Hamiltonian graphs in Abelian 2-groups

Kristijan Tabak
Rochester Institute of Technology, Zagreb Campus

Combinatorics 2022, Mantua (Italy), May 30 - June 3, 2022

This work has been fully supported by Croatian Science Foundation under the projects 6732 and 9752
Graphs defined on Abelian groups of exponent 2
Graphs defined on Abelian groups of exponent 2

\[ E_{2k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]
Graphs defined on Abelian groups of exponent 2

\[ E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]

elementary abelian group, \(|E_{2^k}| = 2^k\).
Graphs defined on Abelian groups of exponent 2

\[ E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]

elementary abelian group, \(|E_{2^k}| = 2^k\).

Let \( E_{2^s}[E_{2^k}] = \{ T \leq E_{2^k} \mid T \cong E_{2^s} \} \).
Graphs defined on Abelian groups of exponent 2

\[ E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]

elementary abelian group, \( |E_{2^k}| = 2^k \).

Let \( E_{2^s}[E_{2^k}] = \{ T \leq E_{2^k} \mid T \cong E_{2^s} \} \).

Let \( E_{2^s}[T, H]^{-1} = \{ S \mid T \leq S \leq H, \ S \cong E_{2^s} \} \).
Graphs defined on Abelian groups of exponent 2

\[ E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]

elementary abelian group, \(|E_{2^k}| = 2^k|\).

Let \( E_{2^s}[E_{2^k}] = \{T \leq E_{2^k} \mid T \cong E_{2^s}\} \).

Let \( E_{2^s}[T, H]^{-1} = \{S \mid T \leq S \leq H, \ S \cong E_{2^s}\} \)

\( E_{2^s}\)-subgroups that contain \( T \) and that are also contained in \( H \).
Graphs defined on Abelian groups of exponent 2

\[ E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]

elementary abelian group, \(|E_{2^k}| = 2^k\).

Let \( E_{2^s}[E_{2^k}] = \{ T \leq E_{2^k} \mid T \cong E_{2^s}\} \).

Let \( E_{2^s}[T, H]^{-1} = \{ S \mid T \leq S \leq H, \ S \cong E_{2^s}\} \)

\( E_{2^s}\)-subgroups that contain \( T \) and that are also contained in \( H \).

If \( t \leq s \leq m \), \( H \cong E_{2^m} \), and \( T \cong E_{2^t} \), then
Graphs defined on Abelian groups of exponent 2

\[ E_{2k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1 \]

elementary abelian group, \(|E_{2k}| = 2^k\).

Let \( E_{2s}[E_{2k}] = \{ T \leq E_{2k} \mid T \cong E_{2s} \} \).

Let \( E_{2s}[T, H]^{-1} = \{ S \mid T \leq S \leq H, \ S \cong E_{2s} \} \)

\( E_{2s} \)-subgroups that contain \( T \) and that are also contained in \( H \).

If \( t \leq s \leq m \), \( H \cong E_{2m} \), and \( T \cong E_{2t} \), then

\[ |E_{2s}[T, H]^{-1}| = |E_{2s-t}[H/T]| = |E_{2s-t}[E_{2m-t}]| = \left[ \begin{array}{c} m-t \\ s-t \end{array} \right]_2, \]
Graphs defined on Abelian groups of exponent 2

\[ E_{2k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \ x_i^2 = 1 \]

elementary abelian group, \( |E_{2k}| = 2^k \).

Let \( E_{2s}[E_{2k}] = \{ T \leq E_{2k} \mid T \cong E_{2s} \} \).

Let \( E_{2s}[T, H]^{-1} = \{ S \mid T \leq S \leq H, \ S \cong E_{2s} \} \)

\( E_{2s} \)-subgroups that contain \( T \) and that are also contained in \( H \).

