Polarity transformations of the semipartial geometries $LP(4, q)^{\star}$

Vedran Krčadinac

PMF-MO

29.3.2021.

* This work has been fully supported by the Croatian Science Foundation under the project 9752.

Strongly regular configurations

M. Abreu, M. Funk, V. Krčadinac, D. Labbate, *Strongly regular configurations*, in preparation

Strongly regular configurations

M. Abreu, M. Funk, V. Krčadinac, D. Labbate, *Strongly regular configurations*, in preparation . . . for a very long time.

Strongly regular configurations

M. Abreu, M. Funk, V. Krčadinac, D. Labbate, *Strongly regular configurations*, in preparation . . . for a very long time.

A configuration with parameters (v_r, b_k) is a finite incidence structure such that:

- there are v points and b lines,
- there are k points on every line and r lines through every point,
- there is at most one line through every pair of points.

Strongly regular configurations

M. Abreu, M. Funk, V. Krčadinac, D. Labbate, *Strongly regular configurations*, in preparation . . . for a very long time.

A configuration with parameters (v_r, b_k) is a finite incidence structure such that:

- there are v points and b lines,
- there are k points on every line and r lines through every point,
- there is at most one line through every pair of points.

B. Grünbaum, *Configurations of points and lines*, American Mathematical Society, Providence, RI, 2009 $\rightsquigarrow$ combinatorial configuration.

Strongly regular configurations

M. Abreu, M. Funk, V. Krčadinac, D. Labbate, *Strongly regular configurations*, in preparation . . . for a very long time.

A configuration with parameters (v_r, b_k) is a finite incidence structure such that:

- there are v points and b lines,
- there are k points on every line and r lines through every point,
- there is at most one line through every pair of points.

B. Grünbaum, *Configurations of points and lines*, American Mathematical Society, Providence, RI, 2009 $\rightsquigarrow$ combinatorial configuration.

Incidence geometers from **Gent** $\rightsquigarrow$ finite partial linear space of order (s, t), $s = k - 1$, $t = r - 1$.

Strongly regular configurations

The point graph of a configuration has the v points as vertices, with two vertices being adjacent if the points are collinear. The line graph is defined dually.

Strongly regular configurations

The point graph of a configuration has the v points as vertices, with two vertices being adjacent if the points are collinear. The line graph is defined dually.

The point and line graphs are regular of degree $r(k-1)$ and $k(r-1)$.

Strongly regular configurations

The point graph of a configuration has the v points as vertices, with two vertices being adjacent if the points are collinear. The line graph is defined dually.

The point and line graphs are regular of degree $r(k-1)$ and $k(r-1)$.

A graph is called strongly regular with parameters $SRG(n, d, \lambda, \mu)$ if it has n vertices, is regular of degree d, and every two vertices have λ common neighbors if they are adjacent and μ common neighbors if they are not adjacent.

Strongly regular configurations

The point graph of a configuration has the v points as vertices, with two vertices being adjacent if the points are collinear. The line graph is defined dually.

The point and line graphs are regular of degree $r(k-1)$ and $k(r-1)$.

A graph is called strongly regular with parameters $SRG(n, d, \lambda, \mu)$ if it has n vertices, is regular of degree d, and every two vertices have λ common neighbors if they are adjacent and μ common neighbors if they are not adjacent.

We are interested in configurations with both the point graph and the line graph strongly regular.

Partial geometries

R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.

A partial geometry $pg(s, t, \alpha)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are exactly α points on ℓ collinear with P.

Partial geometries

R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.

A partial geometry $pg(s,t,\alpha)$ is a configuration with $k = s+1$ and $r = t+1$ such that for every non-incident point-line pair (P,ℓ), there are exactly α points on ℓ collinear with P.

The point graph is a

$$SRG\left(\frac{(s+1)(st+\alpha)}{\alpha},\, s(t+1),\, s-1+t(\alpha-1),\, \alpha(t+1)\right),$$

and the line graph is a

$$SRG\left(\frac{(t+1)(st+\alpha)}{\alpha},\, t(s+1),\, t-1+s(\alpha-1),\, \alpha(s+1)\right).$$

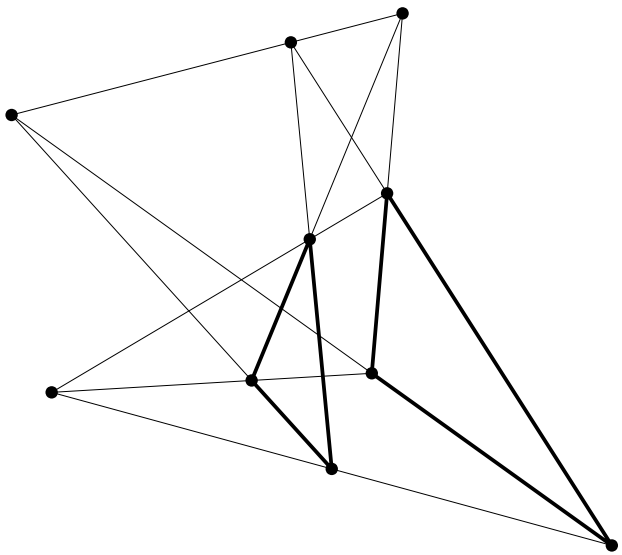
Other examples of such configurations

There are configurations with both associated graphs strongly regular that are **not** partial geometries!

