New results on additive designs

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Definition

A 2-(v, k, λ) *design* is a pair $(V, \mathcal{B})$ such that

- V is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of V (called blocks);
- each 2-subset of V is contained in λ blocks.

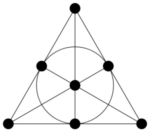

Figure: The Fano plane. 2-$(7, 3, 1)$ design.

- A 2-design is *symmetric* if $|V| = |\mathcal{B}|$.
- A *Steiner system* is a design with $\lambda = 1$.

Definition (Cageggi, Falcone, Pavone, 2017)

A design $(V, \mathcal{B})$ is *additive* under an abelian group G if

- $V \subseteq G$ and
- $\sum_{x \in B} x = 0, \quad \forall B \in \mathcal{B}.$

Examples:

Parameters	Group	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$	$\mathrm{AG}_1(n, p^m)$, points-lines design of $\mathrm{AG}(n, p^m)$
$([n]_2, 3, 1)$	$\mathbb{Z}_2^n$	$\mathrm{PG}_1(n-1, 2)$, points-lines design of $\mathrm{PG}(n-1, 2)$

The number of points of $\mathrm{PG}(n-1, q)$ is denoted by $[n]_q = \frac{q^n - 1}{q - 1}$

Definition (Cameron, 1974. Delsarte, 1976.)

A 2-(v,k,λ) *design over* $\mathbb{F}_q$ is a pair $(V,\mathcal{B})$ such that

- V is the set of points of $\mathrm{PG}(v-1,q)$
- $\mathcal{B}$ is a collection of $(k-1)$-dimensional subspaces $\mathrm{PG}(v-1,q)$ (blocks)
- each line is contained in λ blocks.

Properties:

- (v,k,λ) design over $\mathbb{F}_q$ is a classical $([v]_q,[k]_q,\lambda)$ design
- (v,k,λ) design over $\mathbb{F}_2$ is additive under $\mathbb{Z}_2^v$

Parameters	Description	Reference
$([v]_2,7,7)$	$(v,3,7)$ design over $\mathbb{F}_2$ for all v odd	Thomas, 1987 + Buratti, A.N., 2019
$(8191,7,1)$	$(13,3,1)$ design over $\mathbb{F}_2$	Braun, Etzion, Ostergaard, Vardy, Wassermann, 2017

Definition

$(V, \mathcal{B})$ is additive under an abelian group G if $V \subseteq G$ and $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

- strongly additive if $\mathcal{B} = \{B \in \binom{V}{k} \mid \sum_{x \in B} x = 0\}$
- strictly additive if $V = G$
- almost strictly additive if $V = G \setminus \{0\}$

[Cageggi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\text{PG}_1(n-1, 2)$
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		$\text{AG}_1(n, p^m)$
$(p^2, p, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\text{AG}_1(2, p)$
(v, k, λ)	G	✓			symmetric design
(v, k, λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k - \lambda \nmid k$, prime

Known inifinite families of additive Steiner designs [Cageggi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		$\mathrm{AG}_1(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathrm{PG}_1(n-1, 2)$
$([2]_q, q+1, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\mathrm{PG}_1(2, q)$

New examples [Buratti, A.N., 2023]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(5^3, 5, 1)$	$\mathbb{F}_{5^3}$		✓		not isomorphic to $\mathrm{AG}_1(3, 5)$
$(7^3, 7, 1)$	$\mathbb{F}_{7^3}$		✓		not isomorphic to $\mathrm{AG}_1(3, 7)$
$(p^n, p, 1)$	$\mathbb{F}_{p^n}$		✓		$p \in \{5, 7\}$, $n \geq 3$, not isomorphic to $\mathrm{AG}_1(n, p)$

Definition

A Steiner 2-design is *G-super-regular* if it is

- is strictly additive under an abelian group G (the point set is exactly G) and
- G-regular (any translate of any block is a block as well)

Theorem (Buratti, A.N., 2023)

Let $k \geq 3$, $k \not\equiv 2 \pmod 4$ and $k \neq 2^n \cdot 3 \geq 12$.
There are infinitely many values of v for which there exists a super-regular $(v, k, 1)$ design.

- Group is $G \times \mathbb{F}_q$, where G is a non-binary group of order k and q a power of a prime divisor of k

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Group G is binary when G has exactly one involution. Otherwise we say that G is non-binary group.

