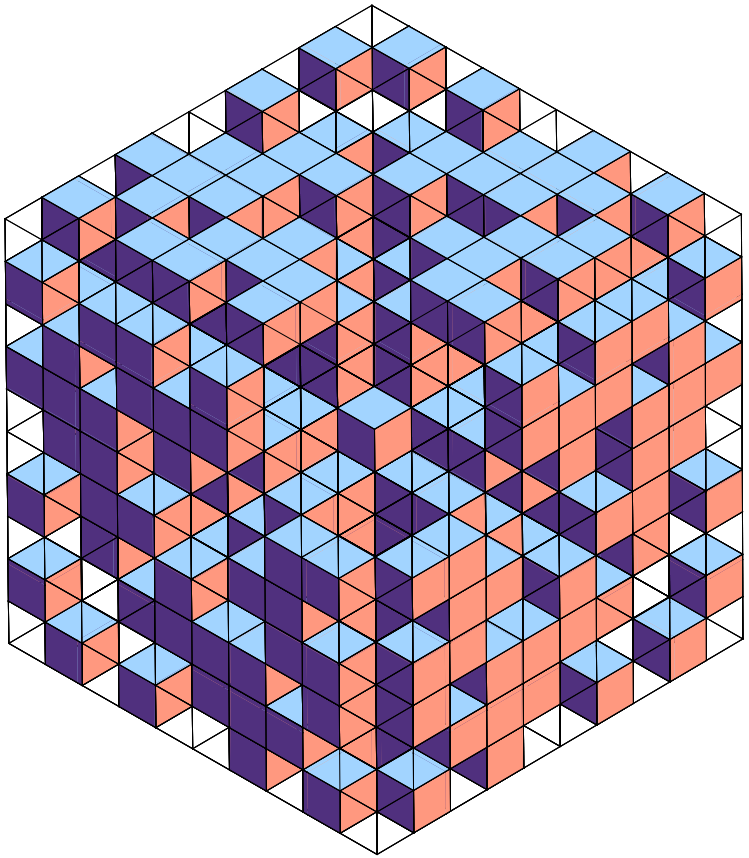
$q = 9$



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Matrica H je invarijanta na razlomljene linearne transformacije s determinantom 1, tj. djelovanje grupe $PSL(2, q)$ na projektivni pravac.

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Matrica H je trodimenzionalna Hadamardova matrica reda $q + 1$ za svaku neparnu prim potenciju q. Ako je $q \equiv 3 \pmod 4$, onda je H prava trodimenzionalna Hadamardova matrica sa svim "šnitama" ekvivalentnim Paleyevoj matrici tipa I.

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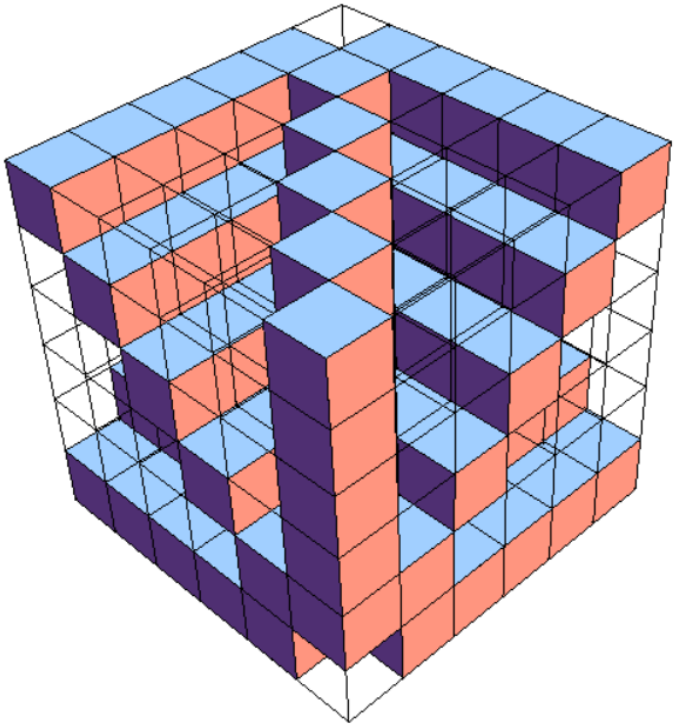
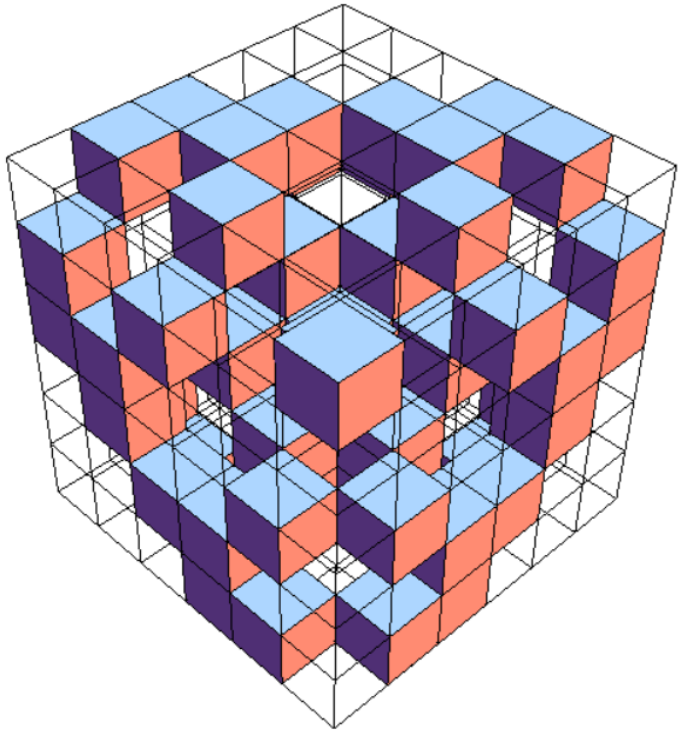
E. Spence, *Hadamard matrices from relative difference sets*, J. Combinatorial Theory Ser. A **19** (1975), no. 3, 287–300.

