On higher-dimensional combinatorial designs*

Vedran Krčadinac

PMF-MO

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The ACCO project

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Croatian Science Foundation

Algorithmic Constructions of Combinatorial Objects (ACCO)

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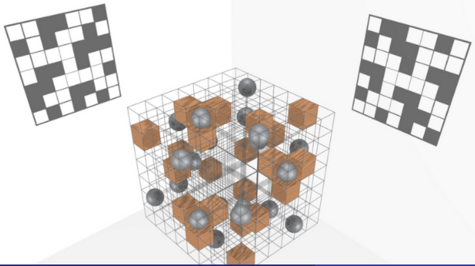
Algorithmic
Constructions
of
Combinatorial
Objects

The topic of this research project are constructions of combinatorial objects with additional algebraic structure, such as quasi-symmetric designs, schematic designs, q-analogs of designs, difference sets, (semi)partial geometries, and generalisations. Results in algebraic combinatorics impose restrictions on the parameters and properties of such objects that can be exploited to narrow-down the search space and develop specialised algorithms for their construction and classification.

Research objectives

- Development of algorithmic methods for the construction and classification of combinatorial objects with strong algebraic structure. These methods utilise known algebraic and combinatorial properties of the objects to handle larger parameters and problems that have been out of reach with traditional construction methods.
- Widening of theoretical knowledge about combinatorial objects that are the topic of research. Interesting theorems are often discovered and proved on the basis of available examples. It is expected that the results of the project will lead to such discoveries.
- Development of a software package, implemented in GAP, for the construction and analysis of combinatorial objects.



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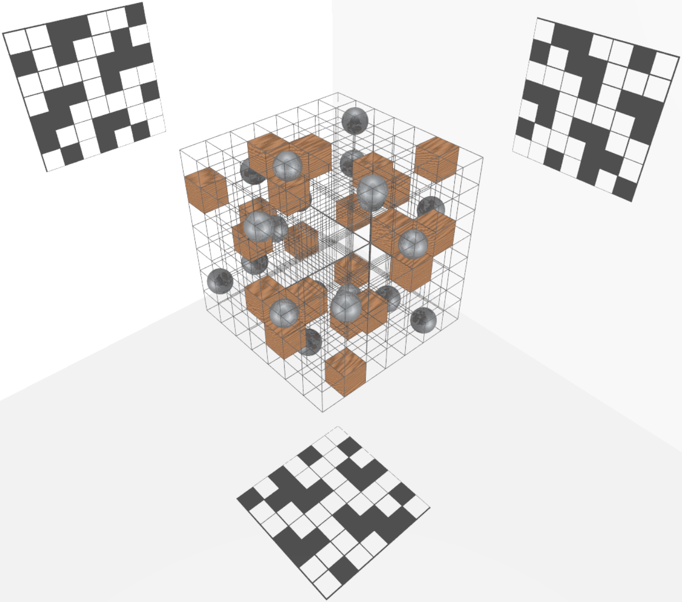

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- V. Krčadinac, M. O. Pavčević, K. Tabak, *Three-dimensional Hadamard matrices of Paley type*, Finite Fields Appl. **92** (2023), 102306. https://doi.org/10.1016/j.ffa.2023.102306
- V. Krčadinac, M. O. Pavčević, K. Tabak, *Cubes of symmetric designs*, Ars Math. Contemp. (2024). https://doi.org/10.26493/1855-3974.3222.e53
- V. Krčadinac, L. Relić, *Projection cubes of symmetric designs*, preprint, 2024. https://arxiv.org/abs/2411.06936
- V. Krčadinac, M. O. Pavčević, *On higher-dimensional symmetric designs*, in preparation, 2024.

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Examples: $\begin{pmatrix} 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

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Smallest unknown order: $v = 668 = 4 \cdot 167$

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An n-dimensional matrix of order v with $\{-1, 1\}$-entries

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- is Hadamard if all $(n-1)$-dimensional parallel slices are orthogonal:

$$\sum_{1\le i_1,\ldots,\widehat{i_j},\ldots,i_n\le v} H(i_1,\ldots,a,\ldots,i_n)H(i_1,\ldots,b,\ldots,i_n) = v^{n-1}\delta_{ab}$$

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Yi Xian Yang, *Proofs of some conjectures about higher-dimensional Hadamard matrices* (Chinese), Kexue Tongbao **31** (1986), no. 2, 85–88.

Warwick de Launey, *(O, G)-designs and applications*, PhD thesis, The University of Sidney, 1987.

Theorem ("Product construction").

Let $h : \{1, \ldots, v\}^2 \to \{-1, 1\}$ be an ordinary Hadamard matrix of order v. Then

$$H(i_1, \ldots, i_n) = \prod_{1 \le j < k \le n} h(i_j, i_k)$$

is an n-dimensional proper Hadamard matrix of order v.

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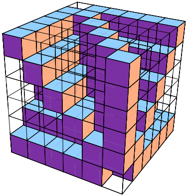

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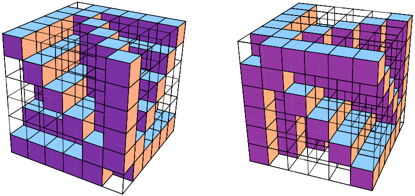

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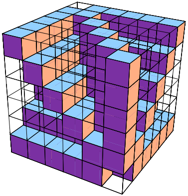

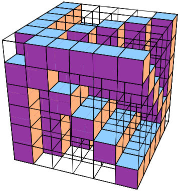

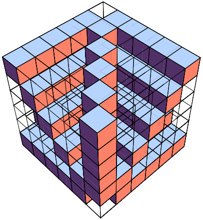

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Concluding questions: (from Y. X. Yang's book)

5. Prove or disprove the existence of three-dimensional Hadamard matrices of orders $4k + 2 \neq 2 \cdot 3^m$.

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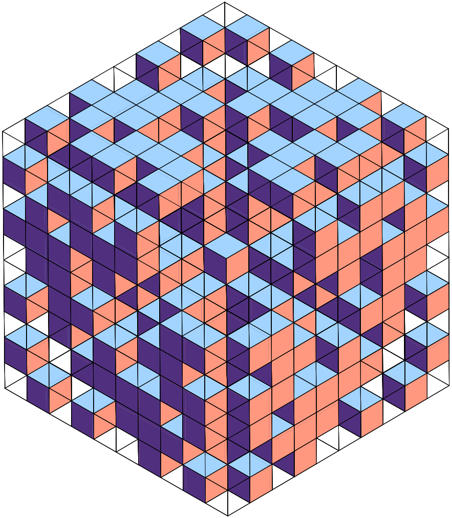

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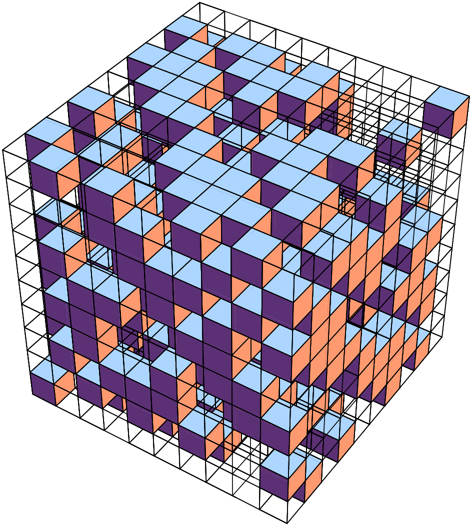

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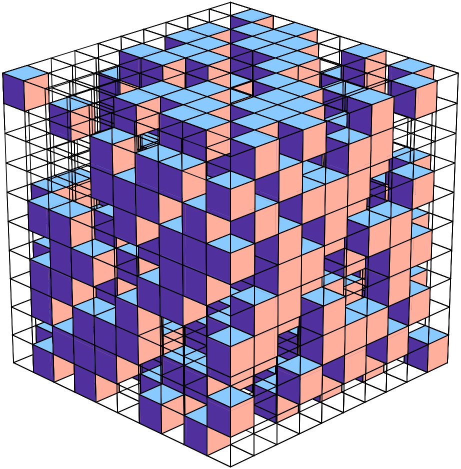

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$H : PG(1,q)^3 \to \{1,-1\}$, $q \equiv 1$ or $3 \pmod 4$,

$$H(x,y,z) = \begin{cases} -1, & \text{if } x = y = z, \\ 1, & \text{if } x = y \neq z \\ & \quad \text{or } x = z \neq y \\ & \quad \text{or } y = z \neq x, \\ \chi(z-y), & \text{if } x = \infty, \\ \chi(x-z), & \text{if } y = \infty, \\ \chi(y-x), & \text{if } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{otherwise.} \end{cases}$$

$PG(1,q) = \mathbb{F}_q \cup \{\infty\}$

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VI. Future Research and Applications

The present exposition suggests a number of unsolved problems and unproven conjectures. Some examples follow.

a) The algebraic approach to the derivation of two-dimensional Hadamard matrices [2]–[7] suggests that a similar procedure may be feasible for three- or higher dimensional matrices.