If \( t \leq s \leq m, \ H \cong E_{2m}, \) and \( T \cong E_{2t}, \) then

\[ |E_{2s}[T, H]^{-1}| = |E_{2s-t}[H/T]| = |E_{2s-t}[E_{2m-t}]| = \begin{bmatrix} m-t \\ s-t \end{bmatrix}_2, \]

where \( H/T \) is a quotient group isomorphic to \( E_{2m-t} \) and \( \begin{bmatrix} a \\ b \end{bmatrix}_2 \) is a Gaussian coefficient.
\((E_{2^{2k}}[E_{2k}], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_{2^k}\),
\((E_2 \mathbb{[} E_2^k], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_2^k\), where \(T \cong E_{2^2} = \mathbb{Z}_2 \times \mathbb{Z}_2\).
\((E_2^2[E_2^k], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_2^k\),

where \(T \cong E_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2\).

Edges \(\mathcal{E}_k\) are defined as follows:
$(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is a graph with vertices $T \leq E_{2^k}$, where $T \cong E_{2^2} = \mathbb{Z}_2 \times \mathbb{Z}_2$.

Edges $\mathcal{E}_k$ are defined as follows:

$$\{T_1, T_2\} \in \mathcal{E}_k \iff T_1 \cap T_2 \cong \mathbb{Z}_2.$$
\((E_{2^2}[E_{2^k}], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_{2^k}\),

where \(T \cong E_{2^2} = \mathbb{Z}_2 \times \mathbb{Z}_2\).

Edges \(\mathcal{E}_k\) are defined as follows:

\[\{T_1, T_2\} \in \mathcal{E}_k \iff T_1 \cap T_2 \cong \mathbb{Z}_2.\]

Theorem: (Ore)
\((E_2^2[E_2^k], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_2^k\),

where \(T \cong E_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2\).

Edges \(\mathcal{E}_k\) are defined as follows:

\[
\{T_1, T_2\} \in \mathcal{E}_k \iff T_1 \cap T_2 \cong \mathbb{Z}_2.
\]

**Theorem: (Ore)**

Let \(G\) be a connected graph with \(n > 3\) vertices. If \(\text{deg}(x) + \text{deg}(y) > n\) for all non-adjacent vertices \(x\) and \(y\), then \(G\) is Hamiltonian.
\((E_{2^k}[E_{2^k}], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_{2^k}\),
where \(T \cong E_{2^2} = \mathbb{Z}_2 \times \mathbb{Z}_2\).

Edges \(\mathcal{E}_k\) are defined as follows:
\[
\{T_1, T_2\} \in \mathcal{E}_k \iff T_1 \cap T_2 \cong \mathbb{Z}_2.
\]

**Theorem: (Ore)**

Let \(G\) be a connected graph with \(n > 3\) vertices. If \(\deg(x) + \deg(y) > n\) for all non-adjacent vertices \(x\) and \(y\), then \(G\) is Hamiltonian.

**Corollary:**
\((E_2^2[E_2^k], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_2^k\),

where \(T \cong E_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2\).

Edges \(\mathcal{E}_k\) are defined as follows:

\[\{T_1, T_2\} \in \mathcal{E}_k \iff T_1 \cap T_2 \cong \mathbb{Z}_2.\]

**Theorem: (Ore)**

Let \(G\) be a connected graph with \(n > 3\) vertices. If \(\deg(x) + \deg(y) > n\) for all non-adjacent vertices \(x\) and \(y\), then \(G\) is Hamiltonian.

**Corollary:**

If \(G = (V, E)\) is \(r\)-regular graph and if \(\deg(x) > \frac{1}{2}|V|\), then \(G\) is Hamiltonian.
\((E_2^2[E_2^k], \mathcal{E}_k)\) is a graph with vertices \(T \leq E_2^k\),

where \(T \cong E_2^2 = \mathbb{Z}_2 \times \mathbb{Z}_2\).

Edges \(\mathcal{E}_k\) are defined as follows:

\[
\{T_1, T_2\} \in \mathcal{E}_k \iff T_1 \cap T_2 \cong \mathbb{Z}_2.
\]

**Theorem: (Ore)**

Let \(G\) be a connected graph with \(n > 3\) vertices. If \(\text{deg}(x) + \text{deg}(y) > n\) for all non-adjacent vertices \(x\) and \(y\), then \(G\) is Hamiltonian.