Pitanja

Yangova 3dH matrica reda 6



Nova 3dH matrica za $q = 5$



Pitanja

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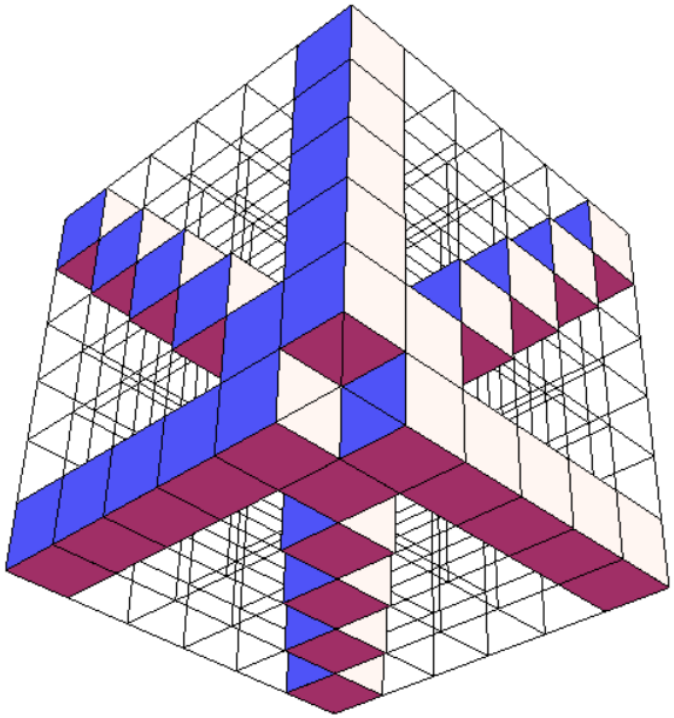
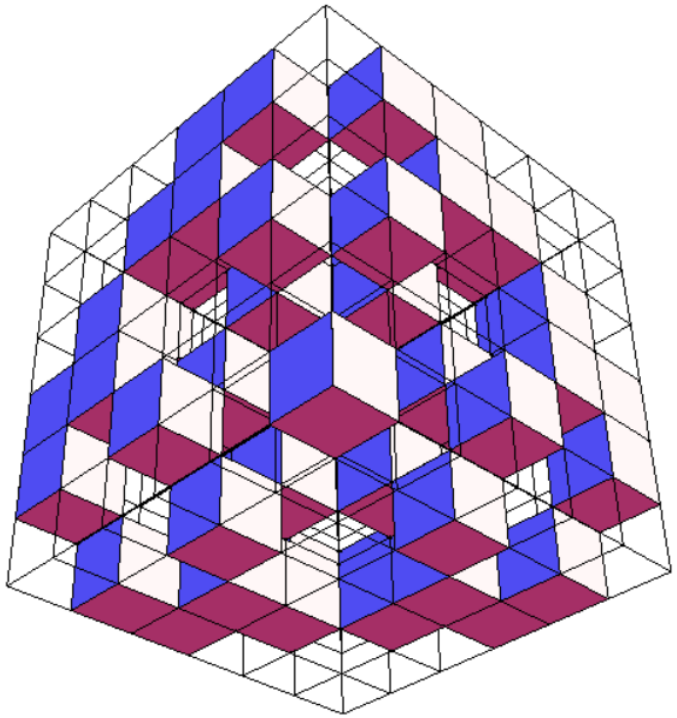


Nova 3dH matrica za $q = 5$



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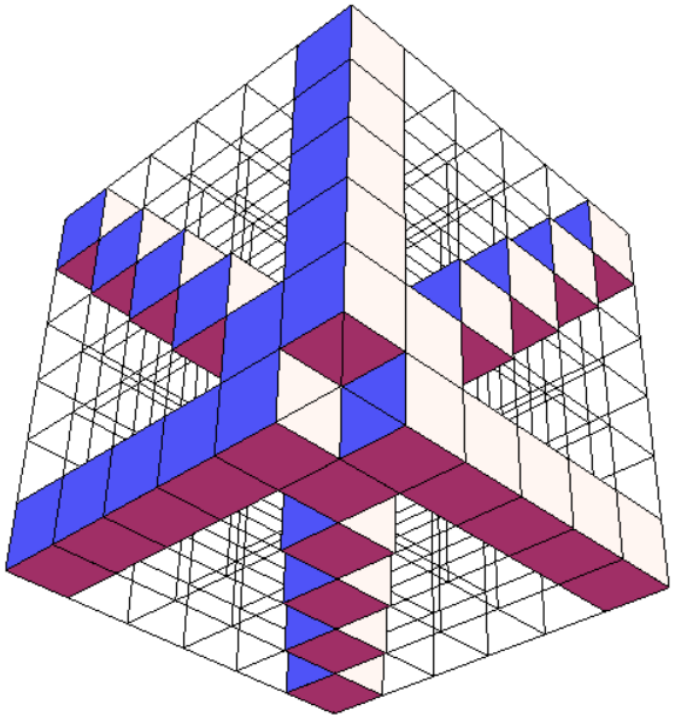

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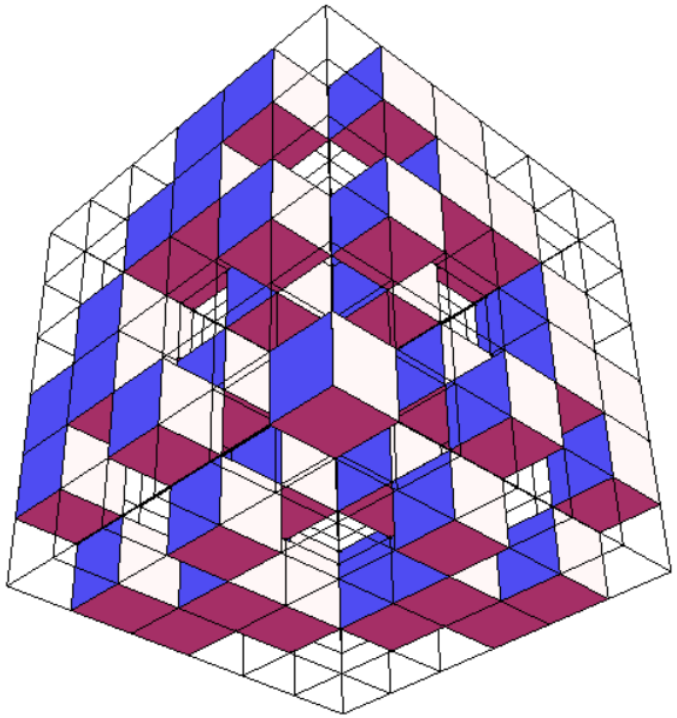

Jesu li ekvivalentne?

