Polarity transformations of semipartial geometries*

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Partial geometries

M. Abreu, M. Funk, V. Krčadinac, D. Labbate. *Strongly regular configurations*, preprint, 2021. https://arxiv.org/abs/2104.04880

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R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.

A partial geometry $pg(s,t,\alpha)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are exactly α points on ℓ collinear with P.

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A partial geometry $pg(s, t, \alpha)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are exactly α points on ℓ collinear with P.

Special cases:

- Steiner 2-designs – $pg(s, t, s + 1)$; duals – $pg(s, t, t + 1)$
- Bruck nets – $pg(s, t, t)$; transversal designs – $pg(s, t, s)$
- generalized quadrangles – $pg(s, t, 1)$

Partial geometries

The point graph of a partial geometry is a

$$SRG\left(\frac{(s+1)(st+\alpha)}{\alpha},\ s(t+1),\ s-1+t(\alpha-1),\ \alpha(t+1)\right),$$

and the line graph is a

$$SRG\left(\frac{(t+1)(st+\alpha)}{\alpha},\ t(s+1),\ t-1+s(\alpha-1),\ \alpha(s+1)\right).$$

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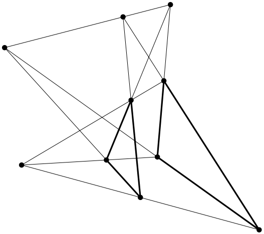
There are configurations with both associated graphs strongly regular that are **not** partial geometries. E.g. the Desargues configuration (10_3):



$\rightsquigarrow$ $SRG(10,6,3,4)$
(complement of the Petersen graph)

Semipartial geometries

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

A semipartial geometry $spg(s, t, \alpha, \mu)$ is a configuration with $k = s + 1$ and $r = t + 1$ such that for every non-incident point-line pair (P, ℓ), there are either 0 or α points on ℓ collinear with P. Furthermore, for every pair of non-collinear points, there are exactly μ points collinear with both.

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The point graph of a semipartial geometry is a

$$SRG\left(1 + \frac{s(t+1)(\mu + t(s+1-\alpha)}{\mu},\ s(t+1),\ s-1+t(\alpha-1),\ \mu\right).$$

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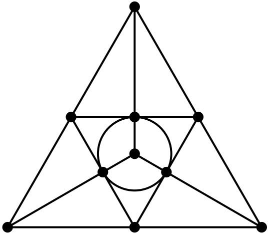
$$SRG\left(1+\frac{s(t+1)(\mu+t(s+1-\alpha))}{\mu},\, s(t+1),\, s-1+t(\alpha-1),\, \mu\right).$$

The Desargues configuration is a $spg(2,2,2,4)$ with both associated graphs strongly regular.

Other such configurations

Another example of a (10_3) configuration:



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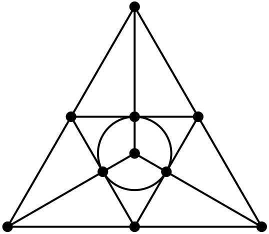

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This configuration is **not** a semipartial geometry and does not belong to other known generalizations of partial geometries such as strongly regular (α, β)-geometries.

N. Hamilton, R. Mathon, *Strongly regular (α, β)-geometries*, J. Combin. Theory Ser. A **95** (2001), no. 2, 234–250.

Non-symmetric examples?

A. E. Brouwer, W. H. Haemers, V. D. Tonchev, *Embedding partial geometries in Steiner designs*, in: *Geometry, combinatorial designs and related structures (Spetses, 1996)*, London Math. Soc. Lecture Note Ser., **245**, Cambridge Univ. Press, Cambridge, 1997, pp. 33–41.

Theorem.

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If the point graph of a (v_r, b_k) configuration is strongly regular, then the configuration is a partial geometry or $v \leq b$.

Corollary.

If both associated graphs of a (v_r, b_k) configuration are strongly regular, then the configuration is a partial geometry or $v = b$.

Strongly regular configurations

Definition.

A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a symmetric (v_k) configuration with the point graph a $SRG(v, k(k-1), \lambda, \mu)$.

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In a $(v_k; \lambda, \mu)$ configuration, the line graph is also a $SRG(v, k(k-1), \lambda, \mu)$. If the incidence matrix is singular, the configuration is a partial geometry.

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We shall call strongly regular configurations with non-singular incidence matrices proper.

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We shall call strongly regular configurations with non-singular incidence matrices proper. This can be recognised from the parameters:

Proposition.

A strongly regular $(v_k; \lambda, \mu)$ configuration that is not a projective plane is proper if and only if $(v-k)(\lambda+1) > k(k-1)^3$ holds.

A family of semipartial geometries

I. Debroey, J. A. Thas, *On semipartial geometries*, J. Comb. Theory A **25** (1978), 242–250.

Family (g) a.k.a. $LP(n, q)$:

- POINTS are lines of the projective space $PG(n, q)$, $n \geq 3$,
- LINES are 2-planes of $PG(n, q)$, and incidence is inclusion.

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$$v = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q, \quad b = \begin{bmatrix} n+1 \\ 3 \end{bmatrix}_q, \quad r = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q.$$

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Lemma.

Two lines of $PG(n, q)$ are coplanar if and only if they intersect.