Other examples of such configurations

There are configurations with both associated graphs strongly regular that are **not** partial geometries!

Desargues configuration (10_3):

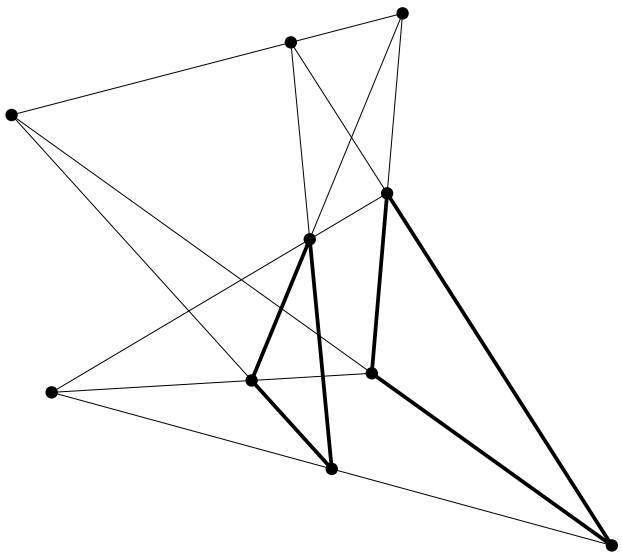


$\rightsquigarrow$ $SRG(10, 6, 3, 4)$
(complement of the Petersen graph)

Other examples of such configurations

There are configurations with both associated graphs strongly regular that are **not** partial geometries!

Desargues configuration (10_3):



$\rightsquigarrow$ $SRG(10, 6, 3, 4)$
(complement of the Petersen graph)

The Desargues configuration is a **semipartial geometry** $spg(2, 2, 2, 4)$.

Other examples of such configurations

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

A semipartial geometry $spg(s, t, \alpha, \mu)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are either 0 or α points on ℓ collinear with P. Furthermore, for every pair of non-collinear points, there are exactly μ points collinear with both.

Other examples of such configurations

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

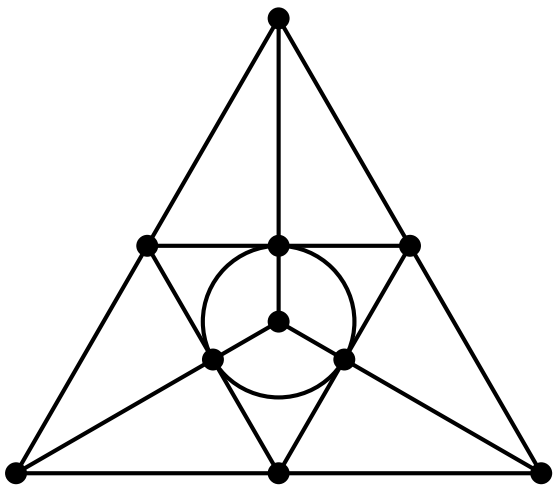
A semipartial geometry $spg(s, t, \alpha, \mu)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are either 0 or α points on ℓ collinear with P. Furthermore, for every pair of non-collinear points, there are exactly μ points collinear with both.

The point graph is a

$$SRG\left(1 + \frac{s(t+1)(\mu + t(s+1-\alpha)}{\mu},\, s(t+1),\, s-1+t(\alpha-1),\, \mu\right).$$

Other examples of such configurations

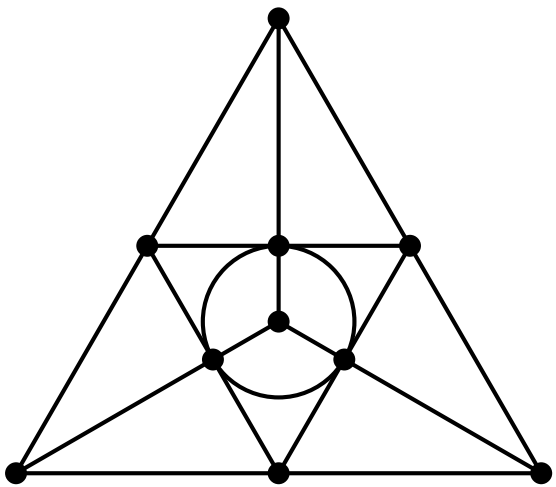
Another configuration (10_3):



$\rightsquigarrow$ $SRG(10, 6, 3, 4)$
(complement of the Petersen graph)

Other examples of such configurations

Another configuration (10_3):



$\rightsquigarrow$ $SRG(10, 6, 3, 4)$
(complement of the Petersen graph)

This configuration is **not** a semipartial geometry and does not belong to other known generalizations of partial geometries such as strongly regular (α, β)-geometries.

N. Hamilton, R. Mathon, *Strongly regular* (α, β)*-geometries*, J. Combin. Theory Ser. A **95** (2001), no. 2, 234–250.

Non-symmetric examples?

Are there non-symmetric examples of such configurations (with $v \neq b$), apart from the partial geometries $pg(s, t, \alpha)$ with $s \neq t$?

Non-symmetric examples?

Are there non-symmetric examples of such configurations (with $v \neq b$), apart from the partial geometries $pg(s, t, \alpha)$ with $s \neq t$?