Pitanja

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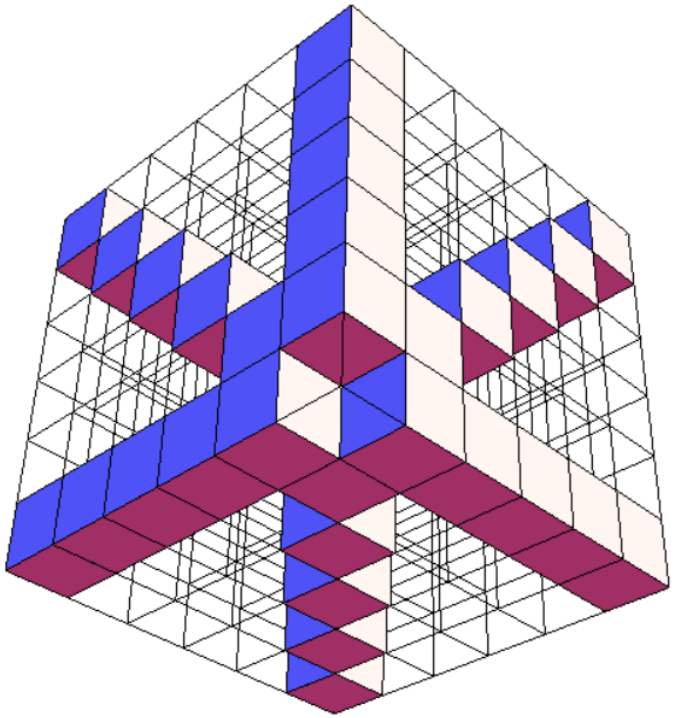

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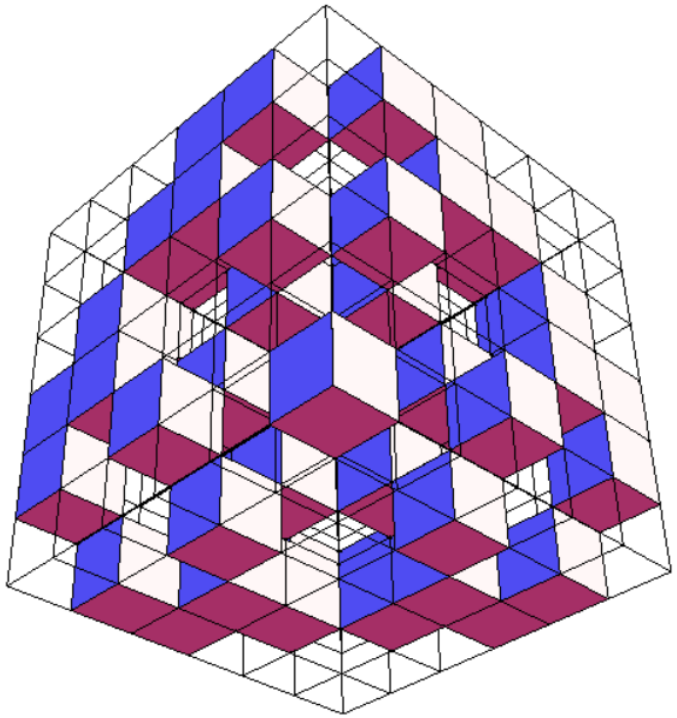

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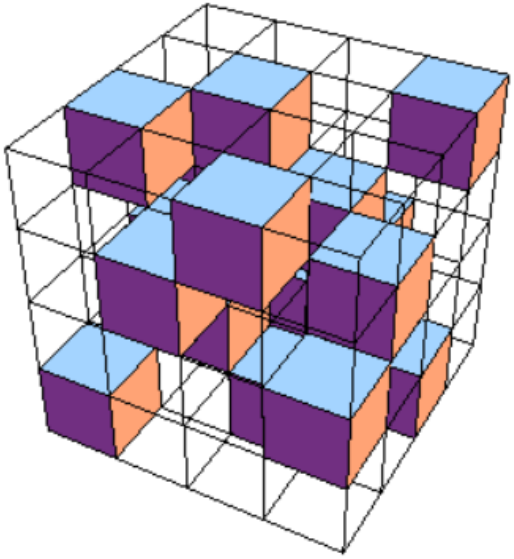
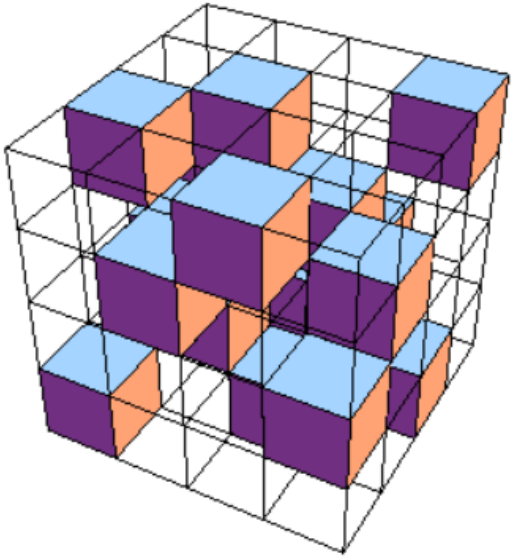
W. de Launey, R. M. Stafford, *Automorphisms of higher-dimensional Hadamard matrices*, J. Combin. Des. **16** (2008), no. 6, 507–544.

Pitanja



$\leftarrow$ Nova 3dH matrica za $q = 3$

Pitanja

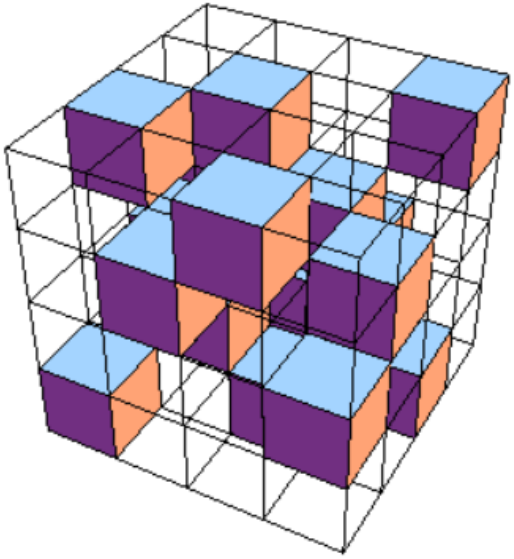


$\leftarrow$ Nova 3dH matrica za $q = 3$

Produktna konstrukcija:

$$\begin{bmatrix} - & + & + & + \\ + & + & + & - \\ + & - & + & + \\ + & + & - & + \end{bmatrix} \quad \begin{bmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{bmatrix} \quad \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}$$

Pitanja



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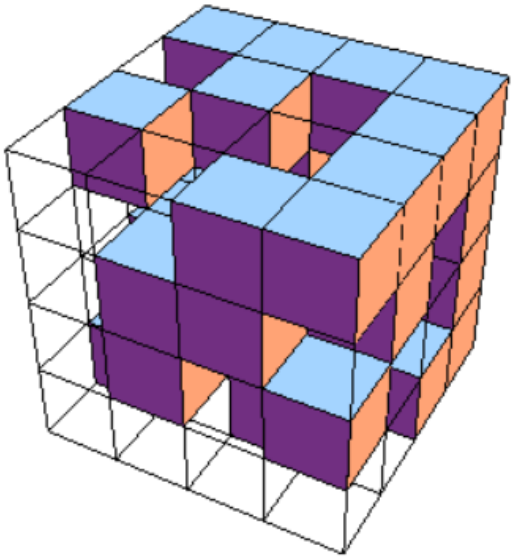

