A census of 2-$(27, 6, 5)$ designs

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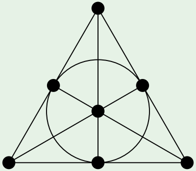
Basic notions

Definition

A (v, k, λ) design is a pair $D = (P, B)$ where P is a set of v points and B is a multiset of k-subsets of P (blocks) such that every 2-subset of P is contained in exactly λ blocks.

- full automorphism group: the group $\mathrm{Aut}(D)$ of all permutations on P leaving B invariant
- automorphism group: any subgroup of $\mathrm{Aut}(D)$

Example of a 2-$(7, 3, 1)$ design



Basic notions

Definition

- Let $(G,+)$ be a group of order v (not necessarily commutative)
- Let $B \subseteq G$ be a subset of G
- Let $Stab(B)$ be the G-stabilizer of B
- Let $Orb(B)$ be the orbit of B under G

The <u>list of differences</u> from B is the multiset

$$\Delta B = \{x - y : x, y \in B, x \neq y\}$$

The <u>list of partial differences</u> is

$$\partial B = \frac{1}{|Stab(B)|} \Delta B$$

Basic notions

Definition

A multiset $\{B_1, ..., B_t\}$ of k-subsets of G is a (v, k, λ)-difference family if every nonidentity element of G occurs λ times in $\partial B_1 \cup \ldots \cup \partial B_t$.

Theorem

If $\{B_1, ..., B_t\}$ *is a family* (v, k, λ) *difference family in* G*, then* $\bigcup_i Orb(B_i)$ *is the block multiset of a* (v, k, λ) *design with point set* G*.*

Basic notions

Definition

We say that a (v, k, λ)-design is 1-rotational if it admits a group of automorphisms acting sharply transitively on all but one point.

Definition

- Let $\infty \notin G$
- Let $B \subseteq G$
- We define the list of partial differences of

$$\partial(B \cup \{\infty\}) = \partial B \cup \frac{|B|}{|Stab(B)|}\{\infty\}$$

Basic notions

Definition

A multiset $\{B_1, ..., B_t\}$ of k-subsets of $G \cup \{\infty\}$ is a 1-rotational (v, k, λ) difference family if every nonzero element of $G \cup \{\infty\}$ occurs λ times in $\partial B_1 \cup \ldots \cup \partial B_t$.

Theorem

If $\{B_1, ..., B_t\}$ *is a* 1*-rotational* (v, k, λ) *difference family in* G*, then* $\bigcup_i Orb(B_i)$ *is the block multiset of a* 1*-rotational* (v, k, λ) *design with point set* $G \cup \{\infty\}$*.*

1-rotational $(27, 6, 5)$ difference families

Previous results

Hanani design[a] and Handbook design[b]

449	66 286 26	6	2	≥ 1	≥ 1	2#88 [958]	
450	27 117 26	6	5	≥ 1	-	D#512* [1042]	VI.16.86
451	14 52 26	7	12	$>$ 1363846	1363486	2#89 [1264]	

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[a] Haim Hanani: *Balanced incomplete block designs and relatex designs.* Discrete Math., 11:255-369, 1975

[b] Colbourn and Dinitz: *Handbook of combinatorial designs.* Discrete Mathematics and its Applications, 2nd edition, 2007

- In this talk $G = \mathbb{Z}_{26}$
- $\infty \notin G$
- we necessarily have 5 base blocks one of which contains ∞

1-rotational $(27, 6, 5)$ difference families

$$\{\quad ,\quad ,\quad ,\quad ,\quad ,\quad \}$$
$$\{\quad ,\quad ,\quad ,\quad ,\quad ,\quad \}$$
$$\{\quad ,\quad ,\quad ,\quad ,\quad ,\quad \}$$
$$\{\quad ,\quad ,\quad ,\quad ,\quad ,\quad \}$$
$$\{\quad ,\quad ,\quad ,\quad ,\quad ,\infty \}$$

1-rotational $(27, 6, 5)$ difference families

$$
\begin{array}{ll}
\{\ \ ,\ \ ,\ \ ,\ \ ,\ \ ,\ \ \ \ \} & \mathit{Stab} = \{0\} \\
\{\ \ ,\ \ ,\ \ ,\ \ ,\ \ ,\ \ \ \ \} & \mathit{Stab} = \{0\} \\
\{\ \ ,\ \ ,\ \ ,\ \ ,\ \ ,\ \ \ \ \} & \mathit{Stab} = \{0\} \\
\{\ \ ,\ \ ,\ \ ,\ \ ,\ \ ,\ \ \ \ \} & \mathit{Stab} = \{0, 13\} \\
\{\ \ ,\ \ ,\ \ ,\ \ ,\ \ ,\ \infty\ \} & \mathit{Stab} = \{0\}
\end{array}
$$

$$30 + 30 + 30 + 15 + 25 = 5 \cdot |\mathbb{Z}_{26} \setminus \{0\} \cup \{\infty\}|$$

