Cubes of designs

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write group G in a group ring $\mathbb{Z}[G]$ as $G = \sum_{s=1}^{v} g_s$ where $g_1 = 1$ (unit in a group G)

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A set $D \subseteq G$ is a (v, k, λ) difference set in a group G if a multiset $\{d_1 d_2^{-1} \mid d_1, d_2 \in D,\ d_1 \neq d_2\}$ contains exactly λ copies of every $g \in G \setminus \{1\}$.

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If X is a set, then δ_X is a characteristic function of a set X defined by

$$\delta_X(x) = \begin{cases} 1, & \text{if } x \in X \\ 0, & \text{otherwise.} \end{cases}.$$

Theorem: Let $G = \sum_{s=1}^{v} g_s, \ g_1 = 1$ be a group of order v with a (v, k, λ) difference set D. Let $A = [a_{ijm}] \in \mathcal{M}_{v\times v\times v}$ be a 3-dimensional matrix defined by $a_{ijm} = \delta_{g_j g_i D}(g_m)$ for all $i, j, m \in [v]$. Then the following holds:

1. A is a cube of a (v, k, λ) symmetric design i.e. $A \in \mathcal{C}(v, k, \lambda)$,
2. A^i_x is an incidence matrix of a symmetric design $(G, \mathcal{D}ev(g_i D))$,
3. $[A^m_y]_t = g_t^{-1} g_m D^{(-1)}$
4. A^m_y is an incidence matrix of a symmetric design $(G, \sum_{t=1}^{v} g_t^{-1} g_m D^{(-1)})$,
5. $[A^m_z]_t = [A^t_x]_m$ for all $m, t \in [v]$,
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5. $[A_z^m]_t = [A_x^t]_m$ for all $m,t \in [v]$,
6. A_z^m is an incidence matrix of a symmetric design $(G, \sum_{t=1}^{v} g_m g_t D)$.

Proposition: Let $A \in \mathcal{C}(v,k,\lambda)$ be a cube constructed via difference set $D \subseteq G$. Let ψ be a permutation of G. Let A^ψ be a 3-dimensional matrix such that $(A^\psi)_{ijm} = \delta_{g_j^\psi g_i D}(g_m)$. Then A is a cube, i.e. $A^\psi \in \mathcal{C}(v,k,\lambda)$ and $[(A^\psi)_y^m]_t = (g_t^{-1})^\psi g_m D^{(-1)}$ and $[(A^\psi)_z^m]_t = g_m^\psi g_t D$.

Proposition: Let $A \in \mathcal{C}(v,k,\lambda)$ be a cube constructed via difference set $D \subseteq G$. Let ψ be a permutation of G. Let A^{ψ} be a 3-dimensional matrix such that $(A^{\psi})_{ijm} = \delta_{g_j^{\psi} g_i D}(g_m)$. Then A is a cube, i.e. $A^{\psi} \in \mathcal{C}(v,k,\lambda)$ and $[(A^{\psi})_y^m]_t = (g_t^{-1})^{\psi} g_m D^{(-1)}$ and $[(A^{\psi})_z^m]_t = g_m^{\psi} g_t D$.

Proposition: Let $A \in \mathcal{C}(v,k,\lambda)$ be a cube constructed via difference set $D \subseteq G$. Let ψ_m be a permutation of $G = \sum_{s=1}^{v} g_s$ given by $\psi_m(g) = g_m g$. Then ψ_m is an isomorphism between A_x^1 and A_x^m. Furthermore, $\psi_m(g_t D) = g_t^{g_m^{-1}} g_m D$, or in terms of rows of incidence matrices, $\psi_m([A_x^1]_t) = [A_x^m]_{\widetilde{t}}$, where $\widetilde{t} = g_t^{g_m^{-1}}$.

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Proposition: Let $A \in \mathcal{C}(v,k,\lambda)$ and $\mathcal{P} = \sum_{s=1}^{v} p_s$ is a set of points of designs A_x^i. Then $[A_y^m]_t = \sum_{s=1}^{v} \delta_{B_{x,t}^s}(p_m)p_s$ and $[A_z^m]_t = [A_x^t]_m = \sum_{s=1}^{v} \delta_{B_{x,m}^t}(p_s)p_s$ for all $m,t \in [v]$, where $[A_y^m]_t$ and $[A_z^m]_t$ are t-th blocks of designs A_y^m and A_z^m respectively.

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Theorem: Let $A \in \mathcal{C}(v,k,\lambda)$. Then for every $i, m \in [v]$ designs A_x^i, A_z^m and A_y^m satisfy the following:

1. $A_x^i = (\mathcal{P}, \sum_{t=1}^{v} B_{x,t}^i) = (\mathcal{P}, \mathcal{B}_x^i)$,
2. $A_z^m = (\mathcal{P}, \sum_{t=1}^{v} B_{x,m}^t) = (\mathcal{P}, \mathcal{B}_z^m)$, meaning that the set of blocks of a design A_z^m is a set of m-th blocks of designs A_x^t for all $t \in [v]$,

3. $A_y^m = (\mathcal{P}_y^m, \sum_{t=1}^{v} \langle p_m \rangle_{A_z^t}) = (\mathcal{P}_y^m, \mathcal{B}_y^m)$, meaning that the t-th block of a design A_y^m is m-th dual block of a design A_z^t.

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Definition: Let $G = \sum_{i=1}^{t} g_1$ where $g_1 = 1$, be a group of order v. Let $(G, \mathcal{B})$ be a (v, k, λ) symmetric design where $\mathcal{B} = \sum_{i=1}^{v} B_i$. Let A_x^1 ba an incidence matrix of a design $(G, \mathcal{B})$. Let A_x^m be an incidence matrix of an incidence structure $(G, g_m\mathcal{B})$, where $g_m\mathcal{B} = \sum_{i=1}^{v} g_m B_i$. A cyclic cube (generated by a symmetric design $(G, \mathcal{B})$) is a 3-dimensional matrix $A = (a_{ijm})$ such that $a_{ijm} = (A_x^i)_{jm}$ for all $i, j, m \in [v]$.

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Proposition: Let A be a cyclic cube generated by a (v, k, λ) symmetric design $(G, \mathcal{B})$, where $G = \sum_{i=1}^{g_i}$ is a group. Then $|\langle T \rangle_{A_x^m}| = |\langle g_m^{-1} T \rangle_{A_x^1}|$, $m \in [v]$. A matrix A_x^m is an incidence matrix of a (v, k, λ) symmetric design for all $m \in [v]$.

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Thank you! Any Q's?