New constructions of higher dimensional Hadamard matrices and SBIBDs*

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(joint work with Mario Osvin Pavčević and Kristijan Tabak)

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Outline

1. What is a higher dimensional design?
2. A brief survey
3. Known constructions
4. Paley-type Hadamard matrices
5. Higher dimensional SBIBDs

What is a higher dimensional design?

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A Hadamard matrix:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

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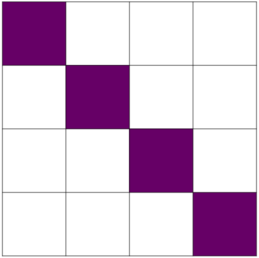

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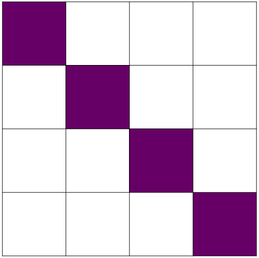

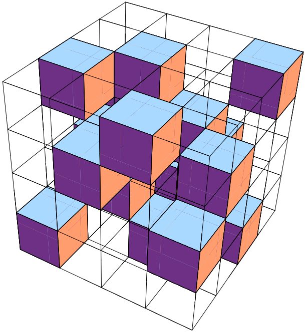

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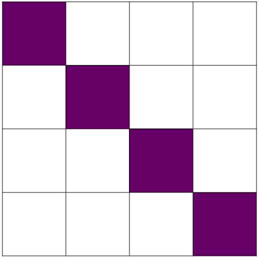

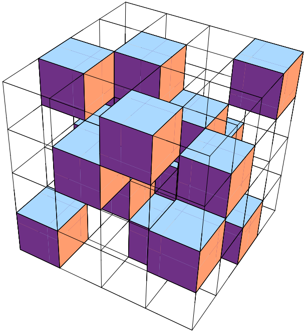

A Hadamard matrix of dimension n and order v is a function

$$H : \{1, \ldots, v\}^n \to \{-1, 1\}$$

such that all $(n-1)$-dimensional parallel layers are mutually orthogonal:

$$\sum_{1 \le i_1, \ldots, \widehat{i_j}, \ldots, i_n \le v} H(i_1, \ldots, a, \ldots, i_n) H(i_1, \ldots, b, \ldots, i_n) = v^{n-1} \delta_{ab}$$

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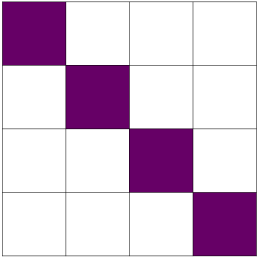
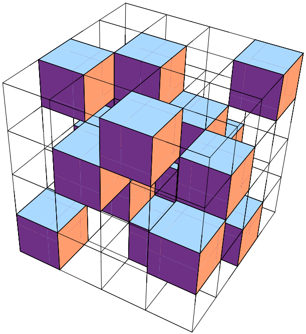




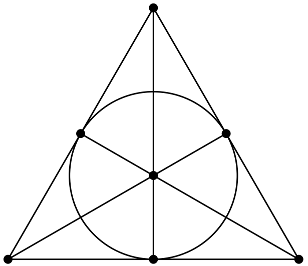
A **proper** Hadamard matrix of dimension n and order v is a function

$$H : \{1, \ldots, v\}^n \to \{-1, 1\}$$

such that all 2-dimensional layers have orthogonal rows and columns (i.e. are Hadamard matrices in the usual sense).

What is a higher dimensional design?

A symmetric design (SBIBD):



What is a higher dimensional design?

