Schematic 4-designs *

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(joint work with Vedran Krčadinac)

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A design of degree d can have at most $\binom{v}{d}$ blocks, i.e. $b \leq \binom{v}{d}$.

D. K. Ray-Chaudhuri, R. M. Wilson, *On t-designs*, Osaka J. Math. **12** (1975), 737–744.

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$t = 2$	many feasible parameters and many known examples

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The blocks of a t-design of degree d and $t \leq 2d - 2$ form a symmetric association scheme with d classes.

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GOAL: determine all **admissible parameters of schematic** 4-(v, k, λ) **designs** with intersection numbers x, y and z ($v \leq 1000$)

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- G_0 is the graph with every vertex of X adjacent to itself and no other adjacencies,
- if x and y are distinct elements of X, there is exactly one graph G_i in which $\{x, y\}$ is an edge,
- for every edge $\{x, y\}$ of G_l, where $x, y \in X$, the number of vertices $z \in X$ such that $\{x, z\}$ is an edge of G_i and $\{y, z\}$ is an edge of G_j depend only of the indices i, j, l. This number is denoted by p^l_{ij} and is called an *intersection number* of the association scheme.

Parameters of the association scheme

Let $(V, \mathcal{B})$ be a 4-(v, k, λ) **design** with three block intersection numbers $x < y < z$.

Let G_1, G_2, G_3 **be graphs with the blocks of $\mathcal{B}$ as vertices** and blocks being adjacent if they intersect in x, y or z points, respectively.

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$\rightsquigarrow$ the graphs G_0, G_1, G_2, G_3 form a symmetric association scheme (for every edge $\{B_1, B_2\}$ of G_ℓ the number of vertices $B_3 \in \mathcal{B}$ such that $\{B_1, B_3\}$ is an edge of G_i and $\{B_2, B_3\}$ is an edge of G_j depends only on the indices i, j, ℓ. This number is denoted by p_{ij}^{ℓ} and is called an *intersection number* of the association scheme.)

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We first determine the degree $n_i = p^0_{ii}$ of vertices in G_i. Obviously, $n_0 = 1$.

Let $B_0 \in \mathcal{B}$ be a fixed block. Then n_1, n_2, n_3 **are the numbers of blocks** B **intersecting** B_0 **in** x, y, **or** z **points**, respectively.

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By double-counting pairs (I, B), where $I \subseteq B_0$ is a set of i points and $B \neq B_0$ is a block containing I, we get the equation

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Here, $\lambda_i = \lambda \cdot \binom{v-i}{4-i} / \binom{k-i}{4-i}$ is the number of blocks through any set of i points, for $i = 0, \ldots, 4$.

The system of equations (1) for $i = 0, 1, 2$ is linear in n_1, n_2, and n_3.

Parameters of the association scheme

Proposition.

The graphs G_1, G_2, G_3 are regular with respective degrees

$$\begin{aligned} n_1 &= \frac{yz(\lambda_0 - 1) + (1 - y - z)k(\lambda_1 - 1) + k(k-1)(\lambda_2 - 1)}{(y-x)(z-x)}, \\ n_2 &= \frac{xz(\lambda_0 - 1) + (1 - x - z)k(\lambda_1 - 1) + k(k-1)(\lambda_2 - 1)}{(x-y)(z-y)}, \\ n_3 &= \frac{xy(\lambda_0 - 1) + (1 - x - y)k(\lambda_1 - 1) + k(k-1)(\lambda_2 - 1)}{(x-z)(y-z)}. \end{aligned} \tag{2}$$

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- $5 \leq k \leq v/2$
- $b \leq \binom{v}{3}$ gives $\lambda \leq \frac{4}{v-3}\binom{k}{4}$

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- compute n_1, n_2, n_3 by (2) for triples (v, k, λ) and for every choice of intersection numbers $0 \leq x < y < z < k$

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- the numbers must also satisfy equation (1) for $i = 3, 4$.