1-rotational $(27, 6, 5)$ difference families

$$
\begin{array}{llllllll}
\{ & , & , & , & , & , & \} & Stab = \{0\} \\
\{ & , & , & , & , & , & \} & Stab = \{0\} \\
\{ & , & , & , & , & , & \} & Stab = \{0\} \\
\{ & d_1, & d_2, & d_3, & d_1 + 13, & d_2 + 13, & d_3 + 13 \} & Stab = \{0, 13\} \\
\{ & , & , & , & , & , & \infty \} & Stab = \{0\}
\end{array}
$$

1-rotational $(27, 6, 5)$ difference families

$$
\begin{array}{rl}
A = \{ & a_0, \quad a_1, \quad a_2, \quad a_3, \quad a_4, \quad a_5 \; \} \\
B = \{ & b_0, \quad b_1, \quad b_2, \quad b_3, \quad b_4, \quad b_5 \; \} \\
C = \{ & c_0, \quad c_1, \quad c_2, \quad c_3, \quad c_4, \quad c_5 \; \} \\
D = \{ & d_0, \quad d_1, \quad d_2, \quad d_0 + 13, \quad d_1 + 13, \quad d_2 + 13 \; \} \\
E = \{ & e_0, \quad e_1, \quad e_2, \quad e_3, \quad e_4, \quad \infty \; \}
\end{array}
$$

Reducing repeating cases

Generally

- $\partial B = \partial(B + g)$
- $Orb(B) = Orb(B + g)$

for all $g \in G$ and $B \subseteq G$

1-rotational $(27, 6, 5)$ difference families

$$
\begin{aligned}
A &= \{\ 0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5\ \} \\
B &= \{\ 0,\ b_1,\ b_2,\ b_3,\ b_4,\ b_5\ \} \\
C &= \{\ 0,\ c_1,\ c_2,\ c_3,\ c_4,\ c_5\ \} \\
D &= \{\ 0,\ d_1,\ d_2,\ 13,\ d_1 + 13,\ d_2 + 13\ \} \\
E &= \{\ 0,\ e_1,\ e_2,\ e_3,\ e_4,\ \infty\ \}
\end{aligned}
$$

Number of parameters: 21

Computer construction

- representative for each base block (of type ABC, D or E)
- all possible lists of partial differences from all the representatives
- combinations of lists of partial differences that complete to a multiset equal to $(\mathbb{Z}_{26} \setminus \{0\}) \cup \{\infty\}$ repeated 5 times
- for each obtained combination look up representative base blocks that produce lists of partial differences in this combination
- number of constructable difference families: $142923488 \sim 1.43 \cdot 10^8$

Notes on the computer construction (work in progress)

- some of the designs obtained from these difference families are isomorphic
- we expect at least 30% of the designs repeated up to isomorphism
- we assume that we would end up with around $\simeq 10^7$ non-isomorphic designs
- random construction yielded about 150000 non-isomorphic designs all with full automorphism group $\mathbb{Z}_{26}$
- rich automorphism groups are rare!

Further construction

For $a = 1, 5, 7, 17$ and b, c, d, e, f distinct:

$$\begin{aligned}
X_0 &= \{ \; 0, \; b, \; c, \; d, \; e, \; f \; \} \\
X_1 &= \{ \; 0, \; 3b, \; 3c, \; 3d, \; 3e, \; 3f \; \} \\
X_2 &= \{ \; 0, \; 9b, \; 9c, \; 9d, \; 9e, \; 9f \; \} \\
Y &= \{ \; 1, \; 3, \; 9, \; 14, \; 16, \; 22 \; \} \\
Z &= \{ \; a, \; 3a, \; 9a, \; 0, \; 13, \; \infty \; \}
\end{aligned}$$

Remark

- $3X_0 = X_1$; $3X_1 = X_2$; $3X_2 = X_0$
- $3Y = Y$
- $3Z = Z$

4140 difference families with this construction

Census of 1-rotational $(27, 6, 5)$ designs

230 non-isomorphic designs

- 1 design with full automorphism group of order 2106
  $(a, b, c, d, e, f) = (1, 7, 19, 21, 22, 25)$
- 229 non-isomorphic designs with full automorphism group of order 78
- Hanani design $(a, b, c, d, e, f) = (1, 3, 6, 10, 15, 22)$
- Handbook design $(a, b, c, d, e, f) = (17, 1, 5, 6, 11, 23)$
- these two are non-isomorphic!

$\geq$ 150000 non-isomorphic designs with full automorphism group $\mathbb{Z}_{26}$

References

Colbourn, Charles J. and Jeffrey H. Dinitz (editors): *Handbook of combinatorial designs*.
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Thank you for your attention!