

# Normalized difference sets tiling - generalizations

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**Norcom 2022, 14th Nordic Combinatorial Conference,  
Tromsø, June 7-9 2022**

This work has been fully supported by  
Croatian Science Foundation under the projects 6732 and 9752



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motivation to make a conjecture that difference sets in a tiling of an abelian group must be normalized



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**Theorem:** Let  $G$  be an abelian group of order  $v$ . Let  $p$  be a prime that divides  $k - \lambda$  and  $(p, v) = 1$  where  $p > \lambda$ . If  $D \in G(v, k, \lambda)_{DS}$  then a map  $\sigma(x) = x^p$  is a multiplier of  $D$ . Furthermore, there is always a difference set from  $\mathcal{Dev}(D)$  that is fixed by a multiplier  $\sigma$ .



**Lemma (shifted orbits):** Let  $G = \langle x \rangle \cong \mathbb{Z}_p$  for an odd prime  $p$  and  $\psi \in \text{Aut}(G)$  is an automorphism of order  $m$ . Then there is a decomposition of  $G^*$  in  $\psi$ -orbits which can be represented in a group

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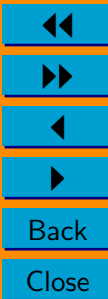
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General abelian case...





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So far, the main assumption is that  $\langle \psi \rangle \curvearrowright \text{Dev}(D) = \{gD \mid g \in G\}$ . This means that  $\psi$  operates on  $\text{Dev}(D)$ .

Also,  $\psi$  fixes some  $gD$ , i.e.  $\text{Fix}(\psi, \text{Dev}(D)) \neq \emptyset$ .





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The question is what if  $G$  is abelian group of order  $v$  and  $D$  is  $(v, k, \lambda)$  difference set such that

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It is possible to prove even more general result



**Theorem:** Let  $G = \prod_{i=1}^s \langle x_i \rangle$ , where  $\langle x_i \rangle \cong \mathbb{Z}_{p_i^{n_i}}$  and  $p_i$  odd prime for  $i \in [s]$ . Let  $G^* = \sum_{j=1}^t g_j^{\langle \varphi_j \rangle}$  for  $\varphi_j \in \text{Aut}(G)$ . If  $G^* = \sum_{j=1}^t h_j g_j^{\langle \varphi_j \rangle}$ , then  $h_j = 1$  for all  $j \in [t]$ .



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$TG(v, k, \lambda)_n$  = tiling of  $G$  made of  $(v, k, \lambda)$  normalized difference sets in  $G$ , all from  $G(v, k, \lambda)_{NDS}$



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Questions?

