On higher-dimensional designs*

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(joint work with Mario Osvin Pavčević and Kristijan Tabak)

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Pushing the Limits of Computational Combinatorial Constructions

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Motivation

$$A_1 = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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$$A_4 = \begin{pmatrix} 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1 \end{pmatrix} \quad A_5 = \begin{pmatrix} 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 0&0&1&1&0&1&0\\ 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1 \end{pmatrix} \quad A_7 = \begin{pmatrix} 0&1&1&0&1&0&0\\ 1&1&0&1&0&0&0\\ 1&0&1&0&0&0&1\\ 0&1&0&0&0&1&1\\ 1&0&0&0&1&1&0\\ 0&0&0&1&1&0&1\\ 0&0&1&1&0&1&0 \end{pmatrix}$$

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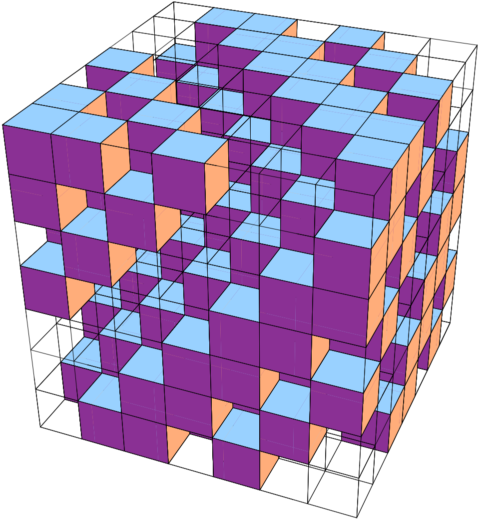

Definitions

A function $C : \{1, \ldots, v\}^n \to \{0, 1\}$ or $\{-1, 1\}$ is an *n-dimensional cube of symmetric (v, k, λ) designs*, or an *n-dimensional Hadamard matrix of order v* if all 2-dimensional slices are incidence matrices of (v, k, λ) designs, or Hadamard matrices of order v.

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The direct product of symmetric groups $(S_v)^n = S_v \times \ldots \times S_v$ acts on the set of all cubes $\mathcal{C}^n(v, k, \lambda)$ by isotopy, i.e. permuting indices:

$$C^\alpha(i_1, \ldots, i_n) = C(\alpha_1^{-1}(i_1), \ldots, \alpha_n^{-1}(i_n)), \text{ for } \alpha = (\alpha_1, \ldots, \alpha_n) \in (S_v)^n.$$

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Additionally, permutations $\gamma \in S_n$ act by conjugation, i.e. changing order of the indices: $C^\gamma(i_1, \ldots, i_n) = C(i_{\gamma^{-1}(1)}, \ldots, i_{\gamma^{-1}(n)})$.

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For Hadamard matrices, multiplication of hyperplanes by -1 is also allowed as an equivalence operation.

Known constructions

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Let $G = \{g_1, \ldots, g_v\}$ be a group of order v and $\chi : G \to \{-1, 1\}$ such that $h = (h_{ij})$, $h_{ij} = \chi(g_i \cdot g_j)$ is a group developed Hadamard matrix. Then $H : \{1, \ldots, v\}^n \to \{-1, 1\}$ is an n-dimensional Hadamard matrix:

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Analogous construction: let $D \subseteq G$ be a (v, k, λ) difference set. Then $A = (a_{ij})$, $a_{ij} = [g_i \cdot g_j \in D]$ is an incidence matrix of a symmetric design. Furthermore, $C : \{1, \ldots, v\}^n \to \{0, 1\}$ is a (v, k, λ) difference cube:

$$C(i_1, \ldots, i_n) = [g_{i_1} \cdots g_{i_n} \in D]$$

Here [.] is the Iverson bracket.

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The product construction:

If $h : \{1,\ldots,v\}^2 \to \{-1,1\}$ is a Hadamard matrix of order v, then

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Question 1. Is there an analogous construction for n-dimensional cubes of symmetric designs?

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Question 2. For parameters $(25, 9, 3)$ there are exactly 78 designs, but no difference sets. Is there a 3-cube of $(25, 9, 3)$ designs?

Known constructions

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A 2-cocycle from G to $\{-1,1\}$ is a function $f : G \times G \to \{-1,1\}$ satisfying $f(a,b)f(ab,c) = f(b,c)f(a,bc), \ \forall a,b,c \in G$.

