Designing and Combining Designs [1]

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Definition of t-designs

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A t-(v, k, λ) *design* $\mathcal{D}$ consists of a v-element set of *points* $\mathcal{P}$ together with a collection $\mathcal{B}$ of k-element subsets called *blocks*, such that every t-element subset of points is contained in exactly λ blocks.

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Following facts

- The number of elements in $\mathcal{B}$ can be computed using the given parameter set t-(v, k, λ) and equals:

$$b = |\mathcal{B}| = \lambda \binom{v}{t} / \binom{k}{t}$$

- Each point lies in exactly r blocks,

$$r = \lambda \binom{v-1}{t-1} / \binom{k-1}{t-1}$$

Examples

Example

For a given v-element set of points $\mathcal{P}$, let $\mathcal{B}$ be the set of all k-element subsets of $\mathcal{P}$,

$$\mathcal{B} = \binom{\mathcal{P}}{k}.$$

We get a k-$(v, k, 1)$ design in a trivial way.

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Example

Let $\mathcal{P} = \{1, 2, 3, 4, 5, 6, 7\}$, and let the set of blocks be

$$\mathcal{B} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{1, 5, 6\}, \{2, 6, 7\}, \{1, 3, 7\}\}$$

These blocks form a 2-$(7, 3, 1)$ design, called the Fano plane.

Another presentation

Definition

Denote the point set of the design $\mathcal{D}$ by $\mathcal{P} = \{1, 2, \ldots, v\}$ and the block set by $\mathcal{B} = \{1, 2, \ldots, b\}$. The 0-1 matrix $A = [a_{ij}]$ of type $v \times b$ with elements

$$a_{ij} = \begin{cases} 1, & \text{if } i \in j \\ 0, & \text{otherwise} \end{cases}$$

is called the incidence matrix of $\mathcal{D}$.

Example

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Smallest open problems

According to the

Handbook of Combinatorial Designs
second edition, 2006
(Tables with existence results by
R. Mathon and **A. Rosa** as well as by
G. B. Khosrovshahi and **R. Laue**),

the smallest parameters for which the existence is still open are:

- For 2-designs: 2-(51,6,1) or 2-(40,10,3) or 2-(39,13,6)
- For symmetric 2-designs: 2-(157,13,1) or 2-(81,16,3)
- For 3-designs: 3-(16,7,5)
- For 4-designs: 4-(13,6,6)

Goal: explicit constructions

Main problem:

Natural size of the search space for a $t-(v,k,\lambda) design : \binom{\binom{v}{k}}{b}$

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Example

In case of 3-$(16,7,5)$ this number equals

$$\binom{\binom{16}{7}}{80} = \binom{11440}{80} = \text{ HUGE}$$

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Techniques (too often not successful):

Computer search; Adding constraints (group actions); Heuristic search; ...

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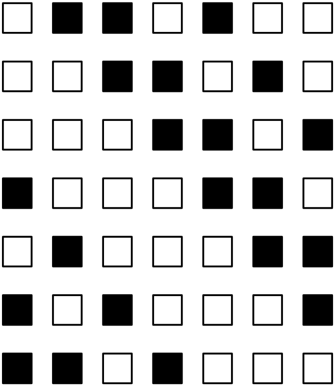
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \square & \blacksquare & \blacksquare & \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \blacksquare & \square & \blacksquare & \square \\ \square & \square & \square & \blacksquare & \blacksquare & \square & \blacksquare \\ \blacksquare & \square & \square & \square & \blacksquare & \blacksquare & \square \\ \square & \blacksquare & \square & \square & \square & \blacksquare & \blacksquare \\ \blacksquare & \square & \blacksquare & \square & \square & \square & \blacksquare \\ \blacksquare & \blacksquare & \square & \blacksquare & \square & \square & \square \end{bmatrix}$$

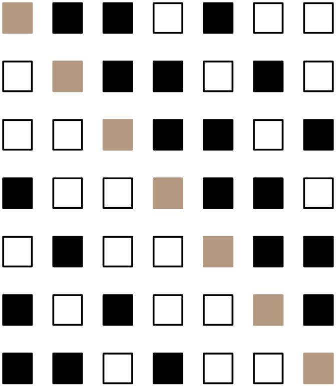
What to do? - Combine designs!