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4. Other generalizations of Hadamard matrices to higher dimensions?

Gnang-Filmus version of higher-dim. Hadamard matrices

Edinah K. Gnang, Yuval Filmus, *On the spectra of hypermatrix direct sum and Kronecker products constructions*, Linear Algebra Appl. **519** (2017), 238–277.

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1. Does the order v have to be divisible by 4?
2. Are there examples with v not of the form 2^m?
3. Apart from the Kronecker product construction, can other known constructions for $n = 2$ be generalized to odd dimensions?

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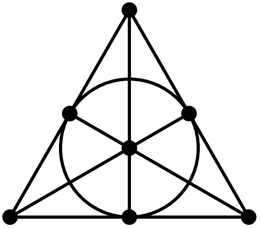
A symmetric (v, k, λ) design is a $v \times v$ matrix with $\{0, 1\}$-entries such that $A \cdot A^\tau = (k - \lambda) I + \lambda J$ holds.

Cubes of symmetric designs

Example: symmetric $(7, 3, 1)$ design (Fano plane)



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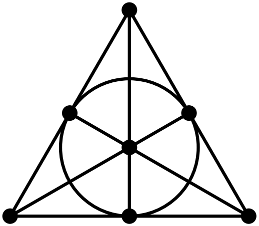

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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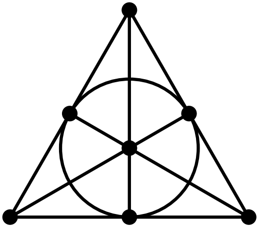

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An *n-dimensional cube of symmetric (v, k, λ) designs* is a function $A : \{1, \ldots, v\}^n \to \{0, 1\}$ such that all 2-dimensional slices are symmetric (v, k, λ) designs. The set of all such objects is denoted $\mathcal{C}^n(v, k, \lambda)$.

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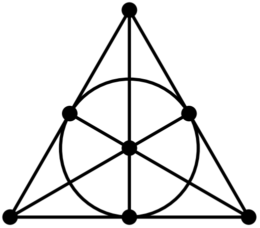

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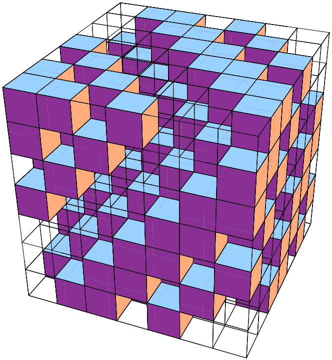
V. Krčadinac, M. O. Pavčević, K. Tabak, *Cubes of symmetric designs*, Ars Math. Contemp. (2024).

Cubes of symmetric designs

Example: 3-cube of $(7,3,1)$ designs ("Fano cube")



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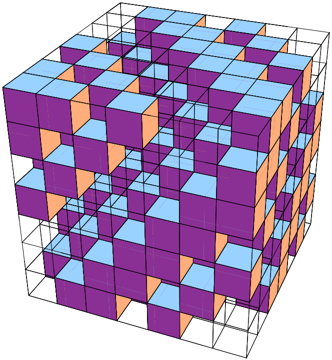

A (v, k, λ) difference set is a k-subset $D \subseteq G$ of an additively written group of order v such that $x - y$, $x, y \in D$ cover $G \setminus \{0\}$ exactly λ times.

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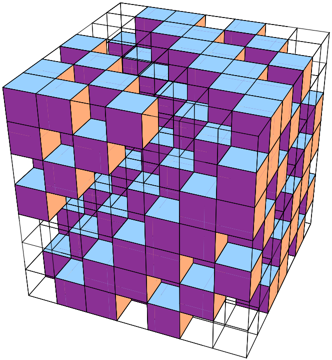

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Example: $D = \{0,1,3\}$ is a $(7,3,1)$ difference set in $G = \mathbb{Z}_7 = \{0,\ldots,6\}$

Cubes of symmetric designs

Theorem ("Difference cubes").

If D is a (v, k, λ) difference set in $G = \{g_1, \ldots, g_v\}$, then

$$A(i_1, \ldots, i_n) = [\, g_{i_1} + \ldots + g_{i_n} \in D\,]$$

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J. Hammer, J. Seberry, *Higher-dimensional orthogonal designs and Hadamard matrices. II*, Proceedings of the Ninth Manitoba Conference on Numerical Mathematics and Computing, Utilitas Math., 1980, pp. 23–29.

Warwick de Launey, *On the construction of n-dimensional designs from 2-dimensional designs*, Australas. J. Combin. **1** (1990), 67–81.

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If $\{D_1, \ldots, D_v\}$ is a family of (v, k, λ) difference sets in $G = \{g_1, \ldots, g_v\}$ that are blocks of a symmetric (v, k, λ) design, then

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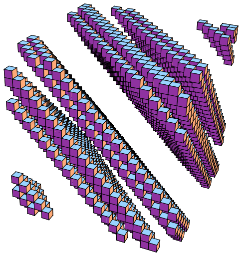
A 3-cube of $(21, 5, 1)$ designs

(projective planes of order 4)



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$$G = \langle a, b \mid a^3 = b^7 = 1,\ ba = ab^2 \rangle$$

$$D_1 = \{1, a, b, b^3, a^2 b^2\}$$
$$D_2 = \{a^2 b^6, b^6, a^2 b^3, a^2 b^4, a\}$$
$$D_3 = \{1, a^2, ab, b^2, b^6\}$$
$$\vdots$$
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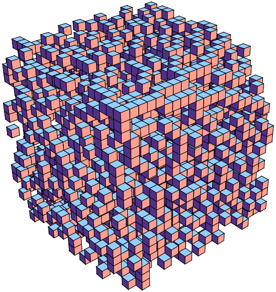

Prescribed Automorphism Groups

PAG

Prescribed Automorphism Groups

Version 0.2.3

Released 2024-05-21

Download .tar.gz

View On GitHub

This project is maintained by Vedran Krcadinac

GAP Package PAG

The PAG package contains functions for constructing combinatorial objects with prescribed automorphism groups.

The current version of this package is version 0.2.3, released on 2024-05-21. For more information, please refer to the package manual. There is also a README file.

Dependencies

This package requires GAP version 4.11

https://vkrcadinac.github.io/PAG/

Cubes of symmetric designs

Theorem.

For every $m \geq 2$ and $n \geq 3$, there are n-cubes of symmetric

$$(4^m,\ 2^{m-1}(2^m-1),\ 2^{m-1}(2^{m-1}-1))$$

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Red design, Green design, Blue design

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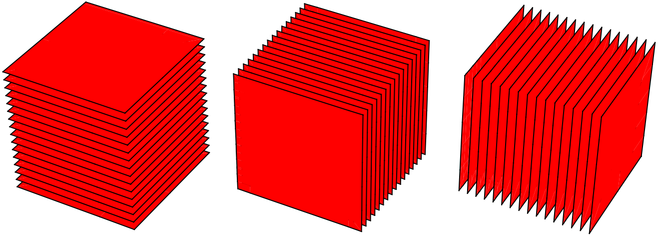

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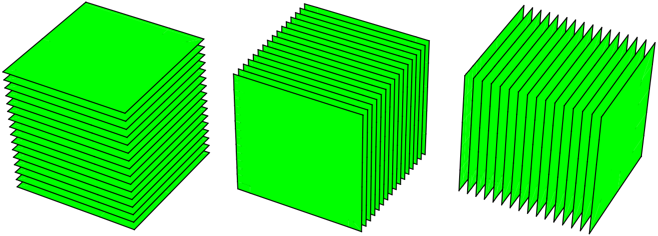

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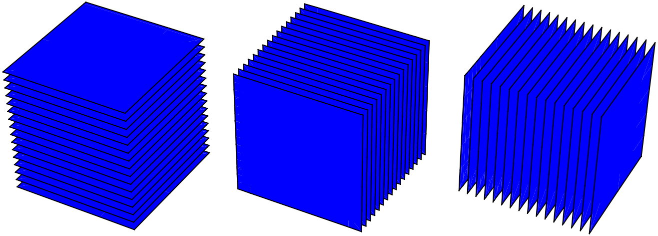