A symmetric design (SBIBD):

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

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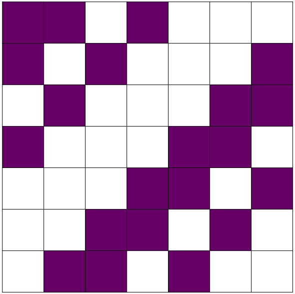

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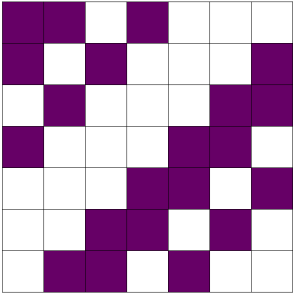

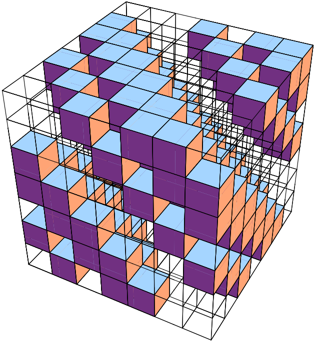

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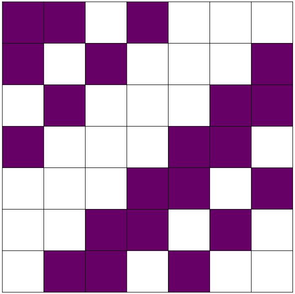

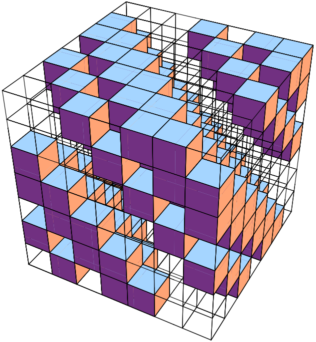

An *n-dimensional cube of symmetric (v, k, λ) designs* is a function

$$C : \{1, \ldots, v\}^n \to \{0, 1\}$$

such that all 2-dimensional layers ("slices") are incidence matrices of (v, k, λ) designs.

A brief survey

Paul J. Shlichta, *Three- and four-dimensional Hadamard matrices*, Bull. Amer. Phys. Soc. **16 (8)** (1971), 825–826.

Paul J. Shlichta, *Higher dimensional Hadamard matrices*, IEEE Trans. Inform. Theory **25** (1979), no. 5, 566–572.

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J. Seberry, *Higher-dimensional orthogonal designs and Hadamard matrices*, Combinatorial mathematics VII (Proc. Seventh Australian Conf., Univ. Newcastle, Newcastle, 1979), pp. 220–223, Lecture Notes in Math. **829**, Springer, Berlin, 1980.

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D. L. Flannery, *Cocyclic Hadamard matrices and Hadamard groups are equivalent*, J. Algebra **192** (1997), no. 2, 749–779.

W. de Launey, D. L. Flannery, K. J. Horadam, *Cocyclic Hadamard matrices and difference sets*, Coding, cryptography and computer security (Lethbridge, AB, 1998), Discrete Appl. Math. **102** (2000), no. 1-2, 47–61.

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Kenneth Ma, *Equivalence classes of n-dimensional proper Hadamard matrices*, Australas. J. Combin. **25** (2002), 3–17.

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K. J. Horadam, *Hadamard matrices and their applications*, Princeton University Press, Princeton, NJ, 2007.

W. de Launey, D. Flannery, *Algebraic design theory*, Mathematical Surveys and Monographs **175**, American Math. Society, Providence, RI, 2011.

Known constructions

Theorem (Y. X. Yang, 1986): Product construction

Let $h : \{1, \ldots, v\}^2 \to \{-1, 1\}$ be a Hadamard matrix of order v. Then

$$H(i_1, \ldots, i_n) = \prod_{1 \le j < k \le n} h(i_j, i_k)$$

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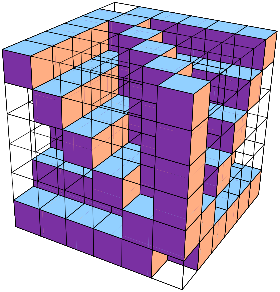

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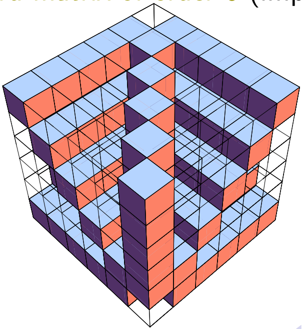

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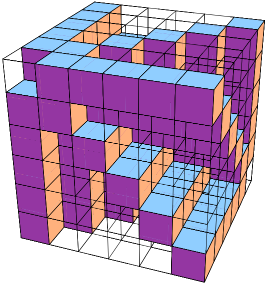