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A cocyclic Hadamard matrix $h = (h_{ij})$ over $G = \{g_1, \ldots, g_v\}$ is of the form $h_{ij} = f(g_i, g_j)\chi(g_i \cdot g_j)$, for some $\chi : G \to \{-1,1\}$.

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Cocyclic Hadamard matrices need not be regular and are conjectured to exist for all orders $v = 4m$ by de Launey and Horadam (1993).

The cocyclic construction:

If $h_{ij} = f(g_i, g_j)\chi(g_i \cdot g_j)$ is a cocyclic Hadamard matrix of order v, then

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There are no nontrivial cocycles for (v, k, λ) designs, because the symbols 0 and 1 in their incidence matrices cannot be exchanged.

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Higher-dimensional Hadamard matrices obtained by both the product construction and the cocyclic construction have the property that **all slices are equivalent**. The same property holds for difference cubes of symmetric designs.

A new construction

V. Krčadinac, M. O. Pavčević, K. Tabak, *Cubes of symmetric designs*, preprint, 2023. http://arxiv.org/abs/2304.05446

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The group cube construction:

If $G = \{g_1, \ldots, g_v\}$ is a group and $\mathcal{D} = \{B_1, \ldots, B_v\}$ is a (v, k, λ) design such that all blocks are difference sets, then

$$H(i_1, \ldots, i_n) = [g_{i_2} \cdots g_{i_n} \in B_{i_1}]$$

is an n-dimensional cube of symmetric designs.

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Proof: If the first index i_1 is fixed, this is the $(n-1)$-dimensional difference cube of B_{i_1}, so all slices are equivalent with its development.

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If $\mathcal{D}$ is the development of its blocks, this is just the difference cube.

Designs with difference sets as blocks

Question 3. Are there symmetric designs with all blocks being (v, k, λ) difference sets in a group G, which are **not** developments?

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These 42 blocks designs give rise to equivalent group cubes.
The group cube is not a difference cube!

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$C_1 =$ difference cube in $\mathbb{Z}_{21}$

$C_2 =$ difference cube in F_{21}

$C_3 =$ non-difference group cube in F_{21}

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$C_1 =$ difference cube in $\mathbb{Z}_{21} \rightsquigarrow |\operatorname{Atop}(C_1)| = 2646$

$C_2 =$ difference cube in $F_{21} \rightsquigarrow |\operatorname{Atop}(C_2)| = 1323$

$C_3 =$ non-difference group cube in $F_{21} \rightsquigarrow |\operatorname{Atop}(C_3)| = 441$

Designs with difference sets as blocks

Question 3. Are there symmetric designs with all blocks being (v, k, λ) difference sets in a group G, which are **not** developments?

Example: $(21, 5, 1)$

$\mathbb{Z}_{21} \rightsquigarrow$ 42 difference sets $\rightsquigarrow$ 2 designs, both are developments

$F_{21} = \mathbb{Z}_7 \rtimes \mathbb{Z}_3 \rightsquigarrow$ 294 difference sets $\rightsquigarrow$ 70 designs $\rightsquigarrow$ 14 designs are left developments, 14 are right developments, and 42 are not developments

These 42 blocks designs give rise to equivalent group cubes.
The group cube is not a difference cube!

$C_1 =$ difference cube in $\mathbb{Z}_{21} \rightsquigarrow |\mathrm{Atop}(C_1)| = 2646$

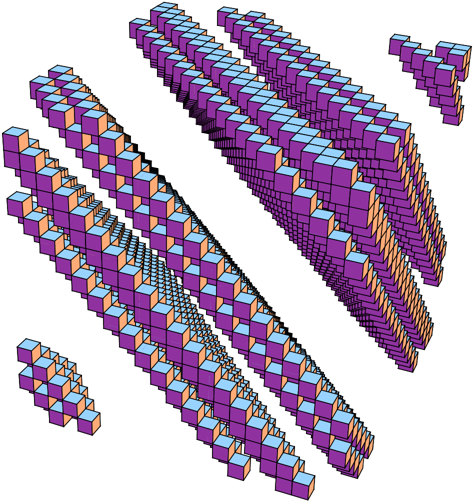
$C_2 =$ difference cube in $F_{21} \rightsquigarrow |\mathrm{Atop}(C_2)| = 1323$

$C_3 =$ non-difference group cube in $F_{21} \rightsquigarrow |\mathrm{Atop}(C_3)| = 441$

However, all slices are equivalent, because there is only one $(21, 5, 1)$ design: the projective plane $PG(2, 4)$.