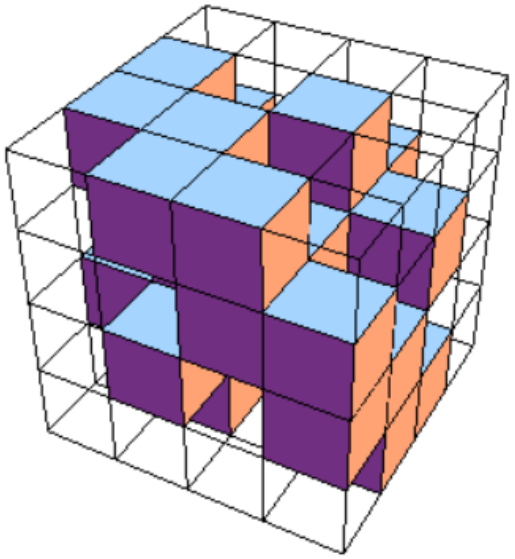

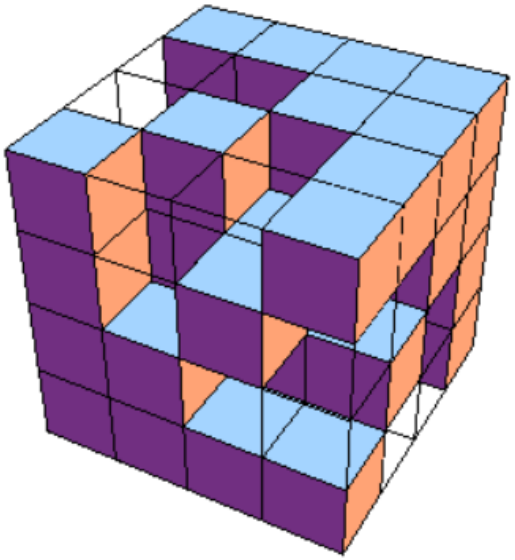

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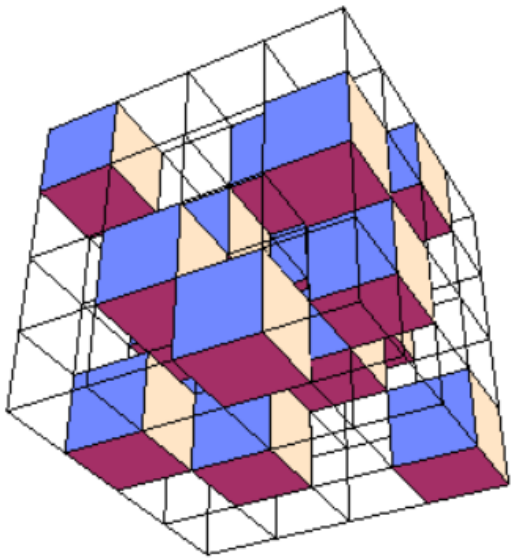

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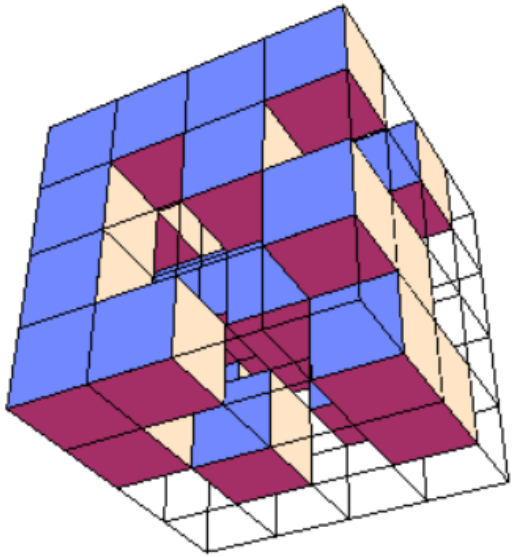
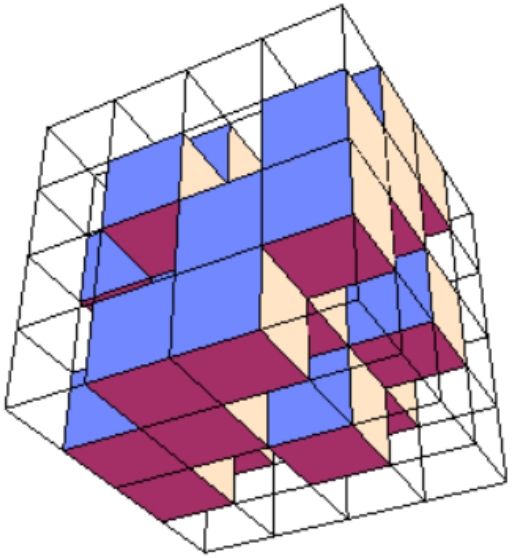
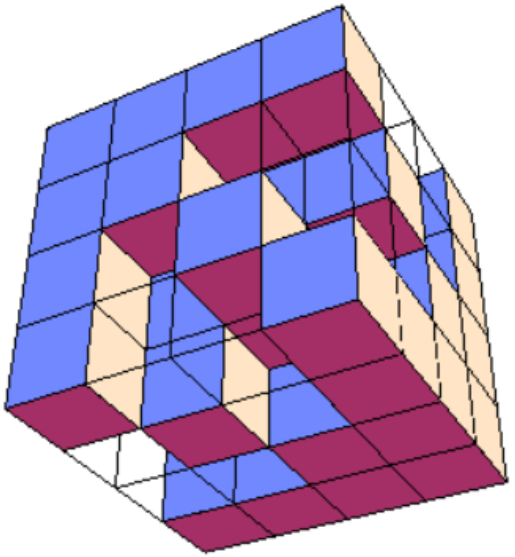
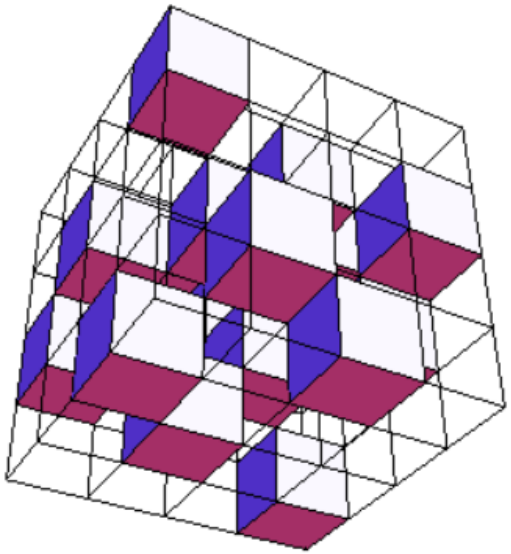




