Cubes of symmetric designs and difference sets*

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Combinatorial Designs and their Applications

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$[X] = 0$ iff X is true, otherwise $[X] = 0$.

Fano plane, motivation for cubes

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$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

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We make a cyclic shift of rows of A_1

Fano plane, cyclic shifts

$$A_1 = \begin{pmatrix} 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1 \end{pmatrix} \quad A_7 = \begin{pmatrix} 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0 \end{pmatrix}$$

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block designs, Hadamard matrices, orthogonal designs, weighing matrices.

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Definition and classes

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An *n-dimensional cube of symmetric* (v, k, λ) *designs* is an n-cube of order v such that all of its slices are incidence matrices of (v, k, λ) designs. The set of all such n-cubes will be denoted by $\mathcal{C}^n(v, k, \lambda)$.

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The orbits of this action are the *isotopy classes* of cubes

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Kramer-Mesner approach + our GAP package *Prescribed Automorphism Groups*

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Theorem.

Let D be a (v, k, λ) difference set in the group G. Order the group elements as $g_1, \ldots, g_v$. Then the function

$$C(i_1, \ldots, i_n) = [g_{i_1} \cdots g_{i_n} \in D] \tag{1}$$

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$$C(i_1, \ldots, i_n) = [g_{i_2} \cdots g_{i_n} \in B_{i_1}] \tag{2}$$

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Then the construction may give non-difference cubes.

The first example comes from $(21, 5, 1)$ symmetric design

There is a $(21, 5, 1)$ design over F_{21} where all blocks are difference sets, but where they don't belong to a development of one difference set.

Group cubes, constructions

Theorem:

For every $m \geq 2$ and $n \geq 3$, the set $\mathcal{C}^n(4^m, 2^{m-1}(2^m - 1), 2^{m-1}(2^{m-1} - 1))$ contains at least two inequivalent group cubes that are not difference cubes.

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Proposition:

Up to equivalence, the set $\mathcal{C}^3(16, 6, 2)$ contains exactly 27 difference cubes and 946 group cubes that are not difference cubes.

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Proposition:

The set $\mathcal{C}^3(21, 5, 1)$ contains exactly three inequivalent group cubes, two of which are difference cubes.

The End

Thank you!