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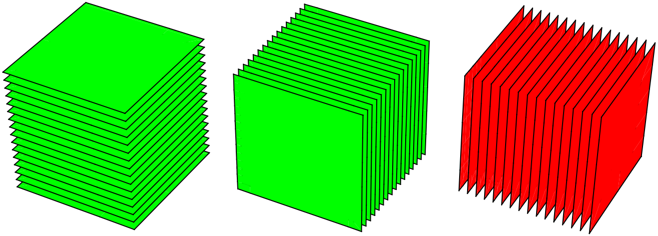

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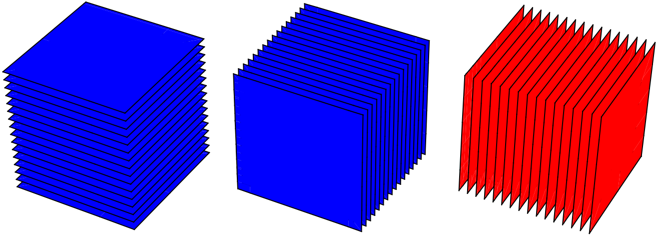

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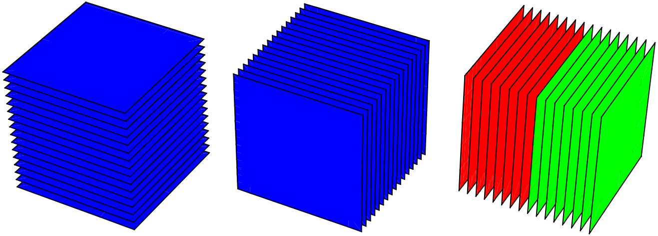

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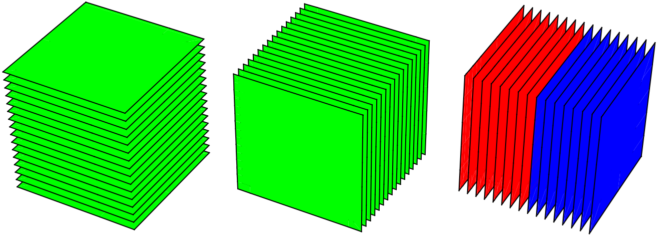

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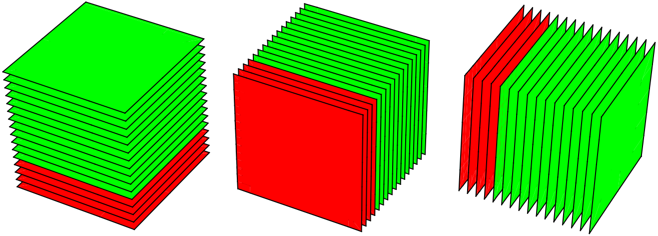

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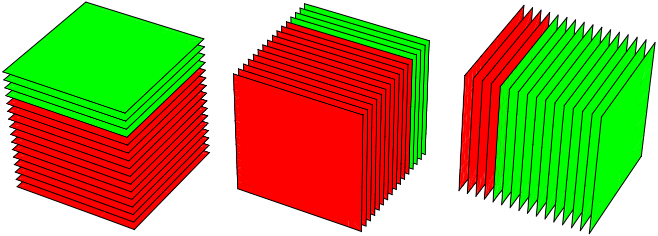

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Proposition.

Up to equivalence, the set $\mathcal{C}^3(16,6,2)$ contains exactly 27 difference cubes and 946 non-difference group cubes. Furthermore, it contains at least 1423 inequivalent non-group cubes.

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The parameters are of Menon type: $(4u^2, 2u^2-u, u^2-u)$. By exchanging $0 \to -1$, the cubes are transformed to n-dimensional Hadamard matrices with inequivalent slices!

Cubes of symmetric designs

Questions:

1. There are exactly 78 symmetric $(25, 9, 3)$ designs, but no difference sets. Are there cubes of $(25, 9, 3)$ designs of dimension $n \geq 3$?

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3. **Is there a product construction for cubes of symmetric designs?**

4. Hadamard matrices coming from Menon designs are of square orders. Are there n-dimensional Hadamard matrices with inequivalent slices of non-square orders?

And now...


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And now. . .

Nobody expects **Room squares**!

Room squares

T. G. Room, *A new type of magic square*, Math. Gaz. **39** (1955), 307.

Thomas Gerald Room

Article Talk

From Wikipedia, the free encyclopedia

Thomas Gerald Room FRS FAA (10 November 1902 – 2 April 1986) was an Australian mathematician who is best known for Room squares. He was a Foundation Fellow of the Australian Academy of Science.[1][2]

Room squares

T. G. Room, *A new type of magic square*, Math. Gaz. **39** (1955), 307.

Let S be a set of $v + 1$ elements, say $S = \{\infty, 1, 2, \ldots, v\}$.

A Room square of order v is a $v \times v$ matrix M such that:

- the entries of M are empty or 2-element subsets of S
- each 2-subset of S appears once in M
- elements of S appear once in every row and column of M

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Example.

$v = 7$

$\infty 1$			26		57	34
45	$\infty 2$			37		16
27	56	$\infty 3$			14	
	13	67	$\infty 4$			25
36		24	17	$\infty 5$		
	47		35	12	$\infty 6$	
		15		46	23	$\infty 7$

Room squares

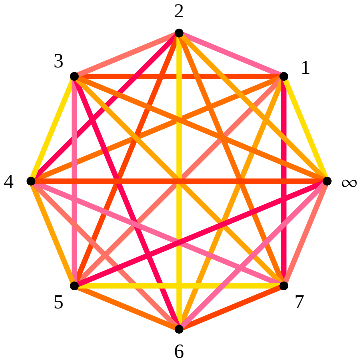
Equivalent objects:

Theorem.

A Room square of order v is equivalent to a pair of orthogonal 1-factorizations of the complete graph K_{v+1}.

Room squares

∞1			26		57	34
45	∞2			37		16
27	56	∞3			14	
	13	67	∞4			25
36		24	17	∞5		
	47		35	12	∞6	
		15		46	23	∞7



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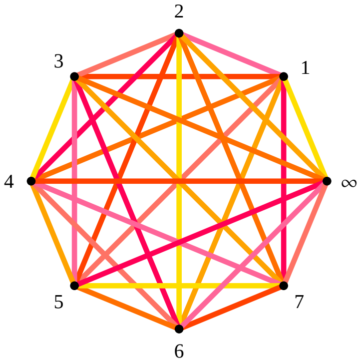
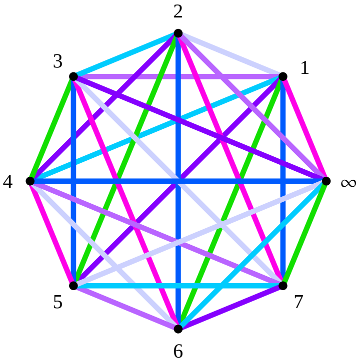
	47		35	12	$\infty 6$	
		15		46	23	$\infty 7$





Room squares

Equivalent objects:

Theorem.

A Room square of order v is equivalent to a pair of orthogonal 1-factorizations of the complete graph K_{v+1}.

Theorem.

A Room square of order v is equivalent to a pair of orthogonal-symmetric latin squares of order v.

Room squares

	1	2	3	4	5	6	7
1	$\infty 1$			26		57	34
2	45	$\infty 2$			37		16
3	27	56	$\infty 3$			14	
4		13	67	$\infty 4$			25
5	36		24	17	$\infty 5$		
6		47		35	12	$\infty 6$	
7			15		46	23	$\infty 7$

1	6	4	3	7	2	5
6	2	7	5	4	1	3
4	7	3	1	6	5	2
3	5	1	4	2	7	6
7	4	6	2	5	3	1
2	1	5	7	3	6	4
5	3	2	6	1	4	7

1	5	2	6	3	7	4
5	2	6	3	7	4	1
2	6	3	7	4	1	5
6	3	7	4	1	5	2
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A Room square of order v exists if and only if v is odd and $v \neq 3, 5$.

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Proof: 1955–1973.