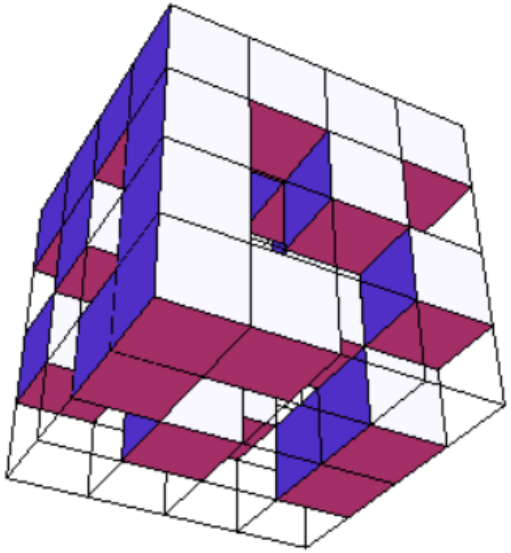


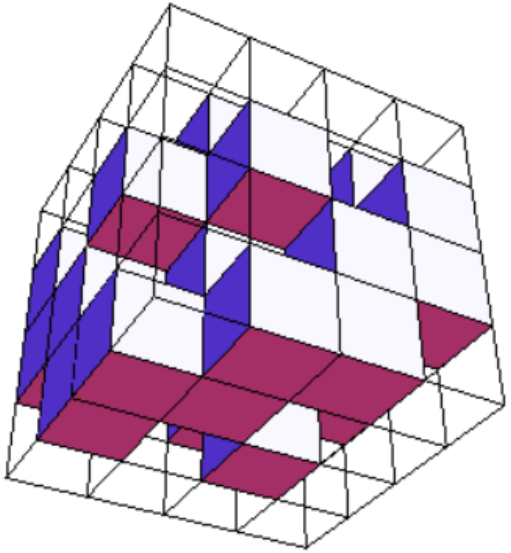
Pitanja

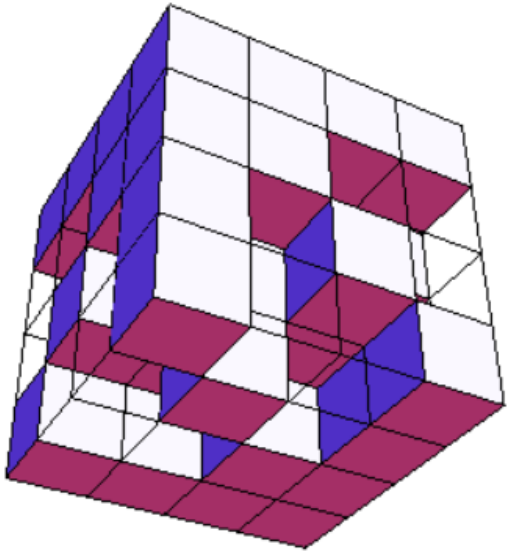


$\leftarrow$ Nova 3dH matrica za $q = 3$

Produktna konstrukcija:







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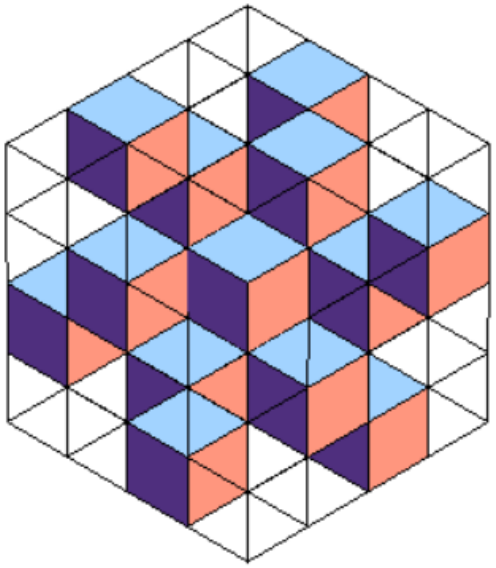
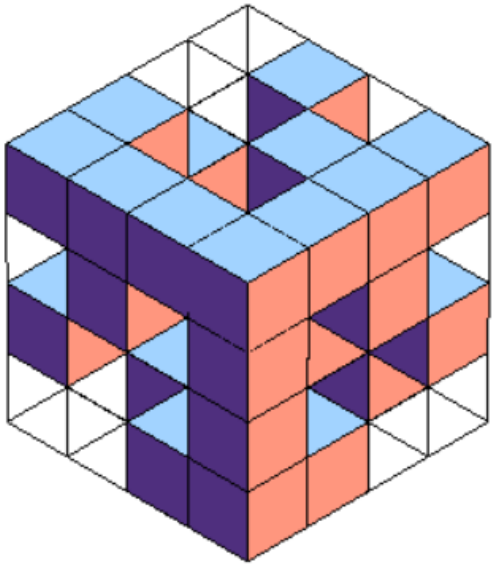
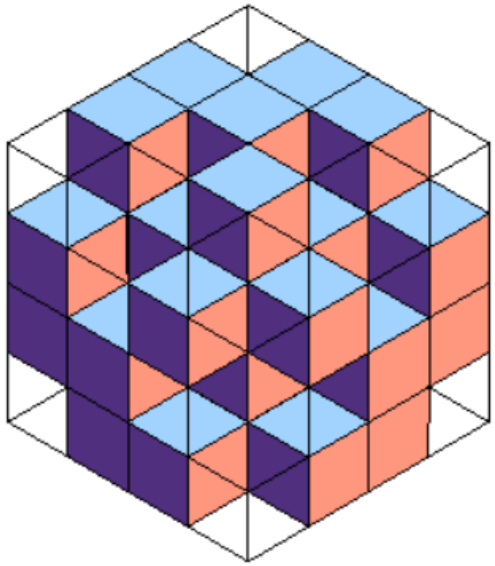
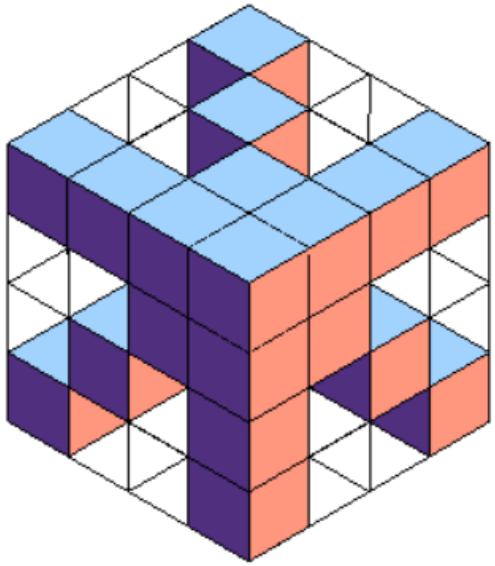