A. E. Brouwer, W. H. Haemers, V. D. Tonchev, *Embedding partial geometries in Steiner designs*, in: *Geometry, combinatorial designs and related structures (Spetses, 1996)*, London Math. Soc. Lecture Note Ser., **245**, Cambridge Univ. Press, Cambridge, 1997, pp. 33–41.

Theorem.

If the point graph of a (v_r, b_k) configuration is strongly regular, then the configuration is a partial geometry or $v \leq b$.

Non-symmetric examples?

Are there non-symmetric examples of such configurations (with $v \neq b$), apart from the partial geometries $pg(s, t, \alpha)$ with $s \neq t$?

A. E. Brouwer, W. H. Haemers, V. D. Tonchev, *Embedding partial geometries in Steiner designs*, in: *Geometry, combinatorial designs and related structures (Spetses, 1996)*, London Math. Soc. Lecture Note Ser., **245**, Cambridge Univ. Press, Cambridge, 1997, pp. 33–41.

Theorem.

If the point graph of a (v_r, b_k) configuration is strongly regular, then the configuration is a partial geometry or $v \leq b$.

Corollary.

If both associated graphs of a (v_r, b_k) configuration are strongly regular, then the configuration is a partial geometry or $v = b$.

Definitions

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

Definitions

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

What about the line graph?

Definitions

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

What about the line graph?

Theorem.

In a $(v_k; \lambda, \mu)$ configuration, the line graph is also a $SRG(v, k(k-1), \lambda, \mu)$. If the incidence matrix is singular, the configuration is a partial geometry.

Definitions

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

What about the line graph?

Theorem.

In a $(v_k; \lambda, \mu)$ configuration, the line graph is also a $SRG(v, k(k-1), \lambda, \mu)$. If the incidence matrix is singular, the configuration is a partial geometry.

We shall call strongly regular configurations with regular incidence matrices proper.

Definitions

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

What about the line graph?

Theorem.

In a $(v_k; \lambda, \mu)$ configuration, the line graph is also a $SRG(v, k(k-1), \lambda, \mu)$. If the incidence matrix is singular, the configuration is a partial geometry.

We shall call strongly regular configurations with regular incidence matrices proper.

Proposition.

A strongly regular $(v_k; \lambda, \mu)$ configuration that is not a projective plane is proper if and only if $(v-k)(\lambda+1) > k(k-1)^3$ holds.

Definitions

Projective planes of order n are partial geometries $pg(n, n, n+1)$ and satisfy equality $(v-k)(\lambda+1) = k(k-1)^3$, but have regular incidence matrices. The associated point and line graphs are complete.

Definitions

Projective planes of order n are partial geometries $pg(n, n, n+1)$ and satisfy equality $(v-k)(\lambda+1) = k(k-1)^3$, but have regular incidence matrices. The associated point and line graphs are complete.

A $(v_k; \lambda, \mu)$ configuration is imprimitive if $\mu = 0$ or $\mu = k(k-1)$ holds.

Definitions

Projective planes of order n are partial geometries $pg(n, n, n+1)$ and satisfy equality $(v-k)(\lambda+1) = k(k-1)^3$, but have regular incidence matrices. The associated point and line graphs are complete.

A $(v_k; \lambda, \mu)$ configuration is imprimitive if $\mu = 0$ or $\mu = k(k-1)$ holds.

First case: $\mu = 0 \iff$ the graphs are disjoint unions of complete graphs
$\iff$ collinearity of points is an equivalence relation
$\iff$ the configuration is a disjoint union of projective planes.

Definitions

Projective planes of order n are partial geometries $pg(n, n, n+1)$ and satisfy equality $(v-k)(\lambda+1) = k(k-1)^3$, but have regular incidence matrices. The associated point and line graphs are complete.

A $(v_k; \lambda, \mu)$ configuration is imprimitive if $\mu = 0$ or $\mu = k(k-1)$ holds.

First case: $\mu = 0 \iff$ the graphs are disjoint unions of complete graphs
$\iff$ collinearity of points is an equivalence relation
$\iff$ the configuration is a disjoint union of projective planes.

Second case: $\mu = k(k-1) \iff$ the graphs are complete multipartite
$\iff$ non-collinearity of points is an equivalence relation
$\iff$ the configuration is an elliptic semiplane.

Definitions

Projective planes of order n are partial geometries $pg(n, n, n+1)$ and satisfy equality $(v-k)(\lambda+1) = k(k-1)^3$, but have regular incidence matrices. The associated point and line graphs are complete.

A $(v_k; \lambda, \mu)$ configuration is imprimitive if $\mu = 0$ or $\mu = k(k-1)$ holds.

First case: $\mu = 0 \iff$ the graphs are disjoint unions of complete graphs
$\iff$ collinearity of points is an equivalence relation
$\iff$ the configuration is a disjoint union of projective planes.

Second case: $\mu = k(k-1) \iff$ the graphs are complete multipartite
$\iff$ non-collinearity of points is an equivalence relation
$\iff$ the configuration is an elliptic semiplane.

P. Dembowski, *Finite geometries*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 44, Springer-Verlag, 1968.

A family of semipartial geometries

We focus on strongly regular configurations that are **proper** and **primitive**, i.e. such that neither collinearity nor non-collinearity of points are equivalence relations. This is equivalent with $0 < \mu < k(k-1)$.