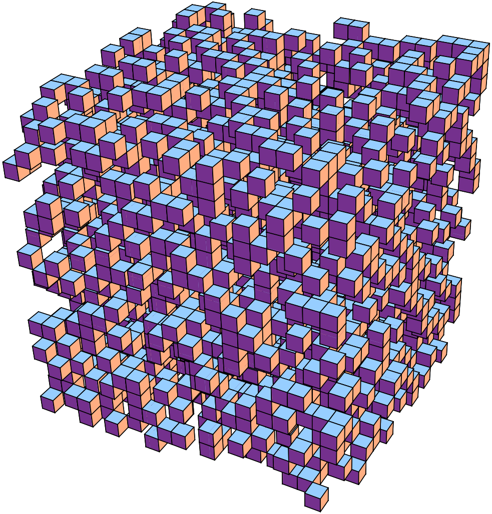
Group 3-cubes

C_1



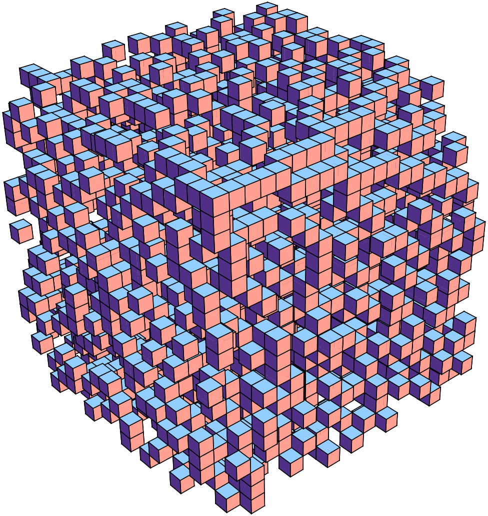
Group 3-cubes

C_2



Group 3-cubes

C_3



PAG

Prescribed Automorphism Groups

0.2.1

4 April 2023

Abstract

PAG is a GAP package for constructing combinatorial objects with prescribed automorphism groups.

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https://vkrcadinac.github.io/PAG/

https://github.com/vkrcadinac/PAG

The PAG package

PAG

Prescribed Automorphism Groups

Version 0.2.0
Released 2023-03-27

Download .tar.gz

View On GitHub

This project is maintained by
Vedran Krcadinac

Hosted on GitHub Pages, based on GitHubPagesForGAP using the Dinky theme

Last updated: 2023-03-27 22:55

GAP Package PAG

The PAG package contains functions for constructing combinatorial objects with prescribed automorphism groups.

The current version of this package is version 0.2.0, released on 2023-03-27. For more information, please refer to the package manual. There is also a README file.

Dependencies

This package requires GAP version 4.11

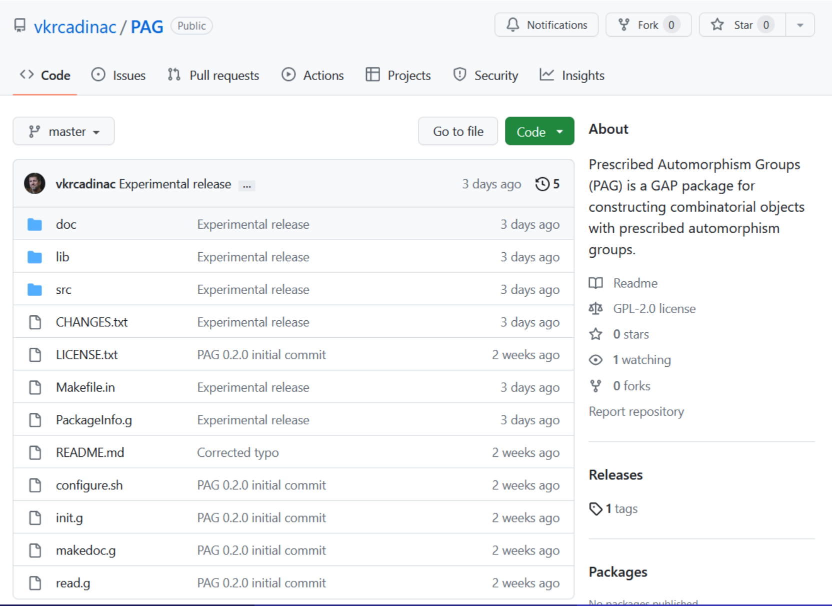
The following other GAP packages are needed:

- GAPDoc 1.5
- images 1.3
- GRAPE 4.8
- DESIGN 1.7

The following additional GAP packages are not required, but suggested:

- AssociationSchemes 2.0
- GUAVA 3.15
- DifSets 2.3.1

The PAG package



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LICENSE.txt	PAG 0.2.0 initial commit	2 weeks ago
Makefile.in	Experimental release	3 days ago
PackageInfo.g	Experimental release	3 days ago
README.md	Corrected typo	2 weeks ago
configure.sh	PAG 0.2.0 initial commit	2 weeks ago
init.g	PAG 0.2.0 initial commit	2 weeks ago
makedoc.g	PAG 0.2.0 initial commit	2 weeks ago
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About

Prescribed Automorphism Groups (PAG) is a GAP package for constructing combinatorial objects with prescribed automorphism groups.

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The PAG package

2.6 Cubes of Symmetric Designs

2.6.1 DifferenceCube

▷ `DifferenceCube(G, ds, n)` (function)

Returns the *n*-dimenional difference cube constructed from a difference set *ds* in the group *G*.

2.6.2 GroupCube

▷ `GroupCube(G, dds, n)` (function)

Returns the *n*-dimenional group cube constructed from a symmetric design *dds* such that the blocks are difference sets in the group *G*.

2.6.3 CubeSlice

▷ `CubeSlice(C, x, y, fixed)` (function)

Returns a 2-dimensional slice of the incidence cube *C* obtained by varying coordinates in positions *x* and *y*, and taking fixed values for the remaining coordinates given in a list *fixed*.

2.6.4 CubeSlices

▷ `CubeSlices(C[, x, y][, fixed])` (function)

Group 3-cubes

Example. There are three $(16, 6, 2)$ designs:

$$|\operatorname{Aut}(\mathcal{D}_1)| = 11520, \quad |\operatorname{Aut}(\mathcal{D}_2)| = 768, \quad |\operatorname{Aut}(\mathcal{D}_3)| = 384$$

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D. Peifer, *DifSets, an algorithm for enumerating all difference sets in a group*, Version 2.3.1, 2019. https://dylanpeifer.github.io/difsets

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ID	Structure	Nds	Ndc	dev	Tds	Ngc
1	$\mathbb{Z}_{16}$	0	0	–	0	0
2	$\mathbb{Z}_4^2$	3	3	$\mathcal{D}_1$	192	55
3	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	4	4	$\mathcal{D}_1$	192	83
4	$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	3	3	$\mathcal{D}_1$	192	81
5	$\mathbb{Z}_8 \times \mathbb{Z}_2$	2	2	$\mathcal{D}_1$, $\mathcal{D}_2$	192	106
6	$\mathbb{Z}_8 \rtimes \mathbb{Z}_2$	2	2	$\mathcal{D}_1$	64	34
7	D_{16}	0	0	–	0	0
8	QD_{16}	2	2	$\mathcal{D}_1$	128	50
9	Q_{16}	2	2	$\mathcal{D}_1$	256	71
10	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	2	2	$\mathcal{D}_1$	448	131
11	$\mathbb{Z}_2 \times D_8$	2	2	$\mathcal{D}_1$	192	52
12	$\mathbb{Z}_2 \times Q_8$	2	2	$\mathcal{D}_1$, $\mathcal{D}_3$	704	197
13	$(\mathbb{Z}_4 \times \mathbb{Z}_2) \rtimes \mathbb{Z}_2$	2	2	$\mathcal{D}_1$, $\mathcal{D}_3$	320	77
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Up to equivalence, the set $\mathcal{C}^3(16,6,2)$ contains exactly 27 difference cubes and 946 group cubes that are not difference cubes.

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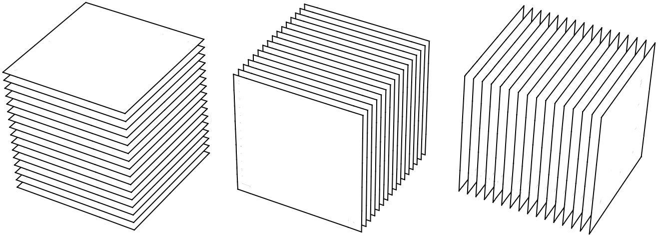
Slices:



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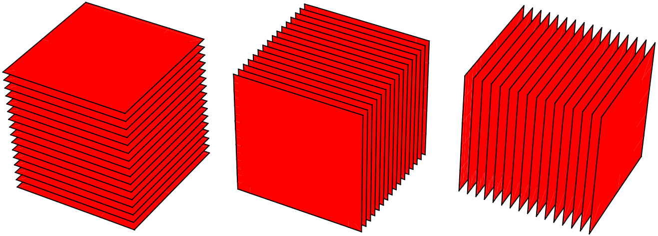

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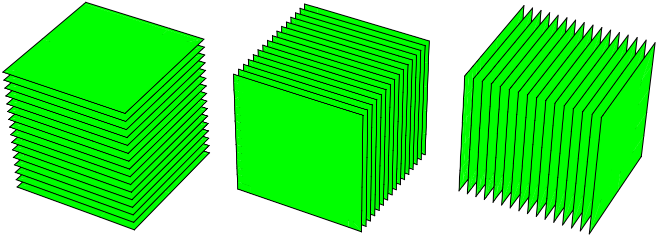

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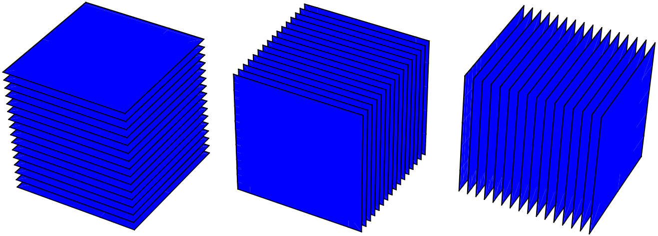

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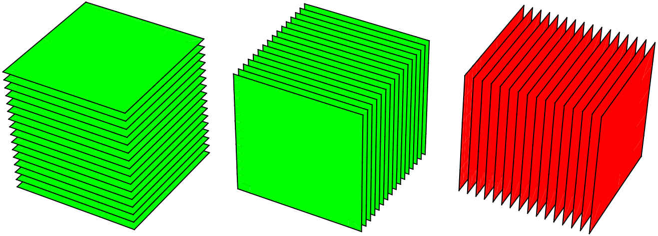

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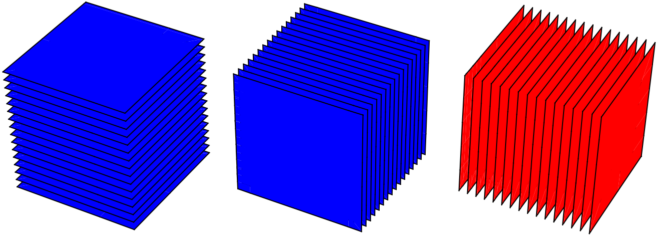

Group 3-cubes

Theorem.

The set $\mathcal{C}^n(4^m, 2^{m-1}(2^m - 1), 2^{m-1}(2^{m-1} - 1))$ contains at least two inequivalent non-difference group cubes constructed in $\mathbb{Z}_2^{2m}$ for every $m \geq 2$ and $n \geq 3$.

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Question 4. Are there examples of n-dimensional Hadamard matrices with inequivalent slices without this restriction?

Group 3-cubes

More examples of $(16, 6, 2)$ cubes...

Group cube in $\mathbb{Z}_8 \times \mathbb{Z}_2$: $\mathcal{D}_3 = \{B_1, \ldots, B_8, B_9, \ldots, B_{16}\}$

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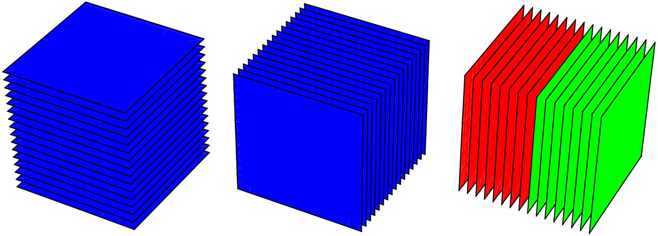

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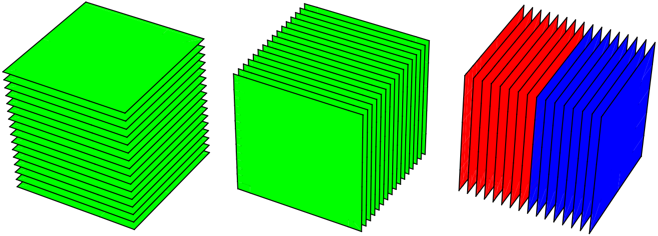