$\leftarrow$ Nova 3dH matrica za $q = 3$

Produktna konstrukcija:







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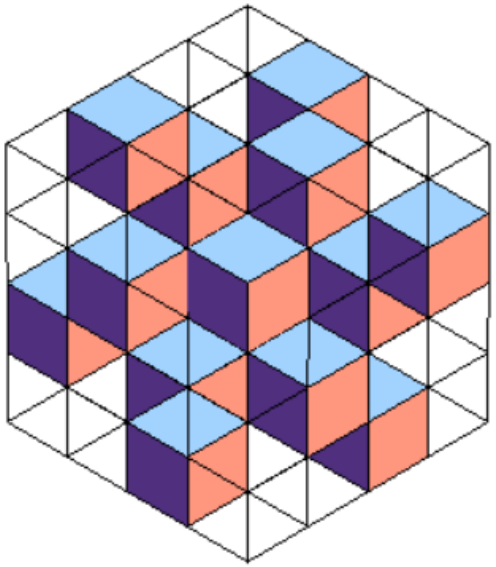
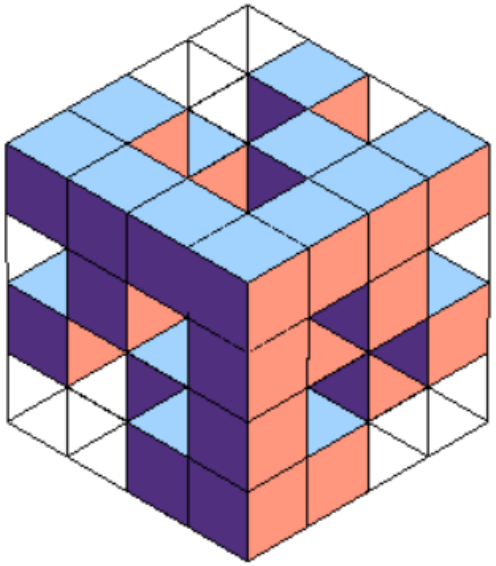
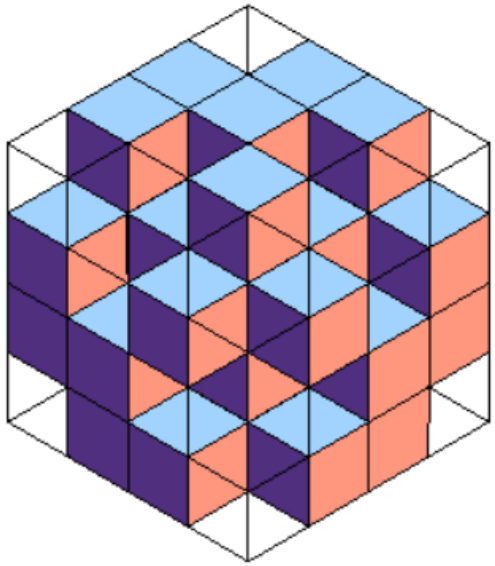


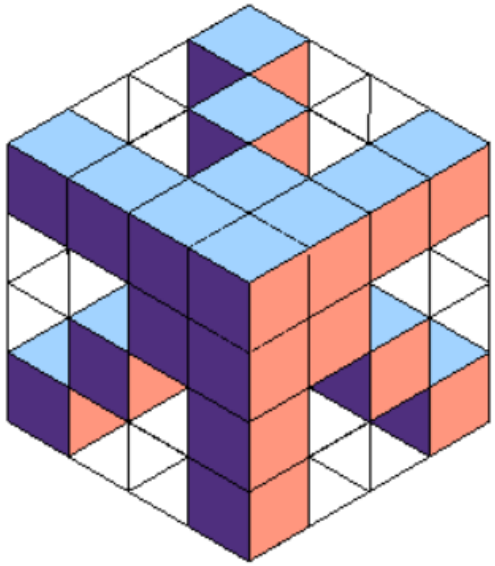
$\leftarrow$ Nova 3dH matrica za $q = 3$

Jesu li ekvivalentne?

Produktna konstrukcija:







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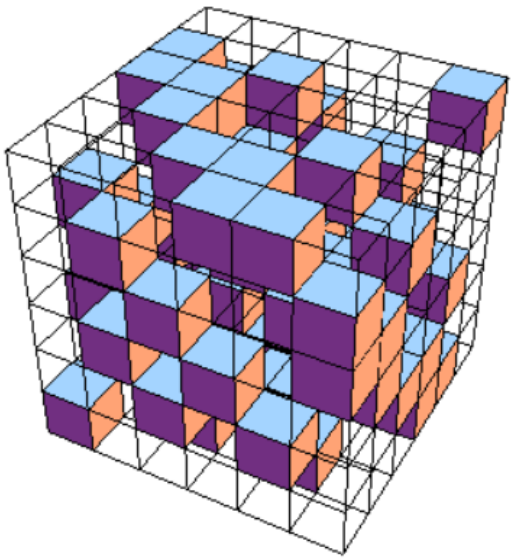
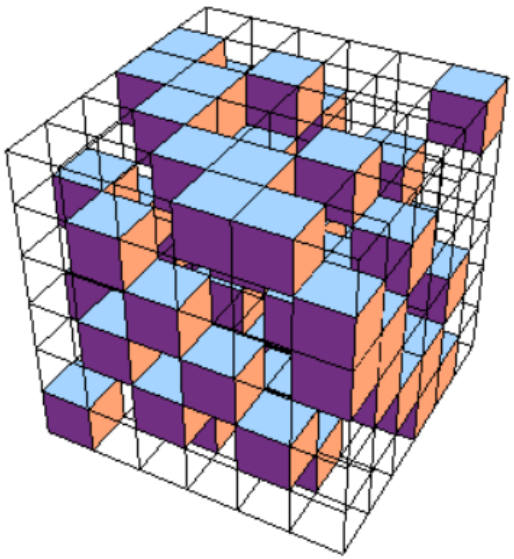