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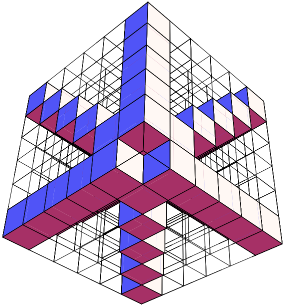

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If h is an n-dimensional Hadamard matrix of order v, then

$$H(i_1, \ldots, i_n, i_{n+1}) = h(i_1, \ldots, i_{n-1},\, i_n + i_{n+1} \bmod v)$$

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Theorem (Y. X. Yang): Digit construction

If h is a 2-dimensional Hadamard matrix of order $v = (2t)^s$, $s > 1$, then

$$H(i_0, \ldots, i_{s-1}, j_0, \ldots, j_{s-1}) =$$

$$= h(i_0 + (2t)i_1 + \ldots + (2t)^{s-1} i_{s-1},\ j_0 + (2t)j_1 + \ldots + (2t)^{s-1} j_{s-1})$$

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Corollary.

If the Hadamard conjecture is true, then n-dimensional Hadamard matrices exist for all even orders v and all $n\geq 4$.

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Existence: $v = 2, 6, 10, 14, 18, 22, 26, 30, 34, 38, 42, 46, 50, 54, 58, 62, \ldots$

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Questions (Y. X. Yang's book)

5. Prove or disprove the existence of 3-dimensional Hadamard matrices of orders $4k + 2 \neq 2 \cdot 3^m$.

6. Construct more 3-dimensional Hadamard matrices of orders $4k + 2$.

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R. E. A. C. Paley, *On orthogonal matrices*, Journal of Mathematics and Physics **12** (1933), 311–320.

Theorem (Paley, 1933)

Let q be an odd prime power. If $q \equiv 3 \pmod 4$, there is a Hadamard matrix of order $v = q + 1$. If $q \equiv 1 \pmod 4$, there is a Hadamard matrix of order $v = 2(q + 1)$.

Paley-type Hadamard matrices

$$\chi : \mathbb{F}_q^* \to \{1, -1\}, \quad \chi(a) = \begin{cases} 1, & \text{if } a \text{ is a square in } \mathbb{F}_q^* \\ -1, & \text{otherwise} \end{cases}$$

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Paley type I matrix: $q \equiv 3 \pmod 4$, indexed by $PG(1,\mathbb{F}_q) = \{\infty\} \cup \mathbb{F}_q$

$$H(x,y) = \begin{cases} -1, & \text{if } x = y = \infty \\ 1, & \text{if } x = y \neq \infty \text{ or } x = \infty \neq y \text{ or } y = \infty \neq x \\ \chi(y-x), & \text{otherwise} \end{cases}$$

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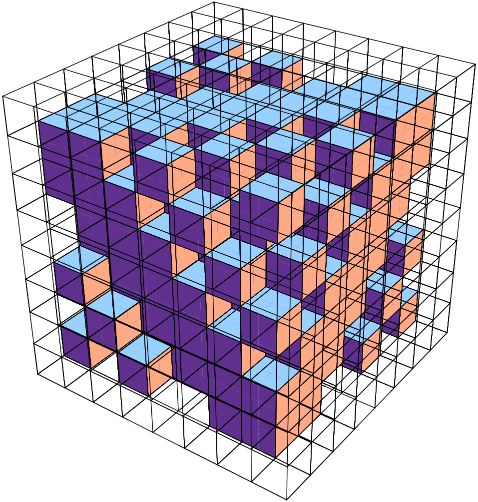
J. Hammer, J. R. Seberry, *Higher-dimensional orthogonal designs and applications*, IEEE Trans. Inform. Theory **27** (1981), no. 6, 772–779.