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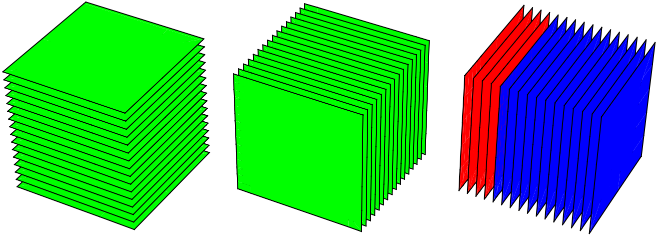

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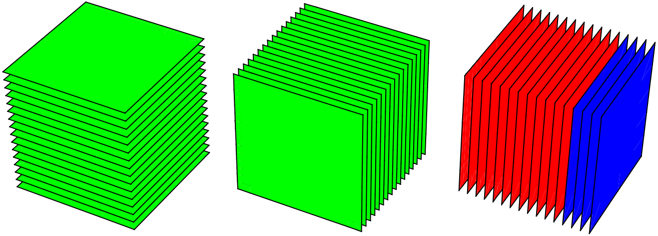

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Larger examples...

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Equivalent difference sets give isomorphic developments (designs), and isotopic difference cubes.

Does the converse hold?

Question 5. If two difference cubes are isotopic, do they necessarily come from equivalent difference sets?

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I know of no examples for dimensions $n \geq 3$.

Non-group cubes

An n-cube $C \in \mathcal{C}^n(v,k,\lambda)$ can be represented as

$$\overline{C} = \{(i_1,\ldots,i_n) \in \{1,\ldots,v\}^n \mid C(i_1,\ldots,i_n) = 1\}.$$

This is an orthogonal array with parameters $OA(kv^{n-1}, n, v, n-1)$.

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By changin n-tuples $(i_1, \ldots, i_n)$ to n-subsets $\{i_1, v + i_2, 2v + i_3, \ldots, (n-1)v + i_n\}$ we get a transversal design. This is an incidence structure of nv points and kv^{n-1} blocks such that the usual notion of isomorphism agrees with paratopy of cubes.

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We can construct such transversal designs by the Kramer-Mesner method. Candidates for prescribed autotopy groups: take a known cube $C \in \mathcal{C}^3(16,6,2)$, compute $\mathrm{Atop}(C)$ and choose a subgroup G.

E. S. Kramer, D. M. Mesner, *t-designs on hypergraphs*, Discrete Math. **15** (1976), no. 3, 263–296.

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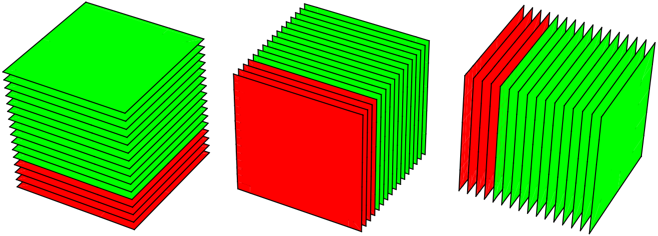

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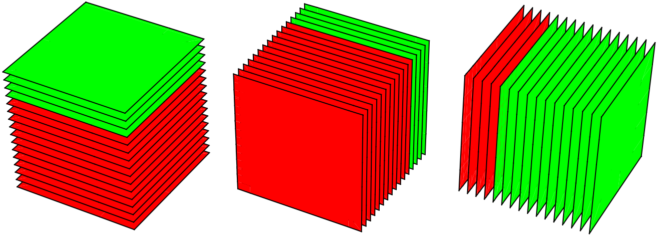

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Question 7. Are there non-group cubes for smaller parameters (v,k,λ)? We tried constructing them by the Kramer-Mesner method, but did not find any examples except for $(16,6,2)$. What about $(15,7,3)$?

Question 8. Are there non-group cubes for larger parameters (v,k,λ)? Our Kramer-Mesner approach was too inefficient. By what computational method can we construct them?

Conference in Dubrovnik

Combinatorial Constructions Conference (CCC) will take place at the Centre for Advanced Academic Studies in Dubrovnik, Croatia.

April 7-13, 2024

Invited Speakers (confirmed):

Eimear Byrne, Ireland	Michael Kiermaier, Germany
Dean Crnković, Croatia	Patric Östergård, Finland
Daniel Horsley, Australia	Kai-Uwe Schmidt, Germany

https://web.math.pmf.unizg.hr/acco/meetings.php

The End

Thanks for your attention!