$\leftarrow$ Nova 3dH matrica za $q = 5$

Pitanja

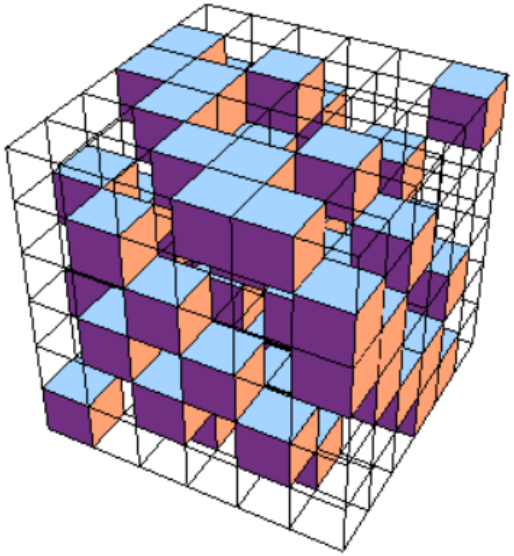


$\leftarrow$ Nova 3dH matrica za $q = 5$

Produktna konstrukcija:

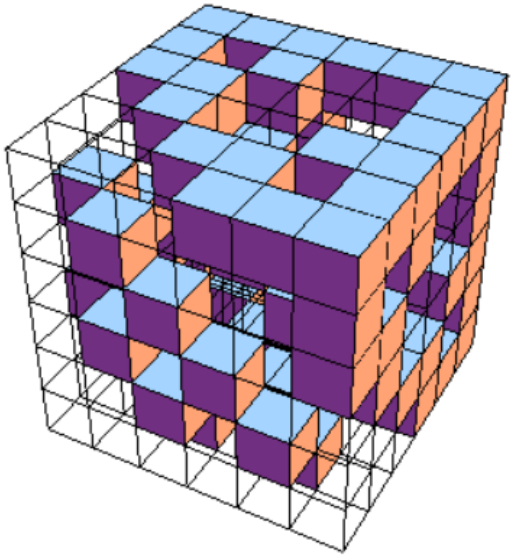
$$\begin{bmatrix} - & + & + & + & + & + \\ + & + & + & - & + & - \\ + & + & + & + & - & - \\ + & - & + & + & - & + \\ + & + & - & - & + & + \\ + & - & - & + & + & + \end{bmatrix} \quad \begin{bmatrix} + & + & + & + & + & + \\ + & - & + & - & - & + \\ + & + & - & - & + & - \\ + & - & - & + & - & + \\ + & - & + & - & + & - \\ + & + & - & + & - & - \end{bmatrix} \quad \begin{bmatrix} - & + & + & + & + & + \\ + & - & + & + & + & + \\ + & + & - & + & + & + \\ + & + & + & - & + & + \\ + & + & + & + & - & + \\ + & + & + & + & + & - \end{bmatrix}$$

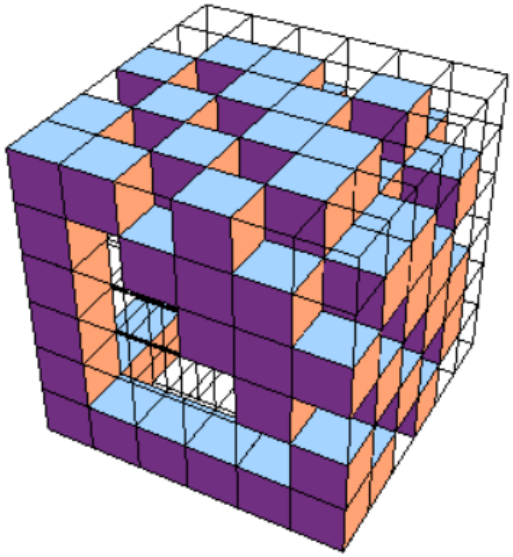
Pitanja

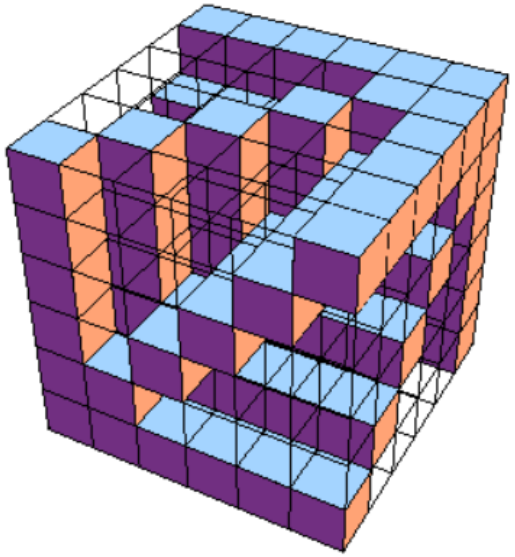


$\leftarrow$ Nova 3dH matrica za $q = 5$

Produktna konstrukcija:







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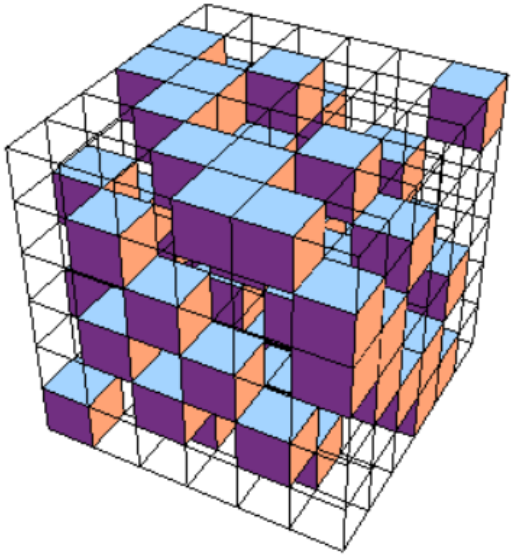
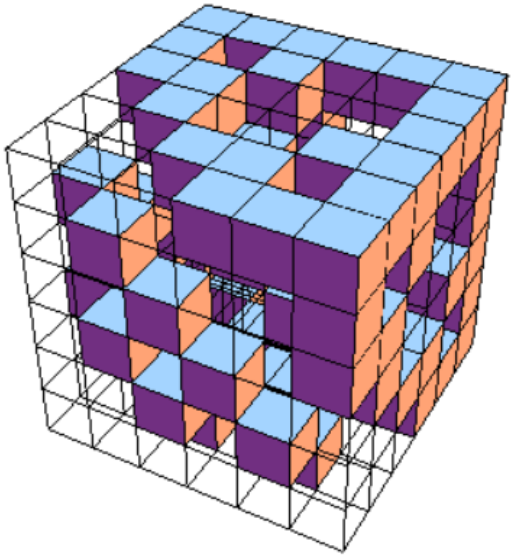
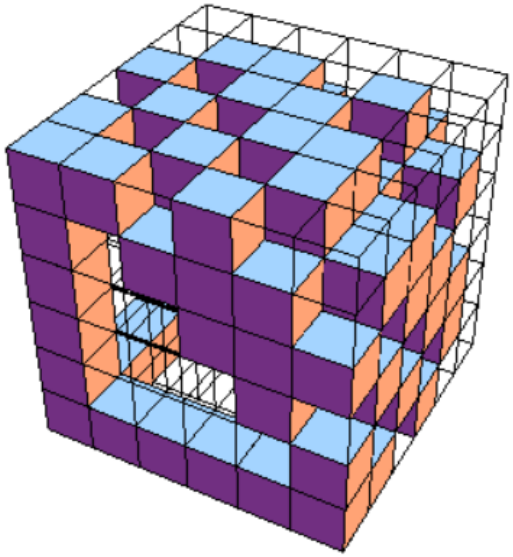


