Coloring incidence graphs of 2-designs

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Definition

A 2-(v, k, λ) design is a pair $(V, \mathcal{B})$ such that

- V is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of V (called blocks);
- each 2-subset of V is contained exactly in λ blocks.

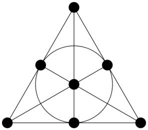


Figure: The Fano plane is a 2-$(7, 3, 1)$ design.

- A 2-design with $\lambda = 1$ is a Steiner system.

The Levi graph or incidence graph $G_{\mathcal{D}}$ of a 2-(v, k, λ) design (or any incidence structure) is a graph with

- one vertex per point
- one vertex per block
- an edge for any incident point block pair

- $n = v + b$, $m = bk$

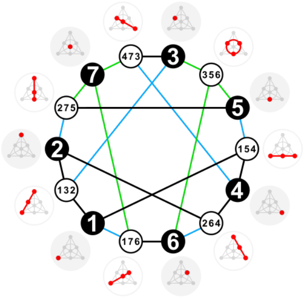


Figure: the Heawood graph is the Levi graph of the Fano plane

- Coloring of a graph G is an assignment of colors to the vertices of G so that adjacent vertices have different colors.
- A coloring is harmonious if each pair of colors appears on at most one pair of adjacent vertices
- The minimum number of colors needed is called the harmonious chromatic number of G, and denoted by $h(G)$

- Araujo-Pardo, Monellano-Ballestreros, Olsen, Rubio-Montiel, *On the harmonious chromatic number of graphs*, `arXiv:2206.04822`

Banff designs

- If $\mathcal{D}$ is a 2-(v, k, λ) design, then

$$h(G_{\mathcal{D}}) \geq v$$

- For which designs is this lower bound attained?

Definition

A (v, k, λ) design $\mathcal{D}$ whose Levi graph has harmonious chromatic number equal to v will be called a Banff design.

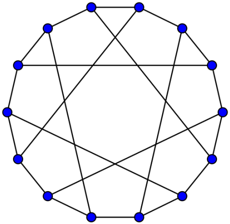


Figure: Levi graph of the Fano plane

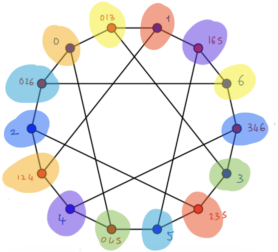
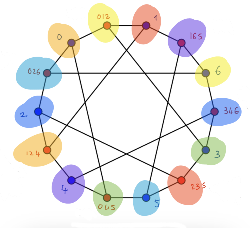


Figure: The Fano plane is a Banff design

Nesting of $2-(7,3,1)$ into a $2-(7,4,2)$

The existence of a $(v,3,1)$ Banff design is equivalent to that of a nesting of a STS(v) into a $(v,4,2)$ design.



Nesting of a $2-(7,3,1)$ into a $2-(7,4,2)$ design

Blocks:

0136, 1240, 2351, 3462, 0453, 1654, 0265

A $(v,3,1)$-design $(V,\mathcal{B})$ can be nested if there is a mapping $\varsigma : \mathcal{B} \to V$ such that $(V, \{B \cup \varsigma(B)\} : B \in \mathcal{B})$ is a $(v,4,2)$-design.

Theorem (Lindner, Rodger, 1987; Stinson, 1985)
There exists a $(v,3,1)$ Banff design if and only if $v \equiv 1 \pmod 6$.

Definition

Let G be an additive group of order v. A (v,k,λ) difference family in G is a set $\mathcal{F}$ of k-subsets of G (called *base blocks* of $\mathcal{F}$) such that the list

$$\Delta\mathcal{F} := \{x - y : x, y \in B, x \neq y, B \in \mathcal{F}\}$$

contains every element of $G \setminus \{0\}$ exactly λ times.

- When the base blocks are pairwise disjoint we speak of a disjoint difference family.
- When $\mathcal{F}$ consists of a single base block B we say that B is a difference set.