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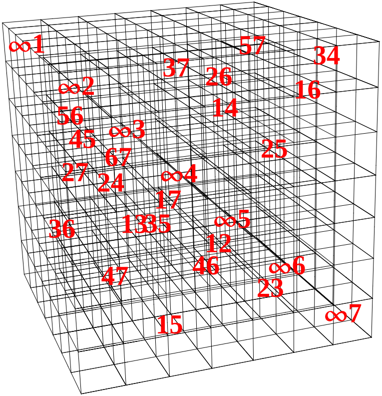

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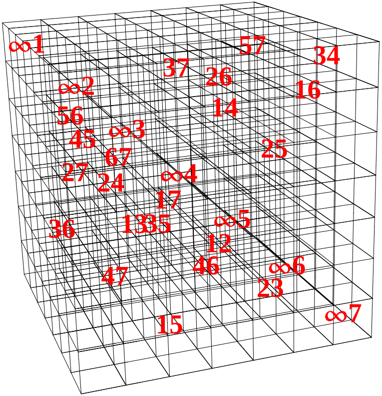

Front view:

∞1	56	24		37		
	∞2	67	35		14	
		∞3	17	46		25
36			∞4	12	57	
	47			∞5	23	16
27		15			∞6	34
45	13		26			∞7

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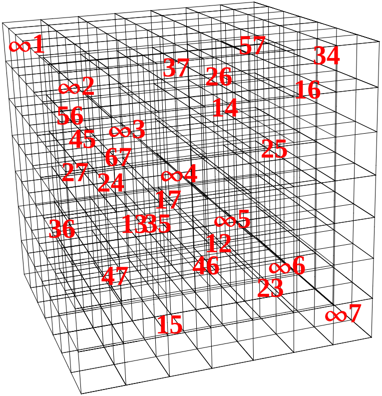

Top view:

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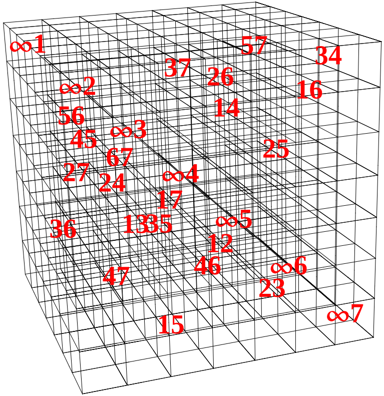

Side view:

26	34		57			∞1
45		16			∞2	37
	27			∞3	14	56
13			∞4	25	67	
		∞5	36	17		24
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∞7	15	23		46		

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Conjecture (W. D. Wallis): $\mu(v) \leq \frac{1}{2}(v-1)$

Idea

Idea: why not use *projections* in the definition of higher-dimensional symmetric designs?

Image: Cousin Ricky

Projection cubes of symmetric designs

An n-cube of (v, k, λ) designs is a function $A : \{1, \ldots, v\}^n \to \{0, 1\}$ such that all 2-dimensional **slices** (**sections**) are symmetric (v, k, λ) designs. The set of all such objects is denoted $\mathcal{C}^n(v, k, \lambda)$.

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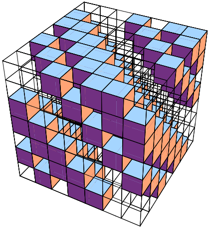

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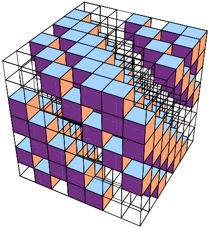
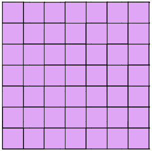
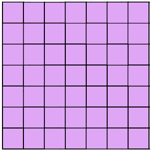


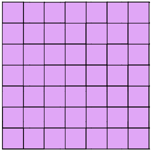
Front view:



Top view:



Side view:



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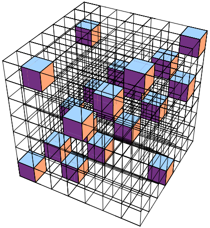

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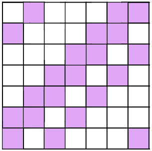

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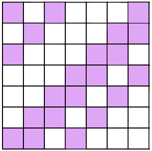

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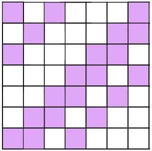

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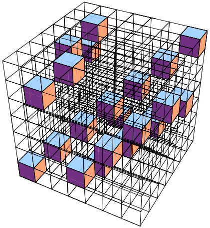

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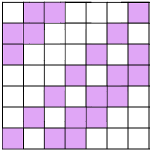

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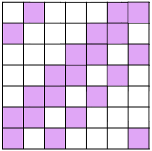

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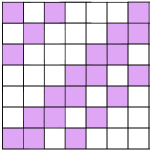

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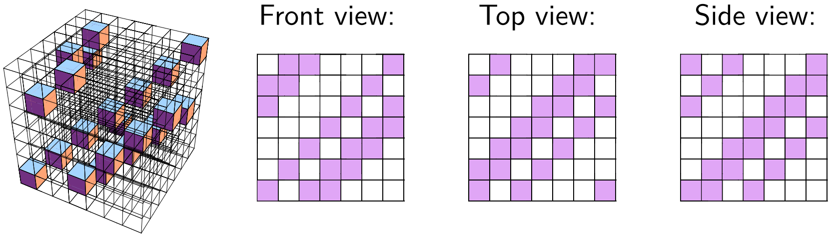


Persistence of Vision Raytracer, Version 3.7 (2013).
http://www.povray.org/

POV-Ray

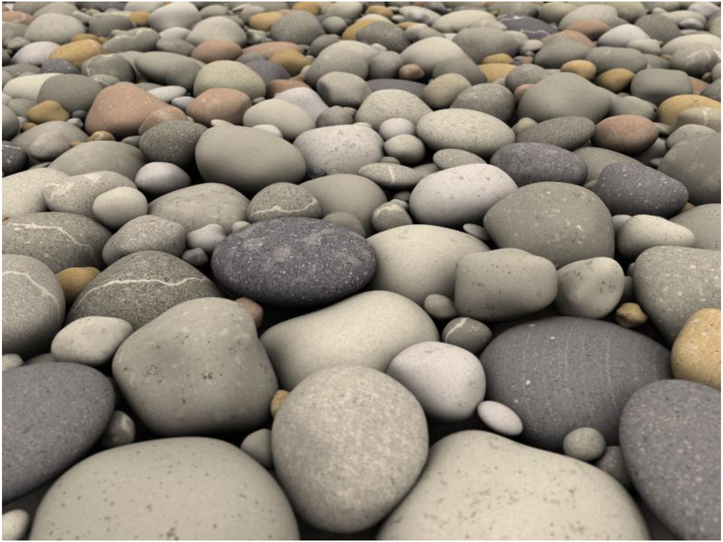
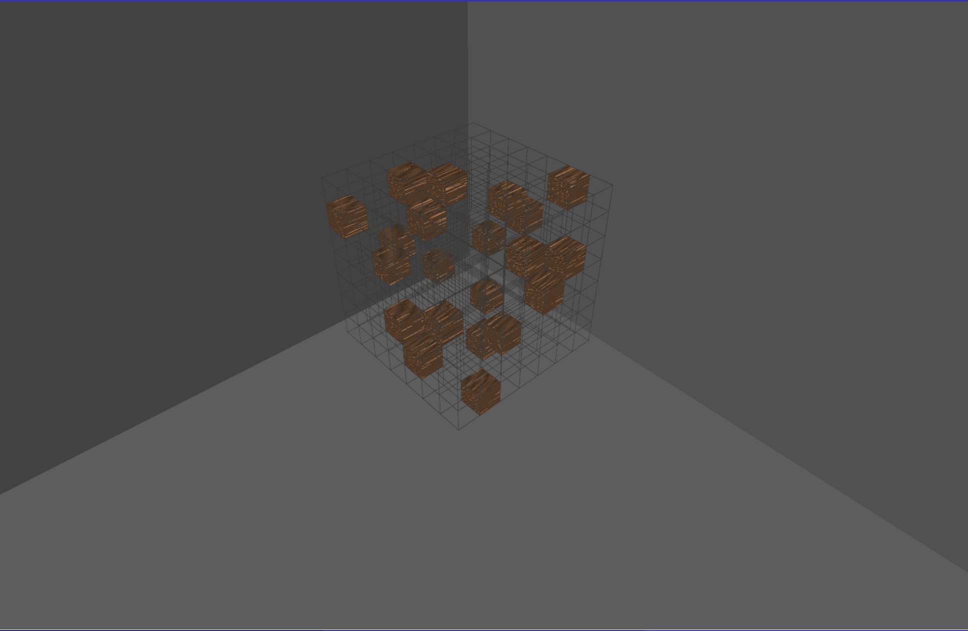


Image: Jonathan Hunt

Introducing. . .



