Normalized difference sets tiling - generalizations

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motivation to make a conjecture that difference sets in a tiling of an abelian group must be normalized

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Definition: Let G be an abelian group and $D \in G(v,k,\lambda)_{DS}$. Let $\sigma : G \to G$ be a map given by $g^\sigma = g^t$ where $(t,v) = 1$. If $D^\sigma = sD$ for some $s \in G$, then σ is called a (numerical) multiplier of D.

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Theorem: Let G be an abelian group of order v. Let p be a prime that divides $k-\lambda$ and $(p,v) = 1$ where $p > \lambda$. If $D \in G(v,k,\lambda)_{DS}$ then a map $\sigma(x) = x^p$ is a multiplier of D. Furthermore, there is always a difference set from $\mathcal{D}ev(D)$ that is fixed by a multiplier σ.

Lemma (shifted orbits): Let $G = \langle x \rangle \cong \mathbb{Z}_p$ for an odd prime p and $\psi \in Aut(G)$ is an automorphism of order m. Then there is a decomposition of G^* in ψ-orbits which can be represented in a group ring $\mathbb{Z}[G]$ as $G^* = \sum_{i=1}^{(p-1)/m} x_i^{\langle\psi\rangle}$ where $x_i^{\langle\psi\rangle} = x_i + x_i^{\psi} + \cdots + x_i^{\psi^{m-1}}$.

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It is possible to prove even more general result

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Theorem: Let $G = \prod_{i=1}^{s} \langle x_i \rangle$, where $\langle x_i \rangle \cong \mathbb{Z}_{p_i^{n_i}}$ and p_i odd prime for $i \in [s]$. Let $G^* = \sum_{j=1}^{t} g_j^{\langle \varphi_j \rangle}$ for $\varphi_j \in Aut(G)$. If $G^* = \sum_{j=1}^{t} h_j g_j^{\langle \varphi_j \rangle}$, then $h_j = 1$ for all $j \in [t]$.

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Questions?