$\Longrightarrow$ **only 17 possibilities**

Parameters of schematic 4-designs

No.	v	k	λ	x	y	z	n_1	n_2	n_3
1	11	5	1	1	2	3	15	20	30
2	23	8	4	0	2	4	15	280	210
3	23	11	48	3	5	7	165	792	330
4	24	8	5	0	2	4	30	448	280
5	31	15	39	5	7	9	168	450	280
6	47	11	8	1	3	5	1386	2475	462
7	71	35	264	14	17	20	1020	2450	1428
8	127	63	915	27	31	35	3472	7938	4464
9	199	99	2328	44	49	54	8820	19602	10780
10	199	99	18624	43	49	55	48125	204248	61250
11	287	143	4935	65	71	77	18744	40898	22152
12	391	195	9264	90	97	104	35308	76050	40740
13	511	255	15939	119	127	135	60960	130050	69088
14	647	323	25680	152	161	170	98532	208658	110124
15	659	329	390874	153	164	175	1011675	4182248	1156200
16	799	399	39303	189	199	209	151240	318402	167160
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We denote **the adjacency matrix of G_i by A_i**. Obviously, $A_0 = I$ (the $b \times b$ identity matrix). The number n_i is an eigenvalue of the matrix A_i corresponding to the all-ones eigenvector $[1, \ldots, 1]^\top$.

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Next, we determine the remaining **eigenvalues of the association scheme**.

Let $p_i(j)$, $j = 0, 1, 2, 3$ be the eigenvalues of A_i.
Then, $p_0(j) = 1$ and $p_i(0) = n_i$ for $i, j \in \{0, 1, 2, 3\}$. The degree of the eigenvalue $p_i(j)$ will be denoted by m_j.

Parameters of the association scheme

Theorem

The association scheme of a 4-(v, k, λ) *design with three intersection numbers* $x < y < z$ *has the following eigenvalues:*

$$\begin{aligned} p_1(j) &= \frac{yz\theta_0(j) + (1-y-z)\theta_1(j) + 2\theta_2(j) - (y-k)(z-k)}{(y-x)(z-x)}, \\ p_2(j) &= \frac{xz\theta_0(j) + (1-x-z)\theta_1(j) + 2\theta_2(j) - (x-k)(z-k)}{(x-y)(z-y)}, \\ p_3(j) &= \frac{xy\theta_0(j) + (1-x-y)\theta_1(j) + 2\theta_2(j) - (x-k)(y-k)}{(x-z)(y-z)} \end{aligned} \tag{3}$$

with multiplicities

$$m_j = \begin{cases} \binom{v}{j} - \binom{v}{j-1}, & \text{for } j = 0, 1, 2, \\ b - \binom{v}{2}, & \text{for } j = 3. \end{cases}$$

Parameters of the association scheme

Here, $\theta_i(j)$ are given by

$$\theta_i(j) = \frac{b}{\binom{v}{k}} \binom{v-i-j}{v-k-j} \binom{k-j}{i-j} = \frac{\lambda}{\binom{v-4}{k-4}} \binom{v-i-j}{v-k-j} \binom{k-j}{i-j}.$$

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Also, the intersection numbers of the association scheme p_{ij}^{ℓ} must be non-negative integers. They can be computed from the eigenvalues by the formula

$$p_{ij}^{\ell} = \frac{1}{bn_\ell} \sum_{s=0}^{3} p_i(s) p_j(s) p_\ell(s) m_s. \tag{4}$$

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No.	v	k	λ	x	y	z	n_1	n_2	n_3
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Parameters of schematic 4-designs

$\Longrightarrow$ **There are 11 admissible parameter sets of schematic 4-designs with $v \leq 1000$.**

No.	v	k	λ	x	y	z	Existence
1	11	5	1	1	2	3	$\exists$
2	23	8	4	0	2	4	$\exists$
3	23	11	48	3	5	7	$\exists$
4	24	8	5	0	2	4	$\exists$
5	47	11	8	1	3	5	$\exists$
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10	659	329	390874	153	164	175	?
11	967	483	57720	230	241	252	?

Parameters of schematic 4-designs

Five of the 11 parameter sets in table belong to the following family:

$$\begin{aligned}
v &= 8n^2 - 1, \\
k &= 4n^2 - 1 = (2n-1)(2n+1), \\
\lambda &= 4n^4 - 7n^2 + 3 = (n-1)(n+1)(4n^2-3), \\
x &= 2n^2 - n - 1 = (n-1)(2n+1), \\
y &= 2n^2 - 1, \\
z &= 2n^2 + n - 1 = (n+1)(2n-1).
\end{aligned}$$

Thank you for your attention!