On t-designs with three intersection numbers*

Vedran Krčadinac

(Joint work with Renata Vlahović Kruc)

University of Zagreb, Croatia

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Combinatorial designs

"The concept of combinatorial t-design is to find subsets which approximate the whole space $\binom{V}{k}$ i.e., the set of k-element subsets of a set V of cardinality $|V| = v$."

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The strength of a design is the largest t for which it is a t-design.

The degree of a design

The degree of a design is the number of different block intersection cardinalities:

$$d = \left|\{\, |B_1 \cap B_2| \;:\; B_1, B_2 \in \mathcal{B}, B_1 \neq B_2\}\right|.$$

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Equality $t = 2$ holds for the symmetric designs, characterized by $v = b$. The single intersection number is λ.

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1. $v = 4\lambda + 3$, $k = 2\lambda + 1$ (Hadamard designs)
2. $v = (\lambda + 2)(\lambda^2 + 4\lambda + 2)$, $k = \lambda^2 + 3\lambda + 1$
3. $v = 495$, $k = 39$, $\lambda = 3$

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A. E. Brouwer, H. Van Maldeghem, *Strongly regular graphs*, 2021. $\rightsquigarrow v \leq 100$

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$t = 4$

V. Krčadinac, R. Vlahović Kruc, *Schematic 4-designs*, preprint, 2022.

Schematic designs

Theorem (Cameron, Delsarte, 1973)

The blocks of a design of degree d and strength $t \geq 2d - 2$ form a symmetric association scheme with d classes.

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$d = 2,\ t = 2 \rightsquigarrow$ The block graph of a QSD is strongly regular.

$d = 3,\ t = 4 \rightsquigarrow$ The blocks form a 3-class association scheme.

Designs with $d = 3$, $t = 4$

Theorem (V.K., R. Vlahović Kruc)

The association scheme of a 4-(v,k,λ) design with three intersection numbers $x < y < z$ has the following eigenvalues:

$$p_1(j) = \frac{yz\theta_0(j) + (1-y-z)\theta_1(j) + 2\theta_2(j) - (y-k)(z-k)}{(y-x)(z-x)},$$

$$p_2(j) = \frac{xz\theta_0(j) + (1-x-z)\theta_1(j) + 2\theta_2(j) - (x-k)(z-k)}{(x-y)(z-y)},$$

$$p_3(j) = \frac{xy\theta_0(j) + (1-x-y)\theta_1(j) + 2\theta_2(j) - (x-k)(y-k)}{(x-z)(y-z)},$$

$$\theta_i(j) = \frac{b}{\binom{v}{k}}\binom{v-i-j}{v-k-j}\binom{k-j}{i-j} = \frac{\lambda}{\binom{v-4}{k-4}}\binom{v-i-j}{v-k-j}\binom{k-j}{i-j}$$

with multiplicities $m_j = \begin{cases} \binom{v}{j} - \binom{v}{j-1}, & \text{for } j = 0,1,2, \\ b - \binom{v}{2}, & \text{for } j = 3. \end{cases}$

Designs with $d = 3$, $t = 4$

No.	v	k	λ	x	y	z	$\exists$
1	11	5	1	1	2	3	
2	23	8	4	0	2	4	
3	23	11	48	3	5	7	
4	24	8	5	0	2	4	
5	47	11	8	1	3	5	
6	71	35	264	14	17	20	
7	199	99	2328	44	49	54	
8	391	195	9264	90	97	104	
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QR codes + Assmus-Mattson theorem.

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Chapter 14: Quadratic residue codes and the Assmus-Mattson theorem.

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E. F. Assmus, Jr., H. F. Mattson, Jr., *New* 5*-designs*, J. Combinatorial Theory **6** (1969), 122–151.

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Designs with $d = 3$, $t = 4$

Infinite series of admissible parameters:

$$\begin{aligned} v &= 8n^2 - 1 \\ k &= 4n^2 - 1 = (2n-1)(2n+1) \\ \lambda &= 4n^4 - 7n^2 + 3 = (n-1)(n+1)(4n^2-3) \\ x &= 2n^2 - n - 1 = (n-1)(2n+1) \\ y &= 2n^2 - 1 \\ z &= 2n^2 + n - 1 = (n+1)(2n-1) \end{aligned}$$

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$$p_{33}^3 = \frac{1}{2}(n+1)(2n+3)(4n^2-2n-1)$$

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3. $x > 0$: possible infinite family related to $AG_{n-1}(n + 1, 2)$

The End

Thanks for your attention!