A family of semipartial geometries

We focus on strongly regular configurations that are **proper** and **primitive**, i.e. such that neither collinearity nor non-collinearity of points are equivalence relations. This is equivalent with $0 < \mu < k(k-1)$.

Our introductory examples with parameters $(10_3; 3, 4)$ are part of a family associated with Moore graphs of diameter two, i.e. strongly regular graphs with $\lambda = 0$ and $\mu = 1$.

A family of semipartial geometries

We focus on strongly regular configurations that are **proper** and **primitive**, i.e. such that neither collinearity nor non-collinearity of points are equivalence relations. This is equivalent with $0 < \mu < k(k-1)$.

Our introductory examples with parameters $(10_3; 3, 4)$ are part of a family associated with Moore graphs of diameter two, i.e. strongly regular graphs with $\lambda = 0$ and $\mu = 1$.

A. J. Hoffman, R. R. Singleton, *On Moore graphs with diameters* 2 *and* 3, IBM J. Res. Develop. **4** (1960), 497–504.

Moore graphs have parameters $SRG(k^2+1, k, 0, 1)$ with $k \in \{2, 3, 7, 57\}$.

$k = 2 \rightsquigarrow$ the pentagon
$k = 3 \rightsquigarrow$ the Petersen graph
$k = 7 \rightsquigarrow$ the Hoffman-Singleton graph
$k = 57 \rightsquigarrow$ **?**

A family of semipartial geometries

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

Family (f):

- points are vertices of a Moore graph $SRG(k^2+1, k, 0, 1)$,
- lines are neighborhoods of single vertices.

A family of semipartial geometries

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

Family (f):

- points are vertices of a Moore graph $SRG(k^2+1, k, 0, 1)$,
- lines are neighborhoods of single vertices.

$\rightsquigarrow$ semipartial geometry $spg(k-1, k-1, k-1, (k-1)^2)$

strongly regular $((k^2+1)_k; k(k-2), (k-1)^2)$ configuration

The point graph is the complementary $SRG(k^2+1, k(k-1), k(k-2), (k-1)^2)$.

A family of semipartial geometries

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

Family (f):

- points are vertices of a Moore graph $SRG(k^2+1, k, 0, 1)$,
- lines are neighborhoods of single vertices.

$\rightsquigarrow$ semipartial geometry $spg(k-1, k-1, k-1, (k-1)^2)$

strongly regular $((k^2+1)_k; k(k-2), (k-1)^2)$ configuration

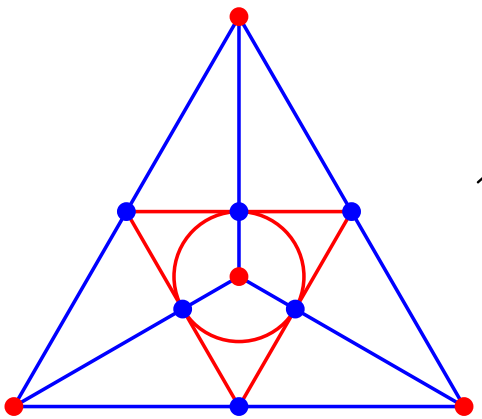
The point graph is the complementary $SRG(k^2+1, k(k-1), k(k-2), (k-1)^2)$.

$k = 3 \rightsquigarrow$ Desargues configuration

semipartial geometry $spg(2, 2, 2, 4)$

strongly regular $(10_3; 3, 4)$ configuration

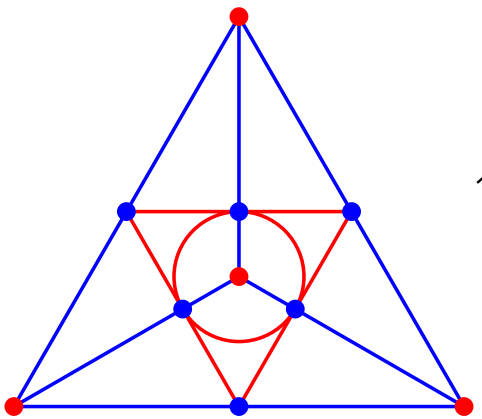
A family of semipartial geometries



$\rightsquigarrow$ strongly regular $(10_3; 3, 4)$ configuration

not a semipartial geometry

A family of semipartial geometries



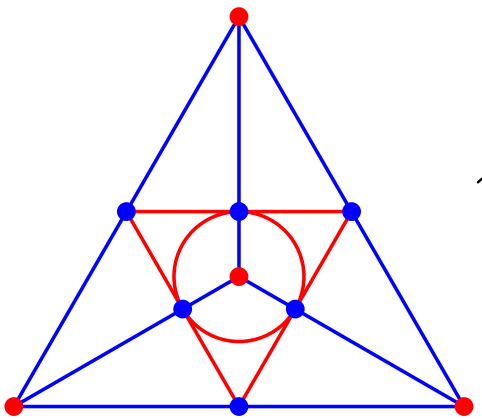
$\rightsquigarrow$ strongly regular $(10_3; 3, 4)$ configuration

not a semipartial geometry

$k = 7 \rightsquigarrow$ semipartial geometry $spg(6, 6, 6, 36)$

strongly regular $(50_7; 35, 36)$ configuration

A family of semipartial geometries



$\rightsquigarrow$ strongly regular $(10_3; 3, 4)$ configuration

not a semipartial geometry

$k = 7 \rightsquigarrow$ semipartial geometry $spg(6, 6, 6, 36)$

strongly regular $(50_7; 35, 36)$ configuration

Proposition.