Paley cube: $q \equiv 3 \pmod 4$, $\chi(0) = -1$

$$H(x_1, \ldots, x_n) = \begin{cases} 1, & \text{if } x_i = \infty \text{ for at least one } i, \\ \chi(x_1 + \ldots + x_n), & \text{otherwise} \end{cases}$$

Paley-type Hadamard matrices

$n = 3,\ q = 7$:



Paley-type Hadamard matrices

V. Krčadinac, M. O. Pavčević, K. Tabak, *Three-dimensional Hadamard matrices of Paley type*, 2023. https://arxiv.org/abs/2305.12415

$H : PG(1,q)^3 \to \{1,-1\}$, $q \equiv 1 \text{ or } 3 \pmod 4$,

$$H(x,y,z) = \begin{cases} -1, & \text{if } x = y = z, \\ 1, & \text{if } x = y \neq z \\ & \text{ or } x = z \neq y \\ & \text{ or } y = z \neq x, \\ \chi(z-y), & \text{if } x = \infty, \\ \chi(x-z), & \text{if } y = \infty, \\ \chi(y-x), & \text{if } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{otherwise} \end{cases}$$

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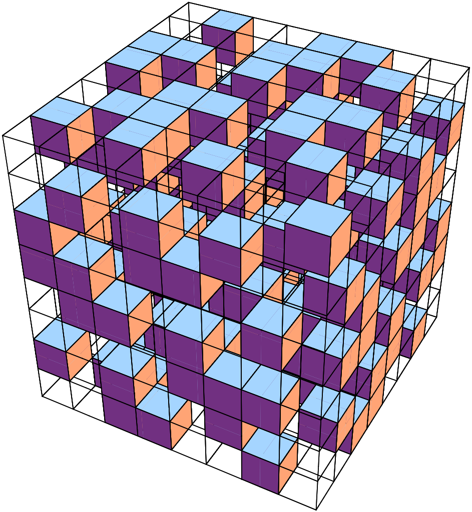
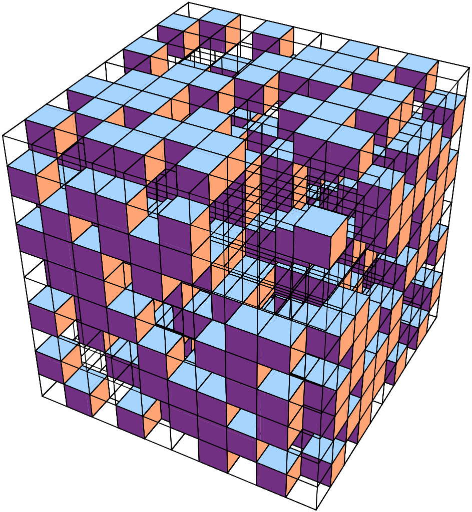


Paley-type Hadamard matrices

$n = 3$, $q = 9$:



Paley-type Hadamard matrices

Theorem (V.K., M.O.Pavčević, K.Tabak, 2023)

$H(x, y, z)$ is a 3-dimensional Hadamard matrix of order $v = q + 1$ for every odd prime power q. If $q \equiv 3 \pmod{4}$, it is proper with all 2-dimensional layers equivalent to the Paley type I matrix.

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Higher dimensional SBIBDs

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Theorem: Difference cubes

Let $G = \{g_1, \ldots, g_v\}$ be a group of order v and $D \subseteq G$ a (v, k, λ) difference set. Then $C : \{1, \ldots, v\}^n \to \{0, 1\}$, $C(i_1, \ldots, i_n) = [g_{i_1} \cdots g_{i_n} \in D]$ is an n-dimensional cube of symmetric (v, k, λ) designs.

Here and in the sequel, [.] is the Iverson bracket.

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For parameters $(25, 9, 3)$ there are exactly 78 designs, but no difference sets.

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For parameters $(25, 9, 3)$ there are exactly 78 designs, but no difference sets. **Are there higher dimensional cubes of $(25, 9, 3)$ designs?**

Higher dimensional SBIBDs

The constructions described so far (known and new) give rise to n-dimensional designs such that **all** 2-**dimensional slices are equivalent**.

Higher dimensional SBIBDs

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Example. There are three $(16, 6, 2)$ designs:

$$|\operatorname{Aut}(\mathcal{D}_1)| = 11520, \quad |\operatorname{Aut}(\mathcal{D}_2)| = 768, \quad |\operatorname{Aut}(\mathcal{D}_3)| = 384$$

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The three designs can be obtained from difference sets in some of the 14 groups of order 16.