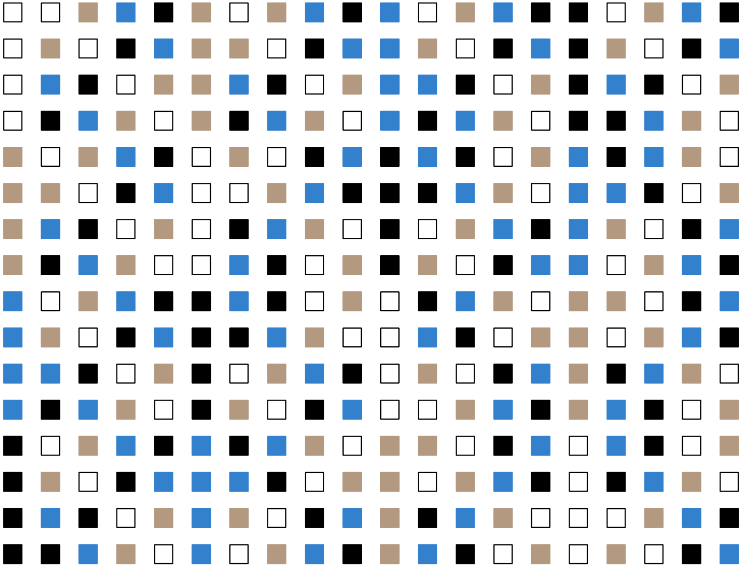
$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \square & \blacksquare & \blacksquare & \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \blacksquare & \square & \blacksquare & \square \\ \square & \square & \square & \blacksquare & \blacksquare & \square & \blacksquare \\ \blacksquare & \square & \square & \square & \blacksquare & \blacksquare & \square \\ \square & \blacksquare & \square & \square & \square & \blacksquare & \blacksquare \\ \blacksquare & \square & \blacksquare & \square & \square & \square & \blacksquare \\ \blacksquare & \blacksquare & \square & \blacksquare & \square & \square & \square \end{bmatrix}$$

$$\begin{array}{ccccccc} \square & \blacksquare & \blacksquare & \square & \blacksquare & \square & \square \\ \square & \square & \blacksquare & \blacksquare & \square & \blacksquare & \square \\ \square & \square & \square & \blacksquare & \blacksquare & \square & \blacksquare \\ \blacksquare & \square & \square & \square & \blacksquare & \blacksquare & \square \\ \square & \blacksquare & \square & \square & \square & \blacksquare & \blacksquare \\ \blacksquare & \square & \blacksquare & \square & \square & \square & \blacksquare \\ \blacksquare & \blacksquare & \square & \blacksquare & \square & \square & \square \end{array}$$

Mosaics







0	0	1	2	3	1	0	1	2	3	2	0	1	2	3	3	0	1	2	3
0	1	0	3	2	1	1	0	3	2	2	1	0	3	2	3	1	0	3	2
0	2	3	0	1	1	2	3	0	1	2	2	3	0	1	3	2	3	0	1
0	3	2	1	0	1	3	2	1	0	2	3	2	1	0	3	3	2	1	0
1	0	1	2	3	0	1	0	3	2	3	2	3	0	1	2	3	2	1	0
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1	2	3	0	1	0	3	2	1	0	3	0	1	2	3	2	1	0	3	2
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2	0	1	2	3	3	2	3	0	1	0	3	2	1	0	1	1	0	3	2
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3	1	0	3	2	2	2	3	0	1	1	0	1	2	3	0	3	2	1	0
3	2	3	0	1	2	1	0	3	2	1	3	2	1	0	0	0	1	2	3
3	3	2	1	0	2	0	1	2	3	1	2	3	0	1	0	1	0	3	2

Constructions

Series of mosaics:

- "Design and its complement"
  (incidence matrix of every 2-design)

$$2\text{-}(v, k, \lambda) \oplus 2\text{-}(v, v-k, b-2r+\lambda)$$

- "Hadamard with diagonal"

$$2\text{-}(v, 1, 0) \oplus 2\text{-}(v, \frac{v-1}{2}, \frac{v-3}{4}) \oplus 2\text{-}(v, \frac{v-1}{2}, \frac{v-3}{4})$$

- "Affine planes"

$$2\text{-}(q^2, q, 1) \oplus \cdots \oplus 2\text{-}(q^2, q, 1)$$

Theorem

Let D be the incidence matrix of a resolvable t-(v, k, λ) design, where the columns have been arranged by parallel classes. Let L be a latin square of order $\frac{v}{k}$ with entries $l_1, \ldots, l_{\frac{v}{k}}$. Then $M := D(I_r \otimes L)$ is a $\frac{v}{k}$-mosaic.