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$\rightsquigarrow$ semipartial geometry $spg(k-1, r-1, q+1, (q+1)^2)$

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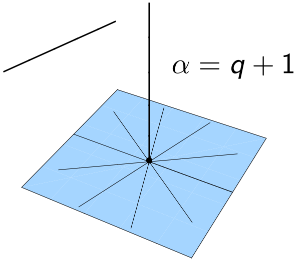

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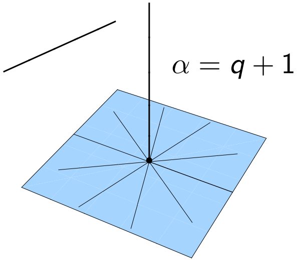

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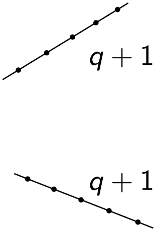

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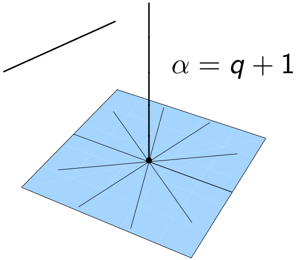

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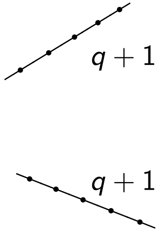

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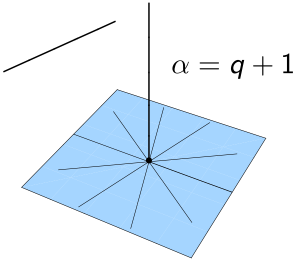

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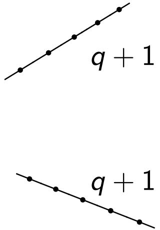

$LP(n, q)$ is a partial geometry $\iff$ $n = 3$

$LP(n, q)$ is symmetric $\iff$ $n = 4$

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We first found three such examples for $q = 2$ by computer. By studying them we discovered a general construction for any prime power q.

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Are there strongly regular configurations with the same parameters that are not semipartial geometries?

We first found three such examples for $q = 2$ by computer. By studying them we discovered a general construction for any prime power q.

The construction is similar to:

D. Jungnickel, V. D. Tonchev, *Polarities, quasi-symmetric designs, and Hamada's conjecture*, Des. Codes Cryptogr. **51** (2009), no. 2, 131–140.

Polarity transformations

Let H_0 be a hyperplane of $PG(4, q)$. As a subgeometry, H_0 is isomorphic to $PG(3, q)$ and possesses a polarity π, i.e. an inclusion-reversing involution. The polarity maps the set of projective lines contained in H_0 onto itself.

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We modify incidence of the POINTS and LINES of $LP(4, q)$ contained in H_0: a POINT L (projective line contained in H_0) is incident with a LINE p (projective plane contained in H_0) if $\pi(L) \subseteq p$. For the remaining pairs (L, p), with L or p not contained in H_0, incidence remains unaltered.

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Theorem.

The new incidence structure $LP(4, q)^\pi$ is a strongly regular configuration with the same parameters that is not a semipartial geometry.

Proof.

The POINT and LINE degrees remain the same and there is at most one LINE through every pair of POINTS.

Polarity transformations

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The POINT graphs of $LP(4,q)^{\pi}$ and $LP(4,q)$ are identical. This follows from the Lemma: if L_1 and L_2 are in H_0, $\pi(L_1)$, $\pi(L_1)$ are contained in a plane p if and only if L_1, L_2 intersect in the point $\pi(p)$ and hence are contained in some plane p'.

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The line graph of $LP(4,q)^\pi$ is changed, but remains strongly regular.

The new configuration $LP(4,q)^\pi$ is not a semipartial geometry: take a plane p in H_0 and a projective line L not in H_0 intersecting the hyperplane in the point $\pi(p)$. Then, (L,p) is a non-incident POINT-LINE pair of $LP(4,q)^\pi$. If $\pi(M) \subseteq p$, then M contains $\pi(p)$ and is coplanar with L, i.e. collinear as a POINT of the configuration. Hence, all q^2+q+1 POINTS on p are collinear with L, whereas in a semipartial geometry the number is always 0 or $\alpha = q+1$.

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We define a **dual transformation** of $LP(4,q)$: take a point P_0 of $PG(4,q)$ and consider the quotient geometry of lines, planes and solids containing P_0. It is isomorphic to $PG(3,q)$ and possesses a polarity π' permuting the planes through P_0 and exchanging the lines and solids through P_0.

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We modify incidence in $LP(4, q)$ for projective lines L and planes p through P_0: they are incident if $L \subseteq \pi'(p)$. Other incidences remain unaltered.

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Theorem.

The new incidence structure $LP(4, q)_{\pi'}$ is the dual of $LP(4, q)^{\pi}$.

The LINE graphs of $LP(4, q)$ and $LP(4, q)_{\pi'}$ are identical. The POINT graph of $LP(4, q)_{\pi'}$ is changed.

Polarity transformations

A **fourth** strongly regular configuration $LP(4,q)^{\pi}_{\pi'}$ is obtained if we take a non-incident point P_0 and hyperplane H_0 and apply both transformations. This configuration has the same LINE graph as $LP(4,q)^{\pi}$ and the same POINT graph as $LP(4,q)_{\pi'}$ and is self-dual.

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Theorem.

For every prime power q, there are at least four strongly regular $(v_k;\lambda,\mu)$ configuration with parameters

$$v = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q, \quad k = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q, \quad \lambda = q^3 + 2q^2 + q - 1, \quad \mu = (q+1)^2.$$

One of them is the semipartial geometry $LP(4,q)$ and the others are not semipartial geometries.

Other constructions of strongly regular configurations

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Theorem.

Let $\mathcal{P}$ be a projective plane of order $n \geq 5$ and A, B, C be three non-collinear points. By deleting all points on the lines AB, AC, BC and all lines through the points A, B, C, there remains a strongly regular $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, and $\mu = (n-3)(n-4)$. This configuration is not an (α, β)-geometry.

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Sporadic examples from difference sets:

- $(13_3; 2, 3)$
- $(63_6; 13, 15)$
- $(96_5; 4, 4)$
- $(120_8; 28, 24)$

The End

Thanks for your attention!