Higher dimensional SBIBDs

ID	Structure	Nds	dev D
1	$\mathbb{Z}_{16}$	0	–
2	$\mathbb{Z}_4^2$	3	$\mathcal{D}_1$
3	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	4	$\mathcal{D}_1$
4	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	3	$\mathcal{D}_1$
5	$\mathbb{Z}_8 \times \mathbb{Z}_2$	2	$\mathcal{D}_1$, $\mathcal{D}_2$
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	2	$\mathcal{D}_1$
7	D_{16}	0	–
8	QD_{16}	2	$\mathcal{D}_1$
9	Q_{16}	2	$\mathcal{D}_1$
10	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	2	$\mathcal{D}_1$
11	$\mathbb{Z}_2 \times D_8$	2	$\mathcal{D}_1$
12	$\mathbb{Z}_2 \times Q_8$	2	$\mathcal{D}_1$, $\mathcal{D}_3$
13	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	2	$\mathcal{D}_1$, $\mathcal{D}_3$
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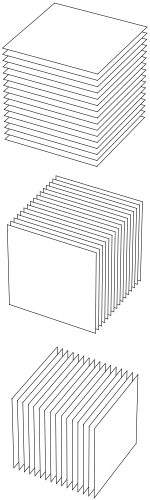
Slices:



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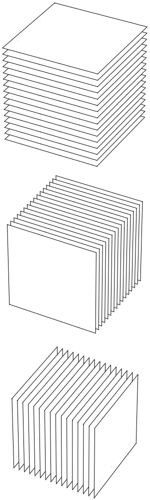
dev $D = \mathcal{D}_1$



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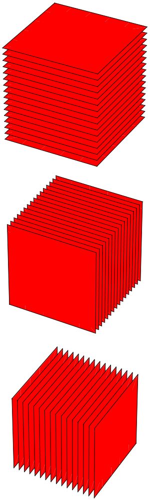

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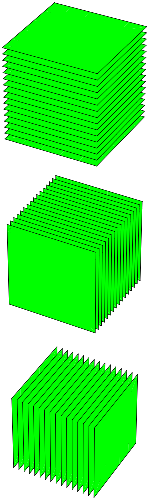
$\text{dev } D = \mathcal{D}_2$



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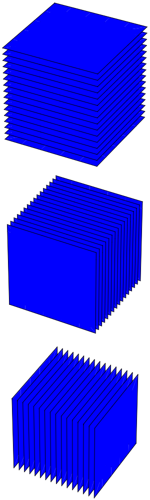

Higher dimensional SBIBDs

Theorem: Group cubes

If $G = \{g_1, \ldots, g_v\}$ is a group and $\mathcal{D} = \{B_1, \ldots, B_v\}$ is a (v, k, λ) design such that all blocks are difference sets, then $C(i_1, \ldots, i_n) = [g_{i_2} \cdots g_{i_n} \in B_{i_1}]$ is an n-dimensional cube of symmetric designs.

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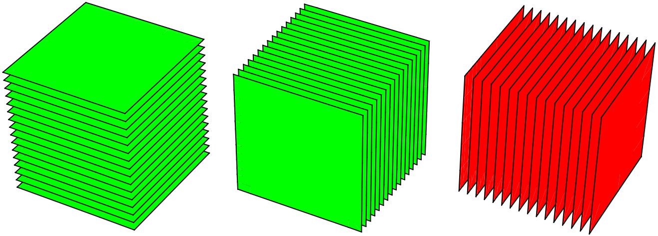

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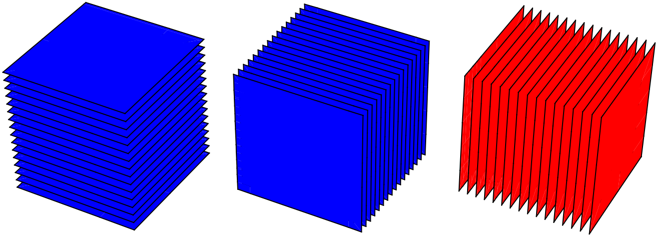