Theorem (Buratti, A.N., 2023)

Let $k \geq 3$, $k \not\equiv 2 \pmod 4$ *and* $k \neq 2^n \cdot 3 \geq 12$.
There are infinitely many values of v *for which there exists a super-regular* $(v, k, 1)$ *design.*

Constructing examples is computationally hard!

k	3	4	5
	$\mathrm{AG}_1(n,3)$	$\mathrm{AG}_1(n,4)$	$\mathrm{AG}_1(n,5)$

k	6	7	8	9	10
	$2^1 \cdot 3$	$\mathrm{AG}_1(n,7)$	$\mathrm{AG}_1(n,8)$	$\mathrm{AG}_1(n,9)$	$2 \pmod 4$

k	11	12	13	14	15
	$\mathrm{AG}_1(n,11)$	$2^2 \cdot 3$	$\mathrm{AG}_1(n,13)$	$2 \pmod 4$	?

- $v = 15 \cdot 5^n, n \geq 10^7$

Theorem (Buratti, A.N., 202?)

- *Every design* $PG_d(n,q)$ *is additive under* $\mathbb{F}_q^{n+1}$.
- *Every design* $PG_d(n,q)$ *is strongly additive under* $\mathbb{Z}_{q^d}^{[n+1]_q}$.

[Cageggi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	$\mathrm{PG}_1(n-1, 2)$
$([2]_q, q+1, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			$\mathrm{PG}_1(2, q)$

Theorem (Buratti, A.N., 202?)

- *A symmetric (v,k,λ) design is strongly additive under $\mathbb{Z}_{k-\lambda}^v$.*
- *Let $\mathcal{D}$ be a cyclic symmetric (v,k,λ) design and let p be a prime dividing $k-\lambda$ but not v. Then $\mathcal{D}$ is additive under $\mathbb{Z}_p^t$ with $t = ord_v(p)$.*

[Cageggi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
(v,k,λ)	G	✓			symmetric design
(v,k,λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k-\lambda \nmid k$, prime

Additivity of cyclic symmetric designs

Definition (Cageggi, Falcone, Pavone, 2017)

A design $(V, \mathcal{B})$ is *additive* under an abelian group G if there exists an injective map

$$f : V \to G$$

such that $f(B)$ is zero-sum for every block $B \in \mathcal{B}$.

Every cyclic symmetric (v, k, λ) design is of the form

$$(\mathbb{Z}_v, \{D + i \mid 0 \leq i \leq v - 1\})$$

where D is a cyclic (v, k, λ) difference set.

An incidences structure $(V, \mathcal{B})$ is *cyclic* if there exists a cyclic permutation on V leaving $\mathcal{B}$ invariant.
A k-subset D of an additive group G is a (G, k, λ) *difference set* if each non-zero element of G is covered λ times by the list of differences of D: $\Delta D \lambda \, (G \setminus \{0\})$

Theorem (Buratti, A.N., 202?)

Let $\mathcal{D}$ be a cyclic symmetric (v, k, λ) design and let p be a prime dividing $k - \lambda$ but not v. Then $\mathcal{D}$ is additive under $\mathbb{Z}_p^t$ with $t = ord_v(p)$.

Proof:

- Let g be a generator of the subgroup of $\mathbb{F}_{p^t}^*$ of order v and consider the injective maps f_1 and f_{-1} defined as follows:

$$f_1 : x \in \mathbb{Z}_v \longrightarrow g^x \in \mathbb{F}_{p^t}, \qquad f_{-1} : x \in \mathbb{Z}_v \longrightarrow g^{-x} \in \mathbb{F}_{p^t}.$$

- Consider the two sums

$$\sigma_1 := \sum_{d \in D} f_1(d) = \sum_{d \in D} g^d, \qquad \sigma_{-1} := \sum_{d \in D} f_{-1}(d) = \sum_{d \in D} g^{-d}$$

- Calculate their product $\sigma_1 \cdot \sigma_{-1} = (k - \lambda) + \lambda \frac{g^v - 1}{g - 1} = 0$
- Therefore

$$\sigma_1 = 0, \quad \text{or} \quad \sigma_{-1} = 0$$

- Since

$$\sum_{b \in B} f_1(b) = \sum_{d \in D} g^{d+i} = \sigma_1 \cdot g^i \quad \text{and} \quad \sum_{b \in B} f_{-1}(b) = \sum_{d \in D} g^{-(d+i)} = \sigma_{-1} \cdot g^{-i}$$

- Either f_1 or f_{-1} is the map we are looking for □

Example

The point-hyperplane design of PG$(2,3)$, the projective plane of order 3, is additive under $\mathbb{Z}_3^3$ that is the additive group of $\mathbb{F}_{3^3}$.