**Corollary:**

If \(G = (V, E)\) is \(r\)-regular graph and if \(\text{deg}(x) > \frac{1}{2}|V|\), then \(G\) is Hamiltonian.

Ore’s Theorem immediately yields that \((E_2^2[E_2^3], \mathcal{E}_3)\) and \((E_2^2[E_2^4], \mathcal{E}_4)\) are Hamiltonian.
Theorem:
Theorem:
A graph \((E_{2^2}[E_{2^k}], \mathcal{E}_k)\) is \(6(2^{k-2} - 1)\)-regular. The inequality
\[
\frac{1}{2}|E_{2^2}[E_{2^k}]| - \deg(V) < 0
\]
holds for all \(V \in E_{2^2}[E_{2^k}]\) if any only if \(k < 5\).
Theorem:
A graph $(E_{2^2}[E_{2^k}], E_k)$ is $6(2^{k-2} - 1)$-regular. The inequality

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - \text{deg}(V) < 0$$

holds for all $V \in E_{2^2}[E_{2^k}]$ if and only if $k < 5$.

Sketch of a proof:
Theorem:
A graph \((E_2[E_2^k], \mathcal{E}_k)\) is \(6(2^{k-2} - 1)\)-regular. The inequality

\[
\frac{1}{2}|E_2[E_2^k]| - \deg(V) < 0
\]

holds for all \(V \in E_2[E_2^k]\) if any only if \(k < 5\).

Sketch of a proof:
Let \(V\) be a vertex and let
**Theorem:**
A graph $(E_{2^2}[E_{2^k}], E_k)$ is $6(2^{k-2} - 1)$-regular. The inequality

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - deg(V) < 0$$

holds for all $V \in E_{2^2}[E_{2^k}]$ if and only if $k < 5$.

**Sketch of a proof:**

Let $V$ be a vertex and let

$$n(V) = \left[ \bigcup_{g \in V^*} E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \right] \setminus \{V\},$$
Theorem:
A graph \((E_2^2[E_{2k}], E_k)\) is \(6(2^{k-2} - 1)\)-regular. The inequality
\[
\frac{1}{2}|E_2^2[E_{2k}]| - \deg(V) < 0
\]
holds for all \(V \in E_2^2[E_{2k}]\) if and only if \(k < 5\).

Sketch of a proof:
Let \(V\) be a vertex and let
\[
n(V) = \left[\bigcup_{g \in V^*} E_2^2[\langle g \rangle, E_{2k}]^{-1}\right] \setminus \{V\},
\]
the collection of all vertices adjacent to \(V\),
Theorem:
A graph \((E_2^2[E_2^k], \mathcal{E}_k)\) is \(6(2^{k-2} - 1)\)-regular. The inequality

\[
\frac{1}{2} |E_2^2[E_2^k]| - \deg(V) < 0
\]

holds for all \(V \in E_2^2[E_2^k]\) if and only if \(k < 5\).

Sketch of a proof:
Let \(V\) be a vertex and let

\[
n(V) = \left[ \bigcup_{g \in V^*} E_2^2[\langle g \rangle, E_2^k]^{-1} \right] \setminus \{V\},
\]

the collection of all vertices adjacent to \(V\),
also
Theorem:
A graph $(E_2^2[E_2^k], E_k)$ is $6(2^{k-2} - 1)$-regular. The inequality

$$\frac{1}{2} |E_2^2[E_2^k]| - \deg(V) < 0$$

holds for all $V \in E_2^2[E_2^k]$ if any only if $k < 5$.

Sketch of a proof:
Let $V$ be a vertex and let

$$n(V) = \left[ \bigcup_{g \in V^*} E_2^2[\langle g \rangle, E_2^k]^{-1} \right] \setminus \{V\},$$

the collection of all vertices adjacent to $V$,

also

$$|E_2^2[\langle g \rangle, E_2^k]^{-1}| = |E_2[E_2^k/\langle g \rangle]| = |E_2[E_2^{k-1}]| = 2^{k-1} - 1.$$
Theorem:
A graph \((E_2^2[E_2^k], \mathcal{E}_k)\) is \(6(2^{k-2} - 1)\)-regular. The inequality
\[
\frac{1}{2}|E_2^2[E_2^k]| - deg(V) < 0
\]
holds for all \(V \in E_2^2[E_2^k]\) if and only if \(k < 5\).