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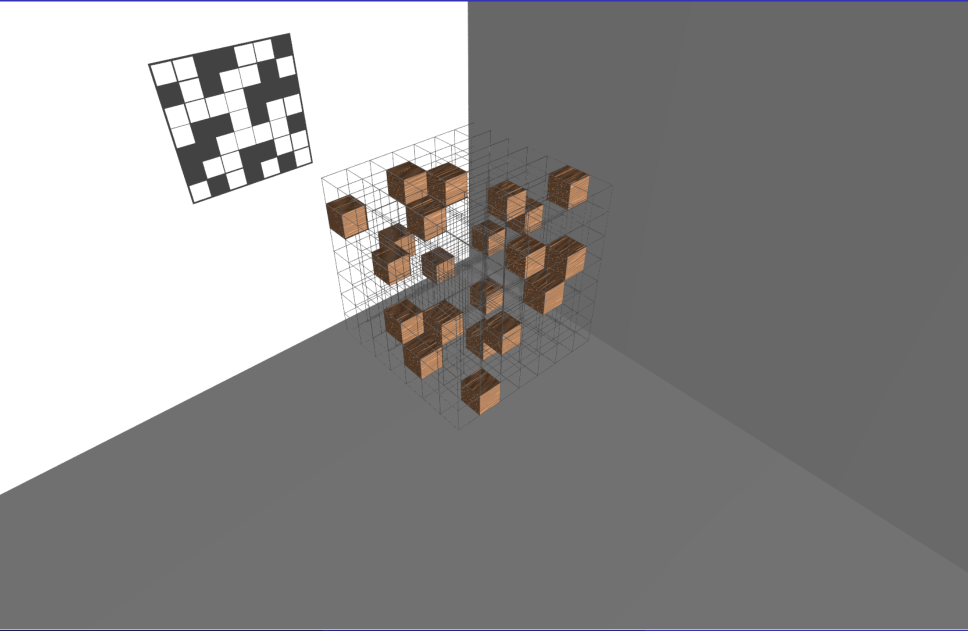

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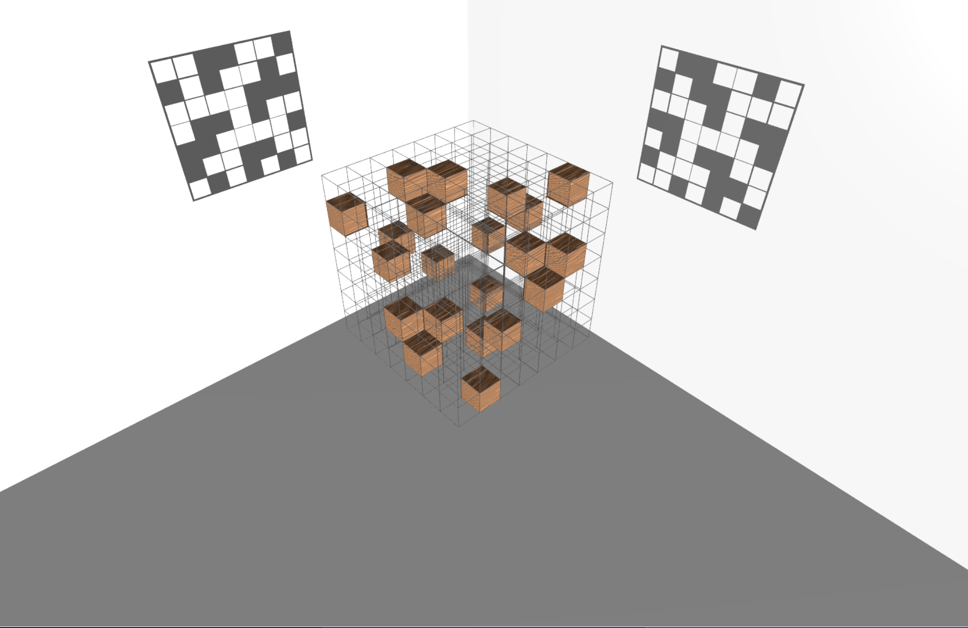

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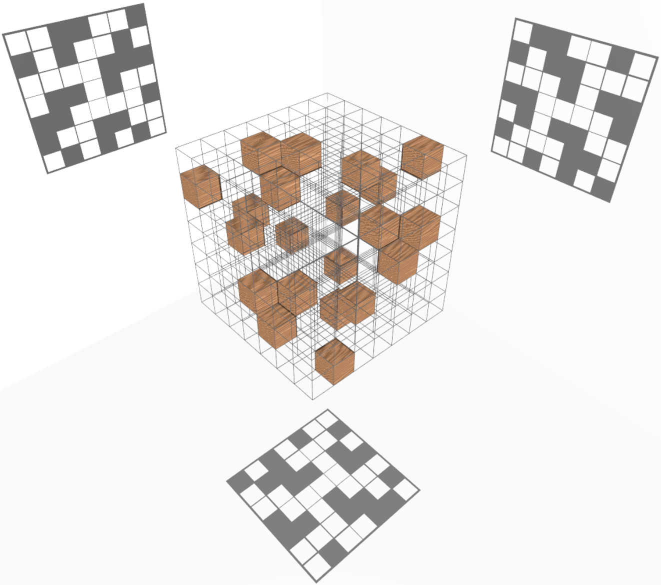

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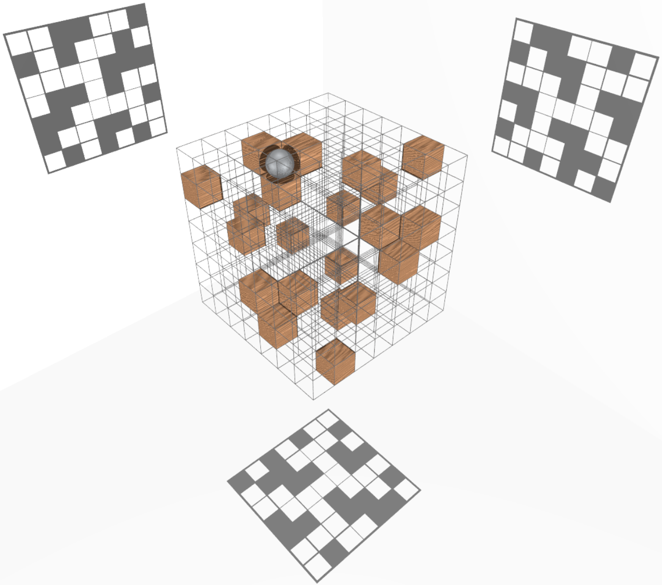

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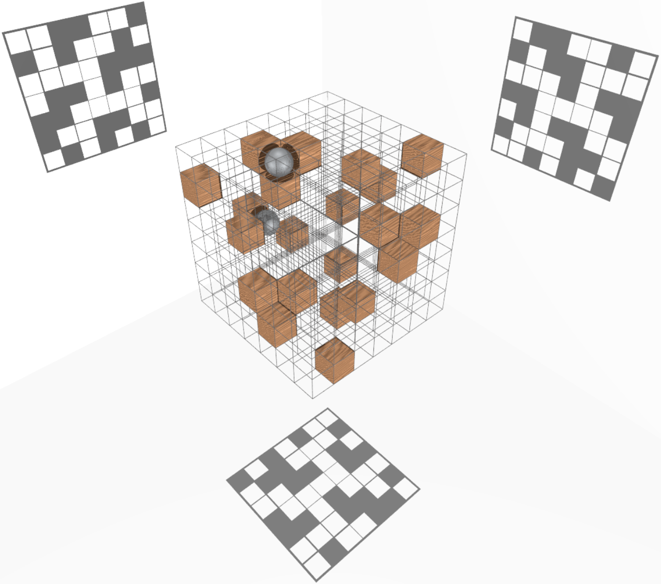