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So, a further construction series is the following:

- "Resolvable designs"

$$\bigoplus_{i=1}^{\frac{v}{k}} t\text{-}(v,k,\lambda)$$

What about **sporadic examples**?

Is there a $2\text{-}(31,6,1) \oplus 2\text{-}(31,10,3) \oplus 2\text{-}(31,15,7)$ mosaic?

- All integrality conditions are satisfied.
- Not easy to find. My favorite open problem!

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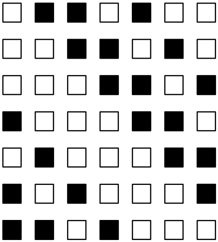
Reference

O. W. Gnilke, M. Greferath, M. O. Pavčević,
Mosaics of combinatorial designs
Des. Codes Cryptogr. 86 (2018), 85–95.

Design cubes

Lets have a look at another way of combining designs.



Take this indicence matrix of the Fano plane as the first layer of a $7 \times 7 \times 7$ cube.

Construction of design cubes

Now, construct the second layer of the cube by permuting the rows cyclically, starting with the second row of the incidence matrix.

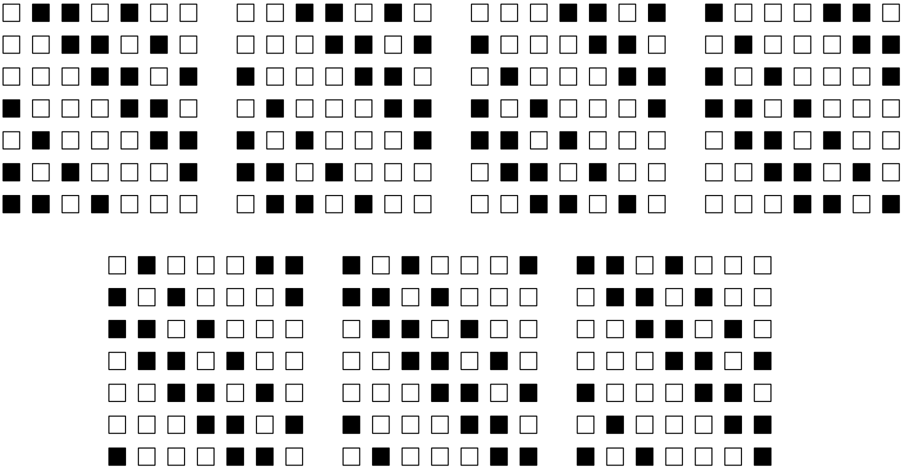
1. layer

□	■	■	□	■	□	□
□	□	■	■	□	■	□
□	□	□	■	■	□	■
■	□	□	□	■	■	□
□	■	□	□	□	■	■
■	□	■	□	□	□	■
■	■	□	■	□	□	□

2. layer

□	□	■	■	□	■	□
□	□	□	■	■	□	■
■	□	□	□	■	■	□
□	■	□	□	□	■	■
■	□	■	□	□	□	■
■	■	□	■	□	□	□
□	■	■	□	■	□	□

Continuing this procedure, you get all the 7 layers:



Definition of design cubes

Definition

A 3-dimensional binary matrix $A = [a_{ijm}]$ of order v is called a **2-(v, k, λ) design cube**, or simply a design cube, if all submatrices

$$\begin{aligned} A_3 &= [a_{ijm}], \quad m \text{ fixed;} \\ A_2 &= [a_{ijm}], \quad j \text{ fixed;} \\ A_1 &= [a_{ijm}], \quad i \text{ fixed;} \end{aligned}$$

are incidence matrices of a symmetric 2-(v, k, λ) design.

Results so far

Theorem

If there is a 2-(v, k, λ) *difference set, then there is a* 2-(v, k, λ) *design cube.*

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Proof.

It can be proven that the construction explained before works always, for all groups of order v, satisfying the desired condition in all 3 "directions" (for fixed i, j and m). $\square$

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Theorem

If there is a 2-(v, k, λ) *difference set, then there is a* 2-(v, k, λ) *design cube.*

Proof.

It can be proven that the construction explained before works always, for all groups of order v, satisfying the desired condition in all 3 "directions" (for fixed i, j and m). □

Main question

What about the other direction of the statement above? Or alternatively, does any design cube come from a difference set?

Thanks for your attention!