- $D = \{1,2,4\}$ is a $(7,3,1)$ difference set

	1	2	4
1	·	6	4
2	1	·	5
4	3	2	·

The development of a (v, k, λ) difference family $\mathcal{F}$ is the multiset

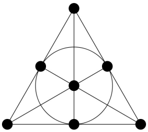
$$dev\mathcal{F} = \{B + g \mid g \in G; B \in \mathcal{F}\}$$

of all possible translates of its base blocks.

The pair $(G, dev\mathcal{F})$ is a (v, k, λ) design admitting an automorphism group isomorphic to G acting sharply transitively on the points.

- The Fano plane is the $\mathbb{Z}_7$-development of D

$$\{D+0, D+1, D+2, D+3, D+4, D+5, D+6\}$$
$$=$$
$$\{\{1,2,4\},\{2,3,5\},\{3,4,6\},\{4,5,0\},\{5,6,1\},\{6,0,2\},\{0,1,3\}\}$$



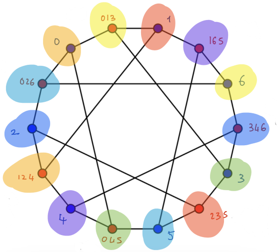
Definition

A Banff difference family is a difference family $\mathcal{F} = \{B_1, \ldots, B_n\}$ such that

- $0 \notin B_i$ for every i,
- it is disjoint,
- and $B_i \cap -B_j = \emptyset$ for every possible pair (i, j).

Theorem

Every Banff difference family generates a Banff design.



Banff difference families

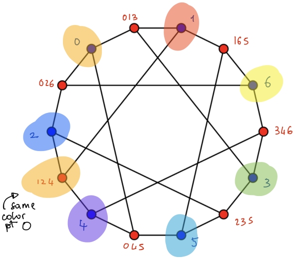
- $D = \{1, 2, 4\}$ is a $(7, 3, 1)$ difference set

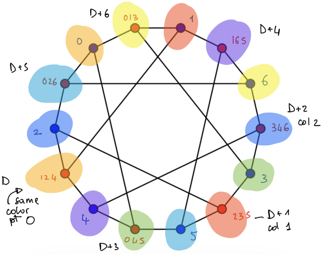
	1	2	4
1	$\cdot$	6	4
2	1	$\cdot$	5
4	3	2	$\cdot$

- The Fano plane is the $\mathbb{Z}_7$-development of D

$$\{D + 0, D + 1, D + 2, D + 3, D + 4, D + 5, D + 6\}$$
$$=$$
$$\{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}, \{0, 1, 3\}\}$$

- the set of points $\mathbb{Z}_7$ is the set of colors
- D has color 0, $D + i$ has color i





Examples of Banff designs

- Start from the cyclic $(85, 4, 1)$ difference family $\mathcal{F} = \{B_1, \ldots, B_7\}$
- It is not a Banff difference family.
- But translating the blocks appropriately we get it: $\{B_1 + t_1, \ldots, B_7 + t_7\}$

starting DF	t_i	Banff DF
$B_1 = \{0, 2, 41, 42\}$	2	$B_1 + 2 = \{2, 4, 43, 44\}$
$B_2 = \{0, 17, 32, 38\}$	7	$B_2 + 7 = \{7, 24, 39, 45\}$
$B_3 = \{0, 18, 27, 37\}$	10	$B_3 + 10 = \{10, 28, 37, 47\}$
$B_4 = \{0, 13, 29, 36\}$	20	$B_4 + 20 = \{20, 33, 49, 56\}$
$B_5 = \{0, 11, 31, 35\}$	19	$B_5 + 19 = \{19, 30, 50, 54\}$
$B_6 = \{0, 12, 26, 34\}$	60	$B_6 + 60 = \{60, 72, 1, 9\}$
$B_7 = \{0, 5, 30, 33\}$	58	$B_7 + 58 = \{58, 63, 3, 6\}$

- 2-$(85, 4, 1)$ generated by this difference family is a Banff design.