Sketch of a proof:
Let \(V\) be a vertex and let
\[
n(V) = \left[ \bigcup_{g \in V^*} E_2^2[\langle g \rangle, E_2^k]^{-1} \right] \setminus \{V\},
\]
the collection of all vertices adjacent to \(V\),
also
\[
|E_2^2[\langle g \rangle, E_2^k]^{-1}| = |E_2[E_2^k/\langle g \rangle]| = |E_2[E_2^{k-1}]| = 2^{k-1} - 1.
\]
Using the inclusion-exclusion formula, the following holds
Using the inclusion-exclusion formula, the following holds

\[
\text{deg}(V) = \sum_{g \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| - \\
\sum_{g \neq h, \ g, h \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1}| + \\
\sum_{g \neq h \neq k \neq g, \ g, h, k \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k \rangle, E_{2^k}]^{-1}| - 1 = \\
\left(3 \choose 1\right)(2^{k-1} - 1) - \left(3 \choose 2\right) \cdot 1 + 1 - 1 = 6(2^{k-2} - 1).
\]
Using the inclusion-exclusion formula, the following holds

\[ \deg(V) = \sum_{g \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| - \sum_{g \neq h, \ g, h \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1}| + \sum_{g \neq h \neq k \neq g, \ g, h, k \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k \rangle, E_{2^k}]^{-1}| - 1 = \]

\[ = \binom{3}{1}(2^{k-1} - 1) - \binom{3}{2} \cdot 1 + 1 - 1 = 6(2^{k-2} - 1). \]

Notice that
Using the inclusion-exclusion formula, the following holds

\[
\deg(V) = \sum_{g \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| -
\]

\[
- \sum_{g \neq h, \ g, h \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1}| +
\]

\[
+ \sum_{g \neq h \neq k \neq g, \ g, h, k \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k \rangle, E_{2^k}]^{-1}| - 1 =
\]

\[
= \binom{3}{1}(2^{k-1} - 1) - \binom{3}{2} \cdot 1 + 1 - 1 = 6(2^{k-2} - 1).
\]

Notice that

\[
|E_{2^2}[E_{2^k}]| = \begin{bmatrix} k \\ 2 \end{bmatrix} = \frac{1}{3}(2^k - 1)(2^{k-1} - 1).
\]
Using the inclusion-exclusion formula, the following holds

\[ \deg(V) = \sum_{g \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| - \sum_{g \neq h, \ g, h \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1}| + \sum_{g \neq h \neq k \neq g, \ g, h, k \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k \rangle, E_{2^k}]^{-1}| - 1 = \]

\[ = \binom{3}{1}(2^{k-1} - 1) - \binom{3}{2} \cdot 1 + 1 - 1 = 6(2^{k-2} - 1). \]

Notice that

\[ |E_{2^2}[E_{2^k}]| = \begin{bmatrix} k \\ 2 \end{bmatrix}_2 = \frac{1}{3}(2^k - 1)(2^{k-1} - 1). \]

Put \( t = 2^{k-2} \).
Therefore,

\[
\frac{1}{2} \left| E_{2^k} \right| - deg(V) = \frac{1}{6} (4t-1)(2t-1) - 6(t-1) = \frac{1}{6} (8t^2 - 42t + 37).
\]
Therefore,

\[
\frac{1}{2}|E_2[E_{2k}]| - \deg(V) = \frac{1}{6}(4t - 1)(2t - 1) - 6(t - 1) = \frac{1}{6}(8t^2 - 42t + 37).
\]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).
Therefore,

\[
\frac{1}{2}|E_2[E_{2k}]| - \deg(V) = \frac{1}{6}(4t - 1)(2t - 1) - 6(t - 1) = \frac{1}{6}(8t^2 - 42t + 37).
\]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).

For \( k \geq 5 \) we have