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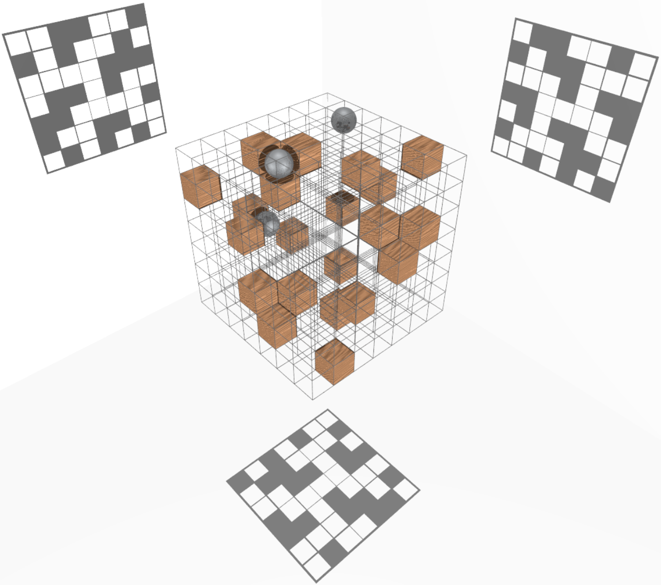

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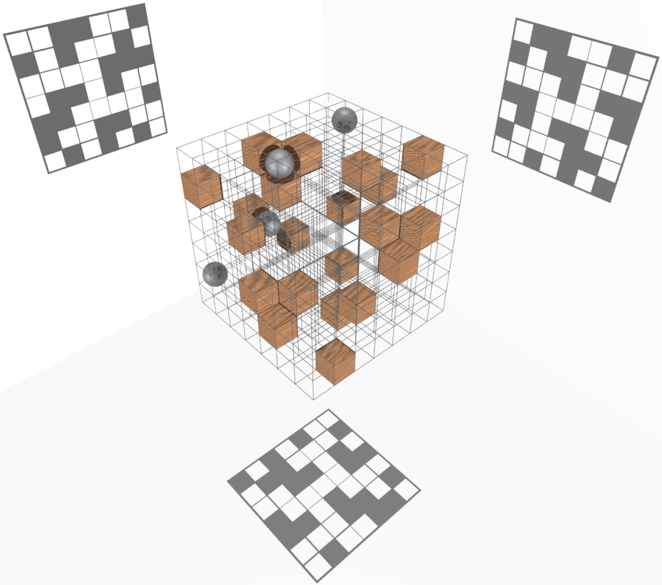

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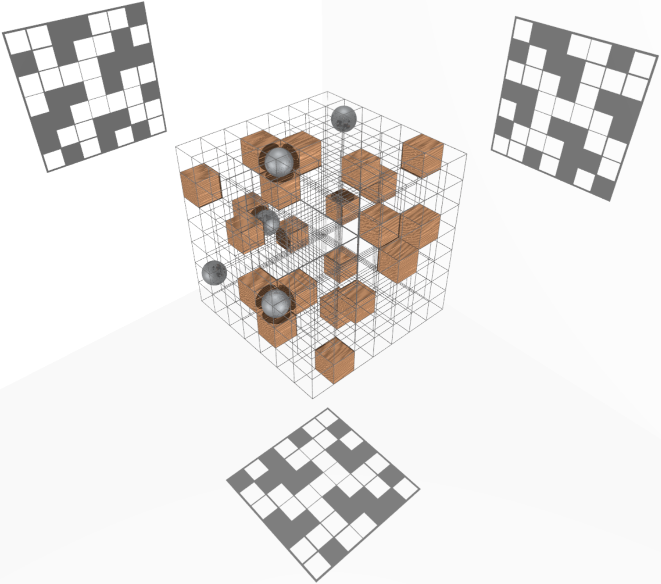

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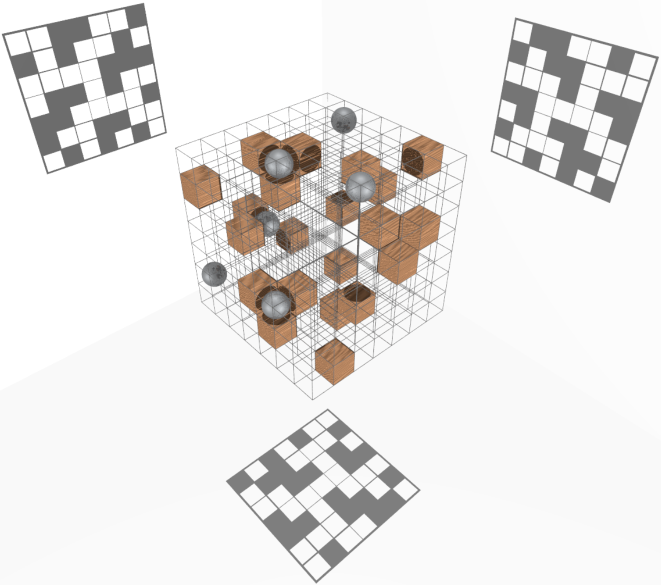

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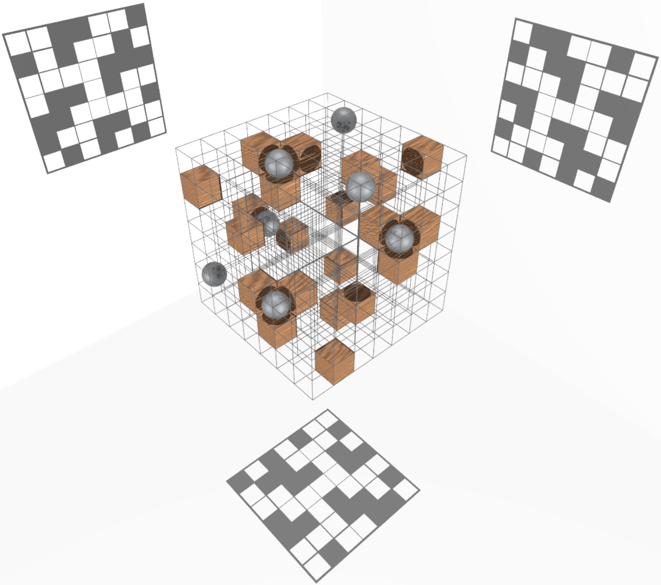

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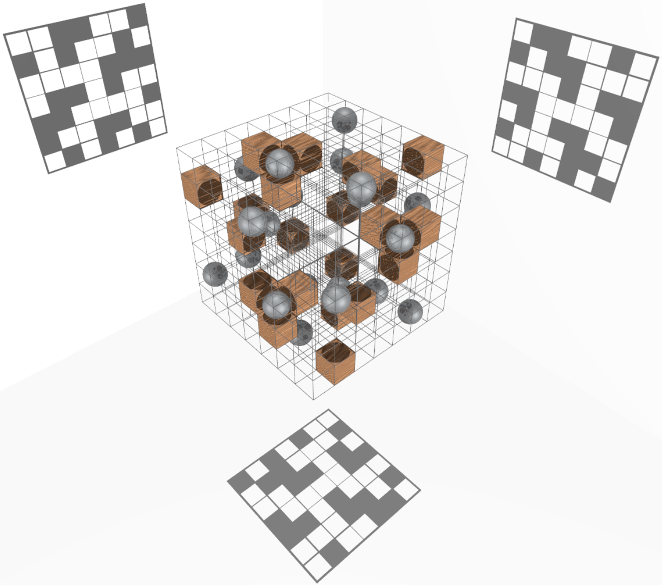

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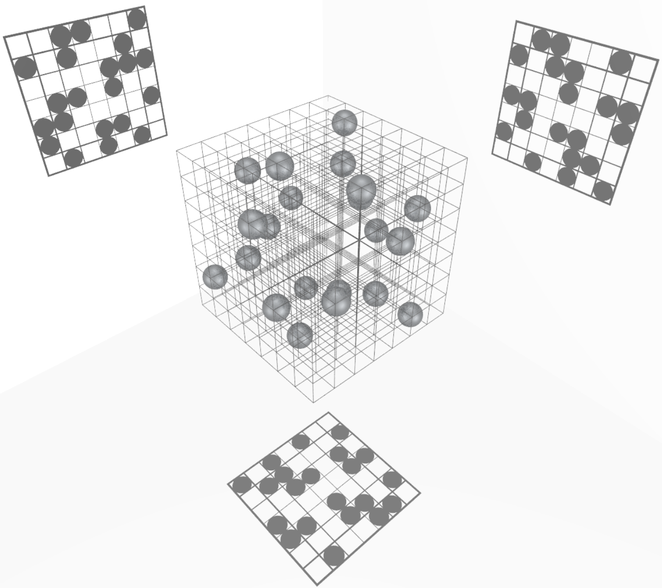

Projection cubes of symmetric designs

What is a projection?