There are at least 211 non-isomorphic $(50_7; 35, 36)$ configurations. Only one of them is a semipartial geometry.

Another family of semipartial geometries

Family (g) of Debroey and Thas:

- POINTS are lines of the projective space $PG(n, q)$, $n \geq 3$,
- LINES are 2-planes of $PG(n, q)$ and incidence is inclusion.

Another family of semipartial geometries

Family (g) of Debroey and Thas:

- POINTS are lines of the projective space $PG(n, q)$, $n \geq 3$,
- LINES are 2-planes of $PG(n, q)$ and incidence is inclusion.

Parameters:

$$v = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q, \quad b = \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q, \quad r = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q.$$

Another family of semipartial geometries

Family (g) of Debroey and Thas:

- POINTS are lines of the projective space $PG(n, q)$, $n \geq 3$,
- LINES are 2-planes of $PG(n, q)$ and incidence is inclusion.

Parameters:

$$v = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q, \quad b = \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q, \quad r = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q.$$

Lemma.

Two lines of $PG(n, q)$ are coplanar if and only if they intersect.

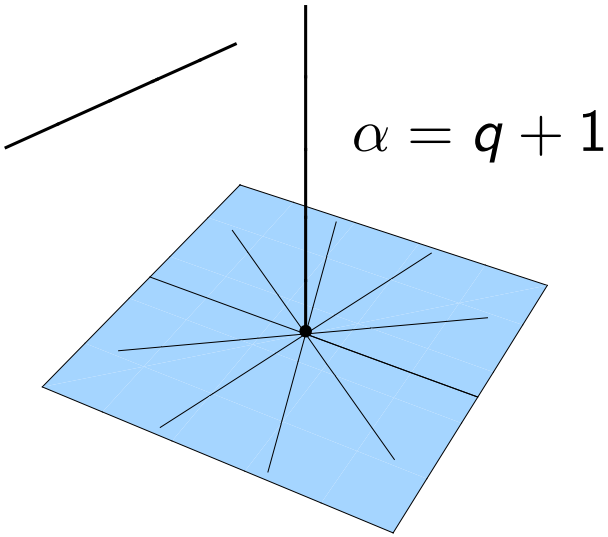
Another family of semipartial geometries

$\rightsquigarrow$ semipartial geometry $spg(k-1, r-1, q+1, (q+1)^2)$

Another family of semipartial geometries

$\rightsquigarrow$ semipartial geometry $spg(k-1, r-1, q+1, (q+1)^2)$

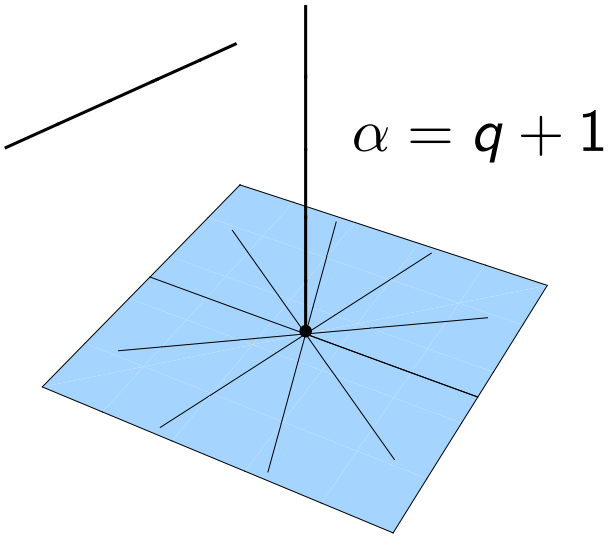
α-condition:



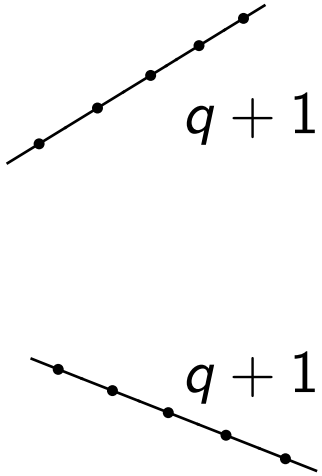
Another family of semipartial geometries

$\rightsquigarrow$ semipartial geometry $spg(k-1, r-1, q+1, (q+1)^2)$

α-condition:



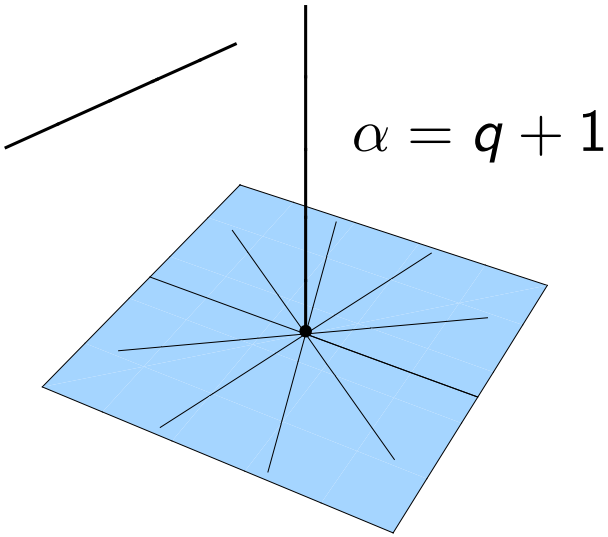
μ-condition:



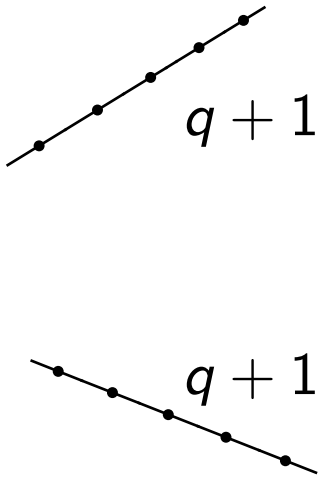
Another family of semipartial geometries

$\rightsquigarrow$ semipartial geometry $spg(k-1, r-1, q+1, (q+1)^2)$

α-condition:



μ-condition:



This semipartial geometry is denoted by $LP(n, q)$ in:

F. De Clerck, H. Van Maldeghem, *Some classes of rank 2 geometries*, Handbook of incidence geometry, 433–475, North-Holland, 1995.

F. De Clerck, *Partial and semipartial geometries: an update*, Discrete Math. **267** (2003), no. 1–3, 75–86.

Another family of semipartial geometries

$LP(n, q)$ is a partial geometry $\iff$ $n = 3$

Another family of semipartial geometries

$LP(n, q)$ is a partial geometry $\iff$ $n = 3$

$LP(n, q)$ is symmetric $\iff$ $n = 4$

Another family of semipartial geometries

$LP(n,q)$ is a partial geometry $\iff$ $n=3$

$LP(n,q)$ is symmetric $\iff$ $n=4$

$LP(4,q) \rightsquigarrow$ semipartial geometry $spg(q(q+1), q(q+1), q+1, (q+1)^2)$

strongly regular $(v_k; \lambda, \mu)$ configuration for

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q, \quad \lambda = q^3 + 2q^2 + q - 1, \quad \mu = (q+1)^2.$$

Another family of semipartial geometries

$LP(n,q)$ is a partial geometry $\iff$ $n=3$

$LP(n,q)$ is symmetric $\iff$ $n=4$

$LP(4,q) \rightsquigarrow$ semipartial geometry $spg(q(q+1), q(q+1), q+1, (q+1)^2)$

strongly regular $(v_k; \lambda, \mu)$ configuration for

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q, \quad \lambda = q^3 + 2q^2 + q - 1, \quad \mu = (q+1)^2.$$

Are there strongly regular configurations with the same parameters that are not semipartial geometries?

Another family of semipartial geometries

$LP(n,q)$ is a partial geometry $\iff$ $n=3$

$LP(n,q)$ is symmetric $\iff$ $n=4$

$LP(4,q) \rightsquigarrow$ semipartial geometry $spg(q(q+1), q(q+1), q+1, (q+1)^2)$

strongly regular $(v_k; \lambda, \mu)$ configuration for

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q, \quad \lambda = q^3 + 2q^2 + q - 1, \quad \mu = (q+1)^2.$$

Are there strongly regular configurations with the same parameters that are not semipartial geometries?

D. Jungnickel, V. D. Tonchev, *Polarities, quasi-symmetric designs, and Hamada's conjecture*, Des. Codes Cryptogr. **51** (2009), no. 2, 131–140.

Polarity transformations

Let H_0 be a hyperplane of $PG(4, q)$. As a subgeometry, H_0 is isomorphic to $PG(3, q)$ and possesses a polarity π, i.e. an inclusion-reversing involution. The polarity maps the set of projective lines contained in H_0 onto itself.

Polarity transformations

Let H_0 be a hyperplane of $PG(4, q)$. As a subgeometry, H_0 is isomorphic to $PG(3, q)$ and possesses a polarity π, i.e. an inclusion-reversing involution. The polarity maps the set of projective lines contained in H_0 onto itself.

We modify incidence of the POINTS and LINES of $LP(n, q)$ contained in H_0: a POINT L (projective line contained in H_0) is incident with a LINE p (projective plane contained in H_0) if $\pi(L) \subseteq p$. For the remaining pairs (L, p), with L or p not contained in H_0, incidence remains unaltered.

Polarity transformations

Let H_0 be a hyperplane of $PG(4, q)$. As a subgeometry, H_0 is isomorphic to $PG(3, q)$ and possesses a polarity π, i.e. an inclusion-reversing involution. The polarity maps the set of projective lines contained in H_0 onto itself.

We modify incidence of the POINTS and LINES of $LP(n, q)$ contained in H_0: a POINT L (projective line contained in H_0) is incident with a LINE p (projective plane contained in H_0) if $\pi(L) \subseteq p$. For the remaining pairs (L, p), with L or p not contained in H_0, incidence remains unaltered.

Theorem.