- Singer $(13,4,1)$ difference set $D = \{0,1,3,9\}$
- $(\mathbb{Z}_{13}, \mathcal{B})$ is cyclic symmetric design with parameters $(13,4,1)$

$$\{D + i \mid 0 \leq i \leq 12\}$$

- Let r be a root of the primitive polynomial $x^3 + 2x^2 + 1$ over $\mathbb{F}_3$
- Taking r as primitive element of $\mathbb{F}_{3^3}$, a generator of the subgroup of $\mathbb{F}_{3^3}^*$ of order 13 is $g = r^2$
- We check

$$\sigma_1 = \sum_{d \in D} g^d = g^0 + g^1 + g^3 + g^9 = r^0 + r^2 + r^6 + r^{18} =$$

$$= (0,0,1) + (1,0,0) + (2,2,0) + (0,1,1) = (0,0,2)$$

- and

$$\sigma_{-1} = \sum_{d \in D} g^{-d} = g^0 + g^{-1} + g^{-3} + g^{-9} = r^0 + r^{-2} + r^{-6} + r^{-18} =$$

$$= (0,0,1) + (0,2,1) + (2,0,2) + (1,1,2) = (0,0,0)$$

- $f_{-1} : x \in \mathbb{Z}_{13} \longrightarrow g^{-x} \in \mathbb{F}_{3^3}$

The point-hyperplane design of PG$(2,3)$ is additive under $\mathbb{Z}_3^3$.

- In other words PG$(2,3)$ can be seen as the design $(V,\mathcal{B})$ where

$$V = \{001, 100, 122, 220, 112, 121, 120, 020, 201, 011, 202, 111, 021\}$$

- and where $\mathcal{B}$ consists of the following zero-sum blocks

$$\{001, 021, 202, 112\}, \quad \{021, 111, 011, 220\}, \quad \{111, 202, 201, 122\},$$
$$\{202, 011, 020, 100\}, \quad \{011, 201, 120, 001\}, \quad \{201, 020, 121, 021\},$$
$$\{020, 120, 112, 111\}, \quad \{120, 121, 220, 202\}, \quad \{121, 112, 122, 011\},$$
$$\{112, 220, 100, 201\}, \quad \{220, 122, 001, 020\}, \quad \{122, 100, 021, 120\},$$
$$\{100, 001, 111, 121\}$$

- There is a $(143, 71, 35)$ difference set $\Rightarrow$ cyclic symmetric $(143, 71, 35)$ design
- The prime divisor of the order $k - \lambda = 71 - 35 = 36 = 2^2 \cdot 3^2$ are 2 and 3
- $ord_{143}(2) = 60$
- $ord_{143}(3) = 15$

Example

The cyclic symmetric $(143, 71, 35)$ design is additive under $\mathbb{Z}_2^{60}$ and under $\mathbb{Z}_3^{15}$ at the same time.

[Cageggi, Falcone, Pavone, 2017]

Parameters	Group	Strongly	Strictly	Almost str.	Description
(v, k, λ)	G	✓			symmetric design
(v, k, λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k - \lambda \nmid k$, prime

- New examples [Buratti, A.N., 2023, 202?]

Parameters	Group	Strongly	Strictly	Al. str.	Description
$(p^n, p, 1)$	$\mathbb{F}_{p^n}$		✓		$p \in \{5,7\}$, $n \geq 3$, not isomorphic to $AG_1(n,p)$
$(v, k, 1)$	$G \times \mathbb{F}_q$		✓		$k \not\equiv 2 \pmod 4$, $k \neq 2^3 \geq 12$
$([n]_q, [d]_q, 1)$	$\mathbb{Z}_{q^d}^{[n+1]_q}$	✓			$PG_d(n,q)$
$([n]_q, [d]_q, 1)$	$\mathbb{F}_q^{n+1}$				$PG_d(n,q)$
(v, k, λ)	$\mathbb{Z}_{k-\lambda}^v$	✓			symmetric design
(v, k, λ)	$\mathbb{Z}_p^t$				cyclic symmetric design, p a prime dividing $k-\lambda$ but not v, $t = ord_v(p)$.
$(4\lambda+3, 2\lambda+1, \lambda)$	$\mathbb{Z}_p^t$				Paley design, $v = 4\lambda+3$ prime, p prime divisor of $\lambda+1$, $t = ord_v(p)$
$(4\lambda+3, 2\lambda+1, \lambda)$	$\mathbb{Z}_2^t$			✓	$v = 2^t - 1$ is a Mersenne prime

Thank you for your attention!