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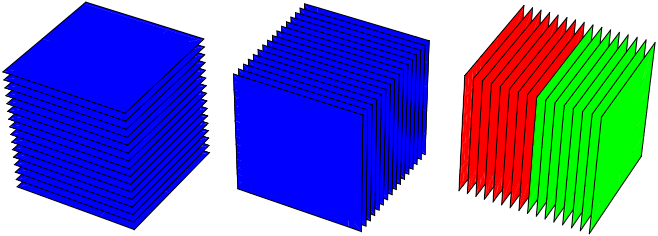

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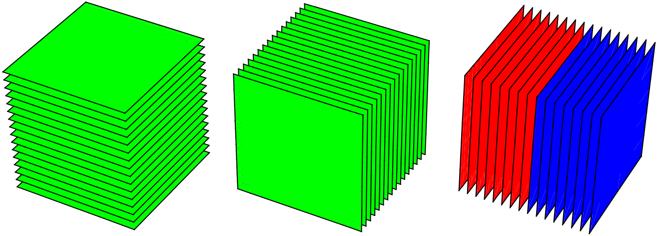

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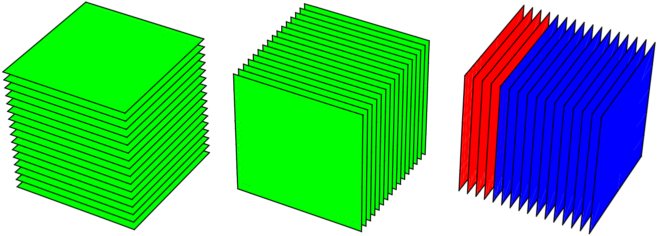

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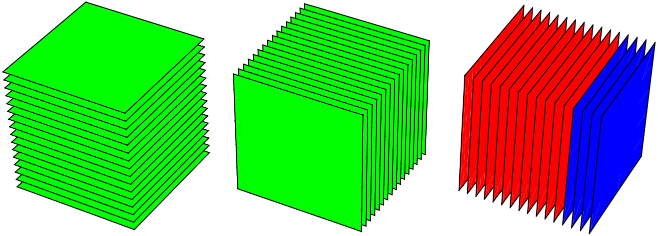

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The parameters are of Menon type. Thus, by exchanging $0 \to -1$ the cubes are transformed to n-dimensional Hadamard matrices with inequivalent slices. These could not have been obtained by previously known construction.

Higher dimensional SBIBDs

Parameters	Nds	Ndc	Ngc
$(27, 13, 6)$	3	2	≥ 7
$(36, 15, 6)$	35	35	≥ 373
$(45, 12, 3)$	2	2	≥ 6
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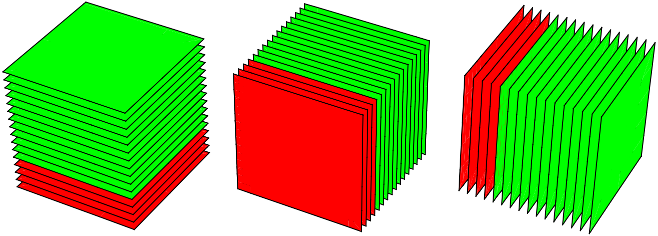

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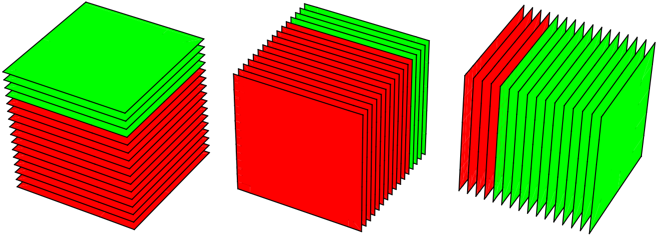

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Non-group cubes?

Proposition.

The set $\mathcal{C}^3(16, 6, 2)$ contains at least 1423 inequivalent non-group cubes.

That's all!

Combinatorial Constructions Conference (CCC) will take place at the Centre for Advanced Academic Studies in Dubrovnik, Croatia.

April 7-13, 2024

Invited speakers:

- Marco Buratti, Italy
- Eimear Byrne, Ireland
- Dean Crnković, Croatia
- Daniel Horsley, Australia
- Michael Kiermaier, Germany
- Patric Östergård, Finland
- Kai-Uwe Schmidt, Germany

https://web.math.pmf.unizg.hr/acco/meetings.php