# Cubes of designs

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write group $G$ in a group ring $\mathbb{Z}[G]$ as $G = \sum_{s=1}^{v} g_s$ where $g_1 = 1$ (unit in a group $G$)

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If $X$ is a set, then $\delta_X$ is a characteristic function of a set $X$ defined by

$$\delta_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}$$

**Theorem:** Let $G = \sum_{s=1}^{v} g_s,\ g_1 = 1$ be a group of order $v$ with a $(v, k, \lambda)$ difference set $D$. Let $A = [a_{ijm}] \in \mathcal{M}_{v\times v\times v}$ be a 3-dimensional matrix defined by $a_{ijm} = \delta_{g_j g_i D}(g_m)$ for all $i, j, m \in [v]$. Then the following holds:

1. $A$ is a cube of a $(v, k, \lambda)$ symmetric design i.e. $A \in \mathcal{C}(v, k, \lambda)$,
2. $A^i_x$ is an incidence matrix of a symmetric design $(G, \mathcal{D}ev(g_i D))$,
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4. $A^m_y$ is an incidence matrix of a symmetric design $(G, \sum_{t=1}^{v} g_t^{-1} g_m D^{(-1)})$,
5. $[A^m_z]_t = [A^t_x]_m$ for all $m, t \in [v]$,
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**Proposition:** Let $A \in \mathcal{C}(v,k,\lambda)$ be a cube constructed via difference set $D \subseteq G$. Let $\psi$ be a permutation of $G$. Let $A^{\psi}$ be a $3$-dimensional matrix such that $(A^{\psi})_{ijm} = \delta_{g_j^{\psi} g_i D}(g_m)$. Then $A$ is a cube, i.e. $A^{\psi} \in \mathcal{C}(v,k,\lambda)$ and $[(A^{\psi})_y^m]_t = (g_t^{-1})^{\psi} g_m D^{(-1)}$ and $[(A^{\psi})_z^m]_t = g_m^{\psi} g_t D$.

**Proposition:** Let $A \in \mathcal{C}(v,k,\lambda)$ be a cube constructed via difference set $D \subseteq G$. Let $\psi$ be a permutation of $G$. Let $A^\psi$ be a 3-dimensional matrix such that $(A^\psi)_{ijm} = \delta_{g_j^\psi g_i D}(g_m)$. Then $A$ is a cube, i.e. $A^\psi \in \mathcal{C}(v,k,\lambda)$ and $[(A^\psi)_y^m]_t = (g_t^{-1})^\psi g_m D^{(-1)}$ and $[(A^\psi)_z^m]_t = g_m^\psi g_t D$.

**Proposition:** Let $A \in \mathcal{C}(v,k,\lambda)$ be a cube constructed via difference set $D \subseteq G$. Let $\psi_m$ be a permutation of $G = \sum_{s=1}^{v} g_s$ given by $\psi_m(g) = g_m g$. Then $\psi_m$ is an isomorphism between $A_x^1$ and $A_x^m$. Furthermore, $\psi_m(g_t D) = g_t^{g_m^{-1}} g_m D$, or in terms of rows of incidence matrices, $\psi_m([A_x^1]_t) = [A_x^m]_{\widetilde{t}}$, where $\widetilde{t} = g_t^{g_m^{-1}}$.

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**Proposition:** Let $A \in \mathcal{C}(v,k,\lambda)$ and $\mathcal{P} = \sum_{s=1}^{v} p_s$ is a set of points of designs $A_x^i$. Then $[A_y^m]_t = \sum_{s=1}^{v} \delta_{B_{x,t}^s}(p_m)p_s$ and $[A_z^m]_t = [A_x^t]_m = \sum_{s=1}^{v} \delta_{B_{x,m}^t}(p_s)p_s$ for all $m,t \in [v]$, where $[A_y^m]_t$ and $[A_z^m]_t$ are $t$-th blocks of designs $A_y^m$ and $A_z^m$ respectively.

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**Theorem:** Let $A \in \mathcal{C}(v,k,\lambda)$. Then for every $i,m \in [v]$ designs $A_x^i$, $A_z^m$ and $A_y^m$ satisfy the following:

1. $A_x^i = (\mathcal{P}, \sum_{t=1}^{v} B_{x,t}^i) = (\mathcal{P}, \mathcal{B}_x^i)$,
2. $A_z^m = (\mathcal{P}, \sum_{t=1}^{v} B_{x,m}^t) = (\mathcal{P}, \mathcal{B}_z^m)$, meaning that the set of blocks of a design $A_z^m$ is a set of $m$-th blocks of designs $A_x^t$ for all $t \in [v]$,

3. $A_y^m = (\mathcal{P}_y^m, \sum_{t=1}^{v} \langle p_m \rangle_{A_z^t}) = (\mathcal{P}_y^m, \mathcal{B}_y^m)$, meaning that the $t$-th block of a design $A_y^m$ is $m$-th dual block of a design $A_z^t$.

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# Cyclic cubes generated by a symmetric design

**Definition:** Let $G = \sum_{i=1}^{t} g_1$ where $g_1 = 1$, be a group of order $v$. Let $(G, \mathcal{B})$ be a $(v, k, \lambda)$ symmetric design where $\mathcal{B} = \sum_{i=1}^{v} B_i$. Let $A_x^1$ ba an incidence matrix of a design $(G, \mathcal{B})$. Let $A_x^m$ be an incidence matrix of an incidence structure $(G, g_m\mathcal{B})$, where $g_m\mathcal{B} = \sum_{i=1}^{v} g_m B_i$. A cyclic cube (generated by a symmetric design $(G, \mathcal{B})$) is a 3-dimensional matrix $A = (a_{ijm})$ such that $a_{ijm} = (A_x^i)_{jm}$ for all $i, j, m \in [v]$.

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**Proposition:** Let $A$ be a cyclic cube generated by a $(v, k, \lambda)$ symmetric design $(G, \mathcal{B})$, where $G = \sum_{i=1}^{g_i}$ is a group. Then $|\langle T\rangle_{A_x^m}| = |\langle g_m^{-1}T\rangle_{A_x^1}|,\ m \in [v]$. A matrix $A_x^m$ is an incidence matrix of a $(v, k, \lambda)$ symmetric design for all $m \in [v]$.

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# Thank you! Any Q's?