The new incidence structure $LP(4, q)^\pi$ is a strongly regular configuration with the same parameters that is not a semipartial geometry.

Polarity transformations

Proof.

The POINT and LINE degrees remain the same and there is at most one LINE through every pair of POINTS.

Polarity transformations

Proof.

The POINT and LINE degrees remain the same and there is at most one LINE through every pair of POINTS.

The POINT graphs of $LP(4,q)^\pi$ and $LP(4,q)$ are identical. This follows from the Lemma: if L_1 and L_2 are in H_0, $\pi(L_1)$, $\pi(L_1)$ are contained in a plane p if and only if L_1, L_2 intersect in the point $\pi(p)$ and hence are contained in some plane p'.

Polarity transformations

Proof.

The POINT and LINE degrees remain the same and there is at most one LINE through every pair of POINTS.

The POINT graphs of $LP(4,q)^\pi$ and $LP(4,q)$ are identical. This follows from the Lemma: if L_1 and L_2 are in H_0, $\pi(L_1)$, $\pi(L_1)$ are contained in a plane p if and only if L_1, L_2 intersect in the point $\pi(p)$ and hence are contained in some plane p'.

The line graph of $LP(4,q)^\pi$ is changed, but remains strongly regular.

Polarity transformations

Proof.

The POINT and LINE degrees remain the same and there is at most one LINE through every pair of POINTS.

The POINT graphs of $LP(4,q)^\pi$ and $LP(4,q)$ are identical. This follows from the Lemma: if L_1 and L_2 are in H_0, $\pi(L_1)$, $\pi(L_1)$ are contained in a plane p if and only if L_1, L_2 intersect in the point $\pi(p)$ and hence are contained in some plane p'.

The line graph of $LP(4,q)^\pi$ is changed, but remains strongly regular.

The new configuration $LP(4,q)^\pi$ is not a semipartial geometry: take a plane p in H_0 and a projective line L not in H_0 intersecting the hyperplane in the point $\pi(p)$. Then, (L,p) is a non-incident POINT-LINE pair of $LP(4,q)^\pi$. If $\pi(M) \subseteq p$, then M contains $\pi(p)$ and is coplanar with L, i.e. collinear as a POINT of the configuration. Hence, all q^2+q+1 POINTS on p are collinear with L, whereas in a semipartial geometry the number is always 0 or $\alpha = q+1$.

Polarity transformations

The composition of two polarities is an isomorphism $\Rightarrow$

Polarity transformations

The composition of two polarities is an isomorphism $\Rightarrow$ configurations obtained by transforming $LP(4, q)$ with different polarities are isomorphic.

Polarity transformations

The composition of two polarities is an isomorphism $\Rightarrow$ configurations obtained by transforming $LP(4, q)$ with different polarities are isomorphic.

We define a **dual transformation** of $LP(4, q)$: take a point P_0 of $PG(4, q)$ and consider the quotient geometry of lines, planes and solids containing P_0. It is isomorphic to $PG(3, q)$ and possesses a polarity π' permuting the planes through P_0 and exchanging the lines and solids through P_0.

Polarity transformations

The composition of two polarities is an isomorphism $\Rightarrow$ configurations obtained by transforming $LP(4, q)$ with different polarities are isomorphic.

We define a **dual transformation** of $LP(4, q)$: take a point P_0 of $PG(4, q)$ and consider the quotient geometry of lines, planes and solids containing P_0. It is isomorphic to $PG(3, q)$ and possesses a polarity π' permuting the planes through P_0 and exchanging the lines and solids through P_0.

We modify incidence in $LP(4, q)$ for projective lines L and planes p through P_0: they are incident if $L \subseteq \pi'(p)$. Other incidences remain unaltered.

Polarity transformations

The composition of two polarities is an isomorphism $\Rightarrow$ configurations obtained by transforming $LP(4,q)$ with different polarities are isomorphic.

We define a **dual transformation** of $LP(4,q)$: take a point P_0 of $PG(4,q)$ and consider the quotient geometry of lines, planes and solids containing P_0. It is isomorphic to $PG(3,q)$ and possesses a polarity π' permuting the planes through P_0 and exchanging the lines and solids through P_0.

We modify incidence in $LP(4,q)$ for projective lines L and planes p through P_0: they are incident if $L \subseteq \pi'(p)$. Other incidences remain unaltered.

Theorem.

The new incidence structure $LP(4,q)_{\pi'}$ is the dual of $LP(4,q)^{\pi}$.

The LINE graphs of $LP(4,q)$ and $LP(4,q)_{\pi'}$ are identical. The POINT graph of $LP(4,q)_{\pi'}$ is changed.

Polarity transformations

A **fourth** strongly regular configuration $LP(4,q)^{\pi}_{\pi'}$ is obtained if we take a non-incident point P_0 and hyperplane H_0 and apply both transformations. This configuration has the same LINE graph as $LP(4,q)^{\pi}$ and the same POINT graph as $LP(4,q)_{\pi'}$ and is self-dual.

Polarity transformations

A **fourth** strongly regular configuration $LP(4,q)^{\pi}_{\pi'}$ is obtained if we take a non-incident point P_0 and hyperplane H_0 and apply both transformations. This configuration has the same LINE graph as $LP(4,q)^{\pi}$ and the same POINT graph as $LP(4,q)_{\pi'}$ and is self-dual.