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For an n-dimensional matrix $C : \{1, \ldots, v\}^n \to \mathbb{F}$ and $1 \le x < y \le n$, the projection $\Pi_{xy}(C)$ is the 2-dimensional matrix with (i_x, i_y)-entry

$$\sum_{1 \le i_1, \ldots, \widehat{i_x}, \ldots, \widehat{i_y}, \ldots, i_n \le v} C(i_1, \ldots, i_n).$$

The sum is taken over all n-tuples $(i_1, \ldots, i_n) \in \{1, \ldots, v\}^n$ with fixed coordinates i_x and i_y in a (semi)field $\mathbb{F}$.

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~~$\mathbb{F}_2$ = binary field ($1+1=0$) $\rightsquigarrow$ examples with different numbers of 1's~~

$\mathbb{F}$ = field of characteristic 0:

Proposition.

The number of incidences (1-entries) of $C \in \mathcal{P}^n(v,k,\lambda)$ is vk.

Projection cubes of symmetric designs

We can interpret $C : \{1,\ldots,v\}^n \to \{0,1\}$ as a characteristic function and identify it with the subset of n-tuples

$$\overline{C} = \{(i_1,\ldots,i_n) \in \{1,\ldots,v\}^n \mid C(i_1,\ldots,i_n) = 1\}$$

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Proposition.

Let $S \subseteq \{1,\ldots,v\}^n$ be a subset of cardinality vk. There exists a cube $C \in \mathcal{P}^n(v,k,\lambda)$ such that $S = \overline{C}$ if and only if the following statements are true for all $1 \le x < y \le n$:

1. for all $i \in \{1,\ldots,v\}$, there are exactly k elements $j \in \{1,\ldots,v\}$ such that $(i,j) \in \Pi_{xy}(S)$,
2. for all $j \in \{1,\ldots,v\}$, there are exactly k elements $i \in \{1,\ldots,v\}$ such that $(i,j) \in \Pi_{xy}(S)$,
3. for all $i, i' \in \{1,\ldots,v\}$, $i \neq i'$, there are exactly λ elements $j \in \{1,\ldots,v\}$ such that $(i,j) \in \Pi_{xy}(S)$ and $(i',j) \in \Pi_{xy}(S)$.

Projection cubes of symmetric designs

Corollary.

If C is a (v, k, λ) projection n-cube, then $\overline{C}$ is an orthogonal array of size vk, degree n, order v, strength 1, and index k, i.e. an $OA(vk, n, v, 1)$.

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$k = 2$: numbers of $\mathcal{P}^n(3, 2, 1)$-cubes up to equivalence

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If a (v,k,λ) projection n-cube with $k \geq 2$ exists, then $n \leq \dfrac{v(v+1)}{2}$.

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$\nu(v,1,0)=\infty, \quad \nu(3,2,1)=5 \quad$ (theorem gives $\nu(3,2,1)\leq 6$)

Higher-dimensional difference sets

Let G be an additively written group of order v. We can index projection cubes with elements of G instead of the integers $\{1, \ldots, v\}$:

$$C : G^n \to \{0,1\}, \qquad \overline{C} \subseteq G^n$$

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An *n-dimensional (v, k, λ) difference set* in G is a set of n-tuples $D \subseteq G^n$ of size k such that $\{d_x - d_y \mid d \in D\} \subseteq G$ are (v, k, λ) difference sets for all $1 \leq x < y \leq n$.

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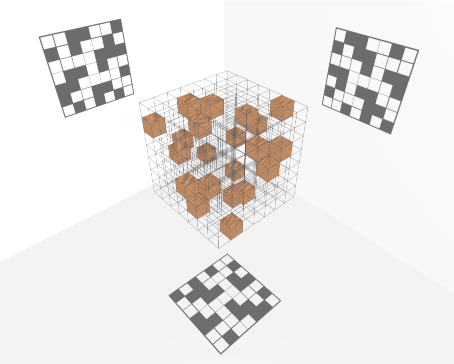
Proposition.

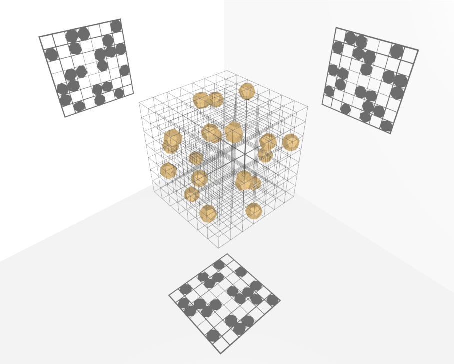
If D is an n-dimensional (v, k, λ) difference set in G, then the development

$$\operatorname{dev} D = \{(d_1 + g, \ldots, d_n + g) \mid g \in G,\ d \in D\}$$

is the representation $\overline{C} \subseteq G^n$ of a projection cube $C \in \mathcal{P}^n(v, k, \lambda)$.

Example. Let $G = \mathbb{Z}_7$. Then $D_1 = \{(0, 1, 3), (0, 2, 6), (0, 4, 5)\}$ and $D_2 = \{(0, 1, 2), (0, 2, 4), (0, 4, 1)\}$ are two 3-dimensional $(7, 3, 1)$ difference set such that $\operatorname{dev} D_1$ and $\operatorname{dev} D_2$ are inequivalent "Fano cubes" in $\mathcal{P}^3(7, 3, 1)$.





Proposition.

Let D be an n-dimensional (v, k, λ) difference set in G. Then the projection cube $\overline{C} = \operatorname{dev} D$ has an autotopy group isomorphic to G acting sharply transitively on each coordinate.

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Proposition.

Let $C \in \mathcal{P}^n(v, k, \lambda)$ be a projection cube with an autotopy group G acting sharply transitively on each coordinate. Then there is an n-dimensional (v, k, λ) difference set $D \subseteq G^n$ such that $\overline{C}$ is equivalent with $\operatorname{dev} D$.

Higher-dimensional difference sets

Theorem ("Higher-dimensional Paley difference sets").

If $q \equiv 3 \pmod 4$ is a prime power, then there exists a q-dimensional difference set with parameters $(q, (q-1)/2, (q-3)/4)$ in the additive group of $\mathbb{F}_q$.

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Example. 7-dimensional $(7,3,1)$ difference set in $\mathbb{Z}_7$:

$$D = \{(0,1,3,2,6,4,5),\ (0,2,6,4,5,1,3),\ (0,4,5,1,3,2,6)\}$$

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Theorem ("Higher-dimensional cyclotomic difference sets").

If q is a prime power such that the 4th powers in $\mathbb{F}_q$ make a $(q, (q-1)/4, (q-5)/16)$ difference set, or the 8th powers in $\mathbb{F}_q$ make a $(q, (q-1)/8, (q-9)/64)$ difference set, then there exists a q-dimensional difference set with the same parameters.

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Theorem ("Higher-dimensional twin prime power difference sets").

If q and $q+2$ are odd prime powers, then there exists a q-dimensional difference set in $G = \mathbb{F}_q \times \mathbb{F}_{q+2}$ with parameters $(4m-1, 2m-1, m-1)$ for $m = (q+1)^2/4$.

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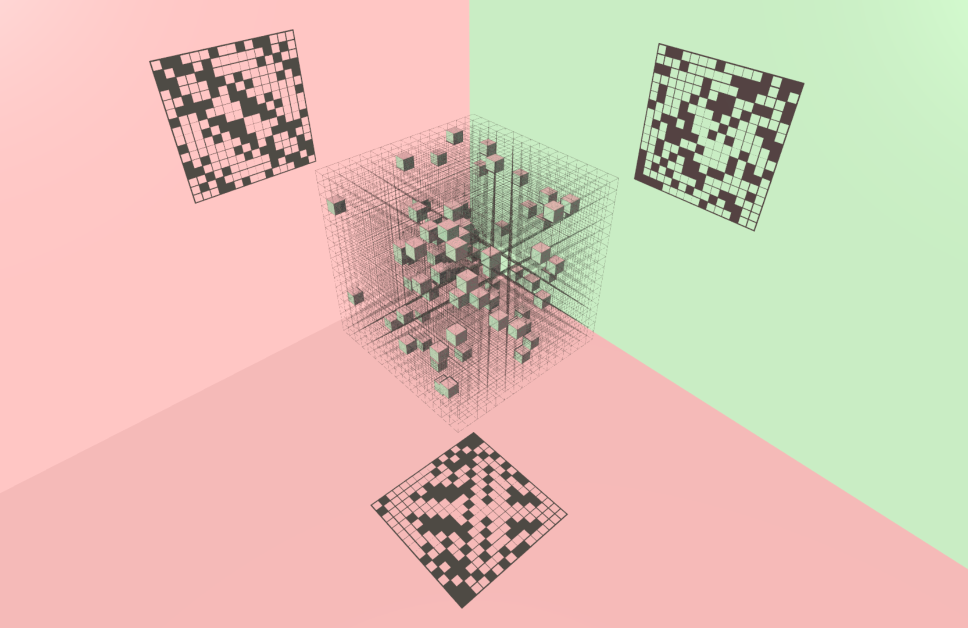
Questions:

1. Examples of projection cubes not coming from difference sets?
2. Examples such that the projections are non-isomorphic designs?