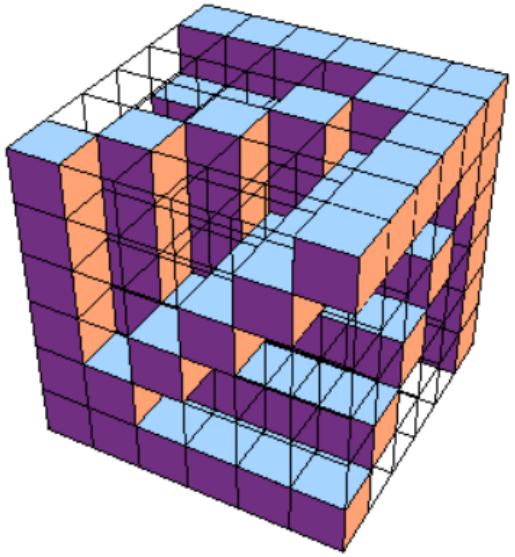
$\leftarrow$ Nova 3dH matrica za $q = 5$

Nisu ekvivalentne!

Produktna konstrukcija:







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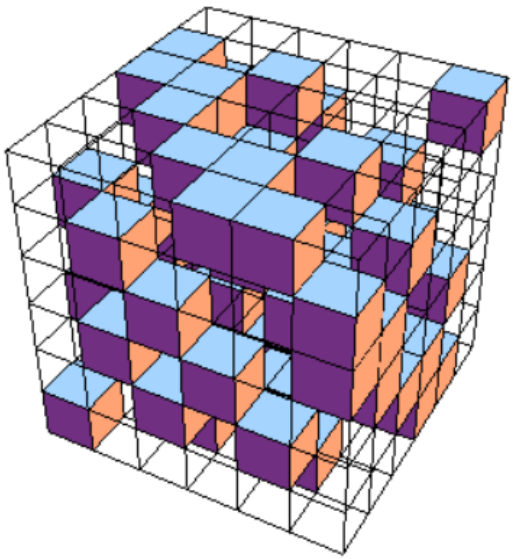


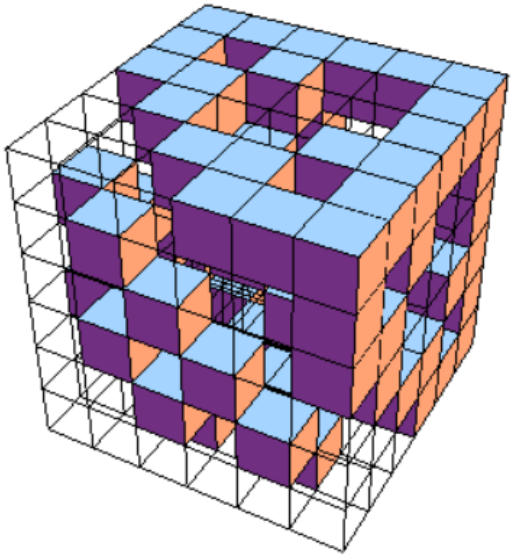
$\leftarrow$ Nova 3dH matrica za $q = 5$

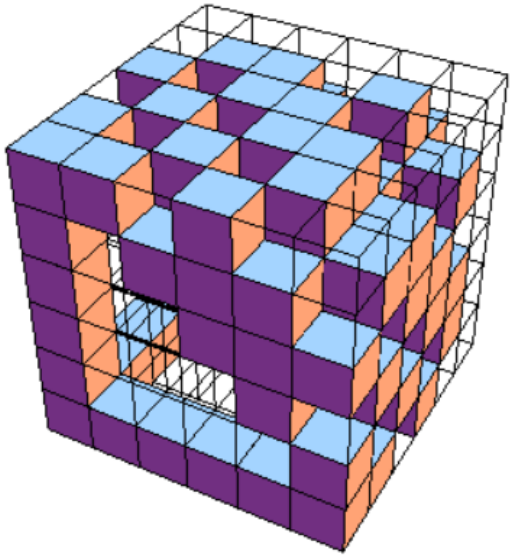
Nisu ekvivalentne!

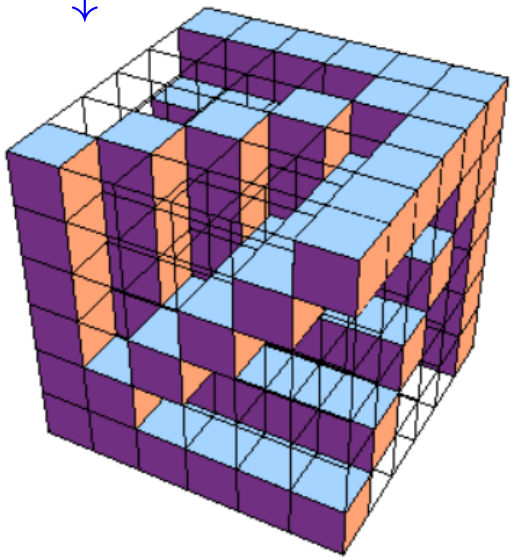
Produktna konstrukcija:

Razlikuje se samo na prostornoj dijagonali od Yangove 3dH matrice

$\downarrow$







Pitanja

Postoje li 3dH matrice reda $v = 22$?

Pitanja

Postoje li 3dH matrice reda $v = 22$?

Može li se Paleyeva konstrukcija generalizirati na n-dimenzionalne matrice?

$$H(x,y,z) = \begin{cases} -1, & \text{ako je } x = y = z, \\ 1, & \text{ako je } x = y \neq z \\ & \text{ili } x = z \neq y \\ & \text{ili } y = z \neq x, \\ \chi(z-y), & \text{ako je } x = \infty, \\ \chi(x-z), & \text{ako je } y = \infty, \\ \chi(y-x), & \text{ako je } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{inače.} \end{cases}$$

Kraj

Hvala na pažnji!