Theorem.

For every prime power q, there are at least four strongly regular $(v_k;\lambda,\mu)$ configuration with parameters

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q, \quad \lambda = q^3 + 2q^2 + q - 1, \quad \mu = (q+1)^2.$$

One of them is the semipartial geometry $LP(4,q)$ and the others are not semipartial geometries.

Examples with parameters different from spg

Are there strongly regular configurations with parameters different from semipartial geometries?

Examples with parameters different from spg

Are there strongly regular configurations with parameters different from semipartial geometries?

Theorem.

Let $\mathcal{P}$ be a projective plane of order $n \geq 5$ and A, B, C be three non-collinear points. By deleting all points on the lines AB, AC, BC and all lines through the points A, B, C, there remains a strongly regular $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, and $\mu = (n-3)(n-4)$. This configuration is not an (α, β)-geometry.

Examples with parameters different from spg

Are there strongly regular configurations with parameters different from semipartial geometries?

Theorem.

Let $\mathcal{P}$ be a projective plane of order $n \geq 5$ and A, B, C be three non-collinear points. By deleting all points on the lines AB, AC, BC and all lines through the points A, B, C, there remains a strongly regular $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, and $\mu = (n-3)(n-4)$. This configuration is not an (α, β)-geometry.

Sporadic examples:

Example.

There is a cyclic $(13_3; 2, 3)$ configuration. It can be obtained from $PG(2, 3)$ by deleting a point from every line.

Examples with parameters different from spg

Example.

There are at least four $(63_6; 13, 15)$ configurations. Two of them are related to the smallest generalized hexagon $GH(2, 2)$.

Examples with parameters different from spg

Example.

There are at least four $(63_6; 13, 15)$ configurations. Two of them are related to the smallest generalized hexagon $GH(2, 2)$.

Example.

There is a flag-transitive $(96_5; 4, 4)$ configuration.

Examples with parameters different from spg

Example.

There are at least four $(63_6; 13, 15)$ configurations. Two of them are related to the smallest generalized hexagon $GH(2, 2)$.

Example.

There is a flag-transitive $(96_5; 4, 4)$ configuration.

Example.

There is a flag-transitive $(120_8; 28, 24)$ configuration. Its 120 lines and the 135 lines of a $pg(7, 8, 4)$ with complementary point graph form a Steiner 2-$(120, 8, 1)$ design.

A. E. Brouwer, W. H. Haemers, V. D. Tonchev, *Embedding partial geometries in Steiner designs*, in: *Geometry, combinatorial designs and related structures (Spetses, 1996)*, London Math. Soc. Lecture Note Ser., **245**, Cambridge Univ. Press, Cambridge, 1997, pp. 33–41.

A table of admissible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
1	$(10_3; 3, 4)$	**2**	**2**
2	$(13_3; 2, 3)$	**1**	**1**
3	$(16_3; 2, 2)$	**1**	**1**
4	$(25_4; 5, 6)$	**0**	**0**
5	$(36_5; 10, 12)$	1	1
6	$(41_5; 9, 10)$	?	?
7	$(45_4; 3, 3)$	**0**	**0**
8	$(49_4; 5, 2)$	**0**	**0**
9	$(49_6; 17, 20)$	1	1
10	$(50_7; 35, 36)$	211	111
11	$(61_6; 14, 15)$	?	?

A table of admissible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
12	$(63_6; 13, 15)$	4	2
13	$(64_7; 26, 30)$	29	11
14	$(81_8; 37, 42)$	?	?
15	$(85_6; 11, 10)$	?	?
16	$(85_7; 20, 21)$	?	?
17	$(96_5; 4, 4)$	1	1
18	$(99_7; 21, 15)$	?	?
19	$(100_9; 50, 56)$	1	1
20	$(105_9; 51, 45)$	?	?
21	$(113_8; 27, 28)$	?	?
22	$(120_8; 28, 24)$	1	1

A table of admissible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
23	$(121_5; 9, 2)$	**0**	**0**
24	$(121_6; 11, 6)$	?	?
25	$(121_9; 43, 42)$	?	?
26	$(121_{10}; 65, 72)$	?	?
27	$(125_9; 45, 36)$	?	?
28	$(136_6; 15, 4)$	?	?
29	$(136_9; 36, 40)$	?	?
30	$(144_{11}; 82, 90)$	1	1
31	$(145_9; 35, 36)$	?	?
32	$(153_8; 19, 21)$	?	?
33	$(155_7; 17, 9)$	4	2

A table of admissible parameters

No.	$(v_k; \lambda, \mu)$	#Cf	#SCf
34	$(169_9; 31, 30)$	?	?
35	$(169_{12}; 101, 110)$	?	?
36	$(171_{11}; 73, 66)$	?	?
37	$(175_6; 5, 5)$	?	?
38	$(181_{10}; 44, 45)$	?	?
39	$(196_{10}; 40, 42)$	?	?
40	$(196_{13}; 122, 132)$	?	?
41	$(196_{13}; 125, 120)$	?	?

The table is based on A. E. Brouwer's tables of strongly regular graphs:

https://www.win.tue.nl/~aeb/graphs/srg/srgtab.html

The End

Thanks for your attention!