Examples of $\mathcal{P}^3(16,6,2)$-cubes



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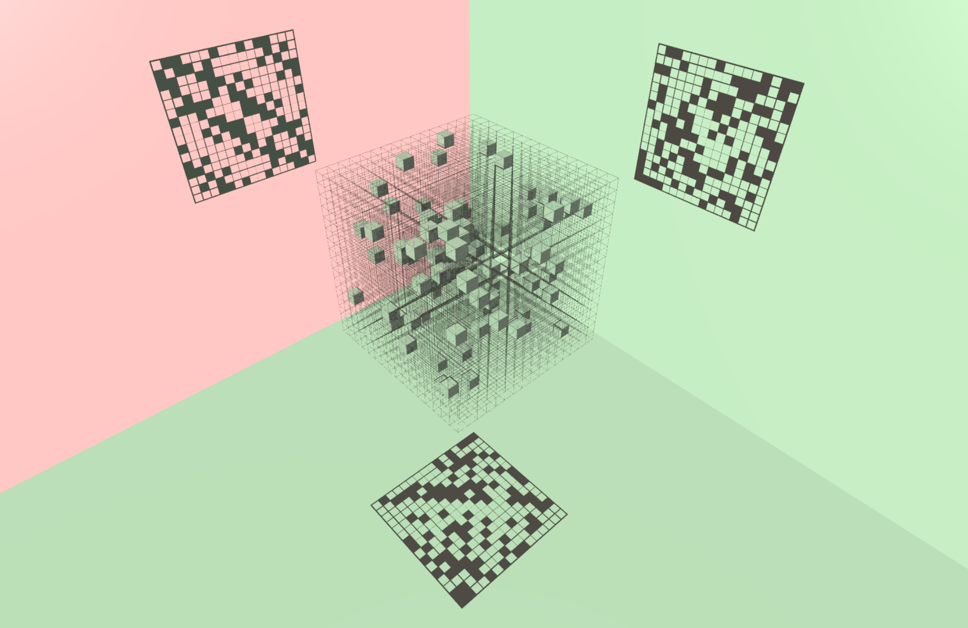

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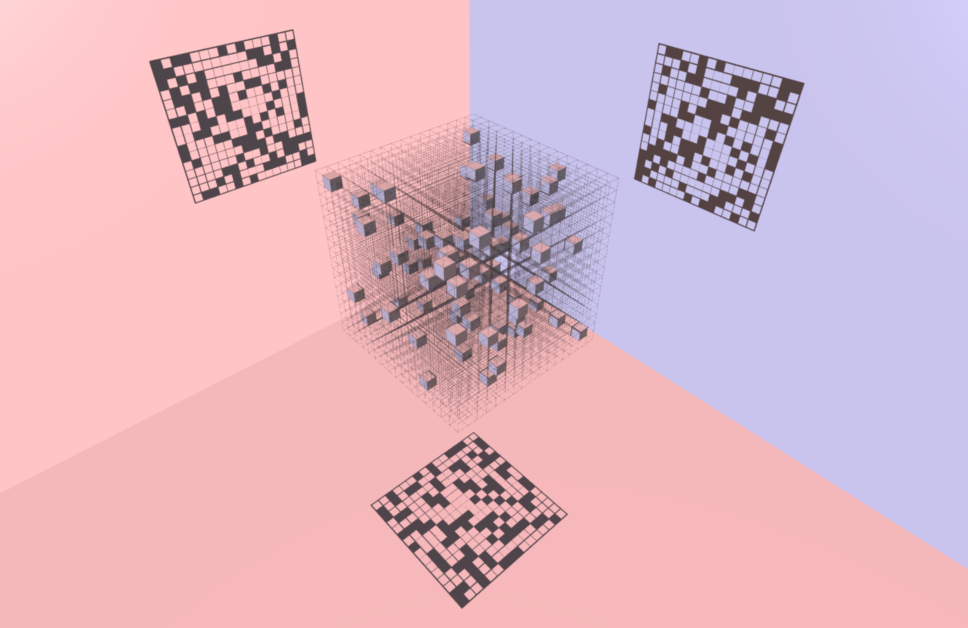

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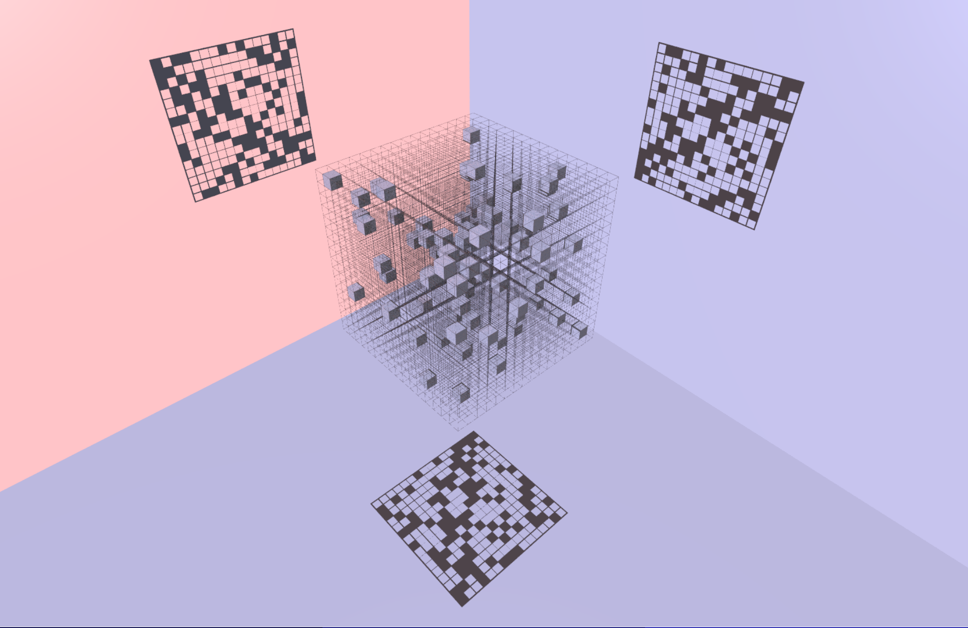

More on $\mathcal{C}^n(v, k, \lambda)$ and $\mathcal{P}^n(v, k, \lambda)$ cubes

V. Krčadinac, M. O. Pavčević, *On higher-dimensional symmetric designs*, in preparation, 2024.

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Proposition.

The total number of $\mathcal{C}^n(3,2,1)$-cubes is $3 \cdot 2^{n-1}$ and they are all isotopic.

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Theorem.

The numbers of inequivalent cubes in $\mathcal{P}^n(7,3,1)$ and $\mathcal{P}^n(7,4,2)$ are given below. In particular, $\nu(7,3,1) = 7$ and $\nu(7,4,2) = 9$.

	n								
(v,k,λ)	2	3	4	5	6	7	8	9	10
$(7,3,1)$	1	13	20	4	3	2	0	0	0
$(7,4,2)$	1	877	884	74	19	9	6	5	0

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Let $(\pi_1,\ldots,\pi_n)$ be an autotopy of $C \in \mathcal{P}^n(v,k,\lambda)$. Then any component π_x uniquely determines all other components.

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Proposition.

Autotopies of cubes in $\mathcal{P}^n(v,k,\lambda)$ have the same number of fixed points on each coordinate. Autotopies of cubes in $\mathcal{C}^n(v,k,\lambda)$ may have different numbers of fixed points.

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We say that $R = \{\varphi_1, \ldots, \varphi_{n-1}\}$ is a regular set of (anti)automorphisms of G if each $\varphi_i : G \to G$ is an automorphism or antiautomorphism, and each difference $\varphi_i - \varphi_j$ is an automorphism or antiautomorphism for $1 \leq i < j \leq n - 1$.

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Let p be the smallest prime divisor of v. Then any cyclic (v, k, λ) difference set extends to p dimensions.

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Work in progress...

The End

Thanks for your attention!