



Cubes of designs

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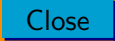
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(joint work with M.O. Pavčević and V. Krčadinac)



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write group G in a group ring $\mathbb{Z}[G]$ as $G = \sum_{s=1}^v g_s$ where $g_1 = 1$ (unit

in a group G)



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$$\delta_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}$$





Theorem: Let $G = \sum_{s=1}^v g_s$, $g_1 = 1$ be a group of order v with a (v, k, λ) difference set D . Let $A = [a_{ijm}] \in \mathcal{M}_{v \times v \times v}$ be a 3-dimensional matrix defined by $a_{ijm} = \delta_{g_j g_i D}(g_m)$ for all $i, j, m \in [v]$. Then the following holds:

1. A is a cube of a (v, k, λ) symmetric design i.e. $A \in \mathcal{C}(v, k, \lambda)$,
2. A_x^i is an incidence matrix of a symmetric design $(G, Dev(g_i D))$,
3. $[A_y^m]_t = g_t^{-1} g_m D^{(-1)}$
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Proposition: Let $A \in \mathcal{C}(v, k, \lambda)$ be a cube constructed via difference set $D \subseteq G$. Let ψ be a permutation of G . Let A^ψ be a 3-dimensional matrix such that $(A^\psi)_{ijm} = \delta_{g_j^\psi g_i D}(g_m)$. Then A is a cube, i.e. $A^\psi \in \mathcal{C}(v, k, \lambda)$ and $[(A^\psi)_y^m]_t = (g_t^{-1})^\psi g_m D^{(-1)}$ and $[(A^\psi)_z^m]_t = g_m^\psi g_t D$.





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Theorem: Let $A \in \mathcal{C}(v, k, \lambda)$. Then for every $i, m \in [v]$ designs A_x^i , A_z^m and A_y^m satisfy the following:

1. $A_x^i = (\mathcal{P}, \sum_{t=1}^v B_{x,t}^i) = (\mathcal{P}, \mathcal{B}_x^i)$,
2. $A_z^m = (\mathcal{P}, \sum_{t=1}^v B_{x,m}^t) = (\mathcal{P}, \mathcal{B}_z^m)$, meaning that the set of blocks of a design A_z^m is a set of m -th blocks of designs A_x^t for all $t \in [v]$,



3. $A_y^m = (\mathcal{P}_y^m, \sum_{t=1}^v \langle p_m \rangle_{A_z^t}) = (\mathcal{P}_y^m, \mathcal{B}_y^m)$, meaning that the t -th block of a design A_y^m is m -th dual block of a design A_z^t .



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Cyclic cubes generated by a symmetric design

Definition: Let $G = \sum_{i=1}^t g_i$ where $g_1 = 1$, be a group of order

v . Let (G, \mathcal{B}) be a (v, k, λ) symmetric design where $\mathcal{B} = \sum_{i=1}^v B_i$. Let

A_x^1 be an incidence matrix of a design (G, \mathcal{B}) . Let A_x^m be an incidence matrix of an incidence structure $(G, g_m \mathcal{B})$, where $g_m \mathcal{B} = \sum_{i=1}^v g_m B_i$. A

cyclic cube (generated by a symmetric design (G, \mathcal{B})) is a 3-dimensional matrix $A = (a_{ijm})$ such that $a_{ijm} = (A_x^i)_{jm}$ for all $i, j, m \in [v]$.





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Proposition: Let A be a cyclic cube generated by a (v, k, λ) symmetric design (G, \mathcal{B}) , where $G = \sum_{i=1}^{g_i}$ is a group. Then $|\langle T \rangle_{A_x^m}| = |\langle g_m^{-1}T \rangle_{A_x^1}|$, $m \in [v]$. A matrix A_x^m is an incidence matrix of a (v, k, λ) symmetric design for all $m \in [v]$.



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Thank you! Any Q's?