Conjecture

For any cyclic $(v, k, 1)$ difference family $\mathcal{F} = \{B_1, \ldots, B_n\}$ there is a suitable n-tuple $(t_1, \ldots, t_n)$ of elements of $\mathbb{Z}_v$ such that $\{B_1 + t_1, \ldots, B_n + t_n\}$ is a $(v, k, 1)$ Banff difference family.

The projective plane PG$(2, q)$ is a $(q^2 + q + 1, q + 1, 1)$ design, which is a development of a Singer difference set.

Theorem

Any projective plane PG$(2, q)$ is a $(q^2 + q + 1, q + 1, 1)$ Banff design.

Proof.

- Let $B = \{b_0, b_1, \ldots, b_q\}$ be the Singer $(q^2 + q + 1, q + 1, 1)$ difference set.
- Consider the subset X of $\mathbb{Z}_{q^2+q+1}$ defined by

$$X = \{\frac{b_i + b_j}{2} \mid 0 \leq i \leq j \leq q\}.$$

- We have $|X| = \frac{(q+1)(q+2)}{2} < q^2 + q + 1$.
- Take an element $t \in \mathbb{Z}_{q^2+q+1} \setminus X$.
- Show that $D := B - t$ is a $(q^2 + q + 1, q + 1, 1)$ Banff difference set.
- $0 \notin D$. Otherwise we would have $t = b_i$ for some i. On the other hand we have $b_i = \frac{b_i + b_i}{2} \in X$ contradicting the choice of t.
- If $B' \cap -B'$ is not empty there would be a pair (i, j) such that $b_i - t = -(b_j - t)$ and then $t = \frac{b_i + b_j}{2} \in X$ contradicting again the choice of t.

Projective plane PG$(2,3)$

- PG$(2,3)$ is a $(13,4,1)$ design, which is the $\mathbb{Z}_{13}$-development of the Singer difference set $B = \{0,1,5,11\}$
- Compute $X = \{\frac{b_i+b_j}{2} \mid 0 \leq i \leq j \leq q\} = \{0,1,3,5,6,7,8,9,11,12\}$.
- Let $t = 2 \in \mathbb{Z}_{13} \setminus X$
- Then $D := B - t = \{3,9,11,12\}$ is a Banff difference set

block	color
$3,9,11,12$	0
$4,10,12,0$	1
$5,11,0,1$	2
$6,12,1,2$	3
$7,0,2,3$	4
$8,1,3,4$	5
$9,2,4,5$	6
$10,3,5,6$	7
$11,4,6,7$	8
$12,5,7,8$	9
$0,6,8,9$	10
$1,7,9,10$	11
$2,8,10,11$	12

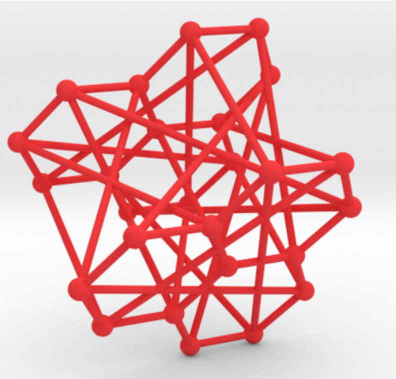


Figure: Levi graph of PG$(3,2)$, shapeways

Theorem
There exists $(q, k, 1)$ Banff design for any prime power $q \equiv 1 \pmod{k(k-1)}$ sufficiently large.

- k = 4
- Applying the above theorem with $k = 4$ we can say that there exists a $(q, 4, 1)$ Banff set for any prime power $q = 12n + 1 > 9,152,353$.
- Computer search: For any prime power $q \equiv 1 \pmod{12}$, $q \leq 9,152,353$, there exists $(q, 4, 1)$ Banff design.

Corollary
For any prime power $q \equiv 1$ (mod 12), there exists $(q, 4, 1)$ Banff design.

Theorem (Buratti, Kreher, Stinson, 2024)
For any $v \equiv 1, 4$ (mod 12), there exists $(v, 4, 1)$ Banff design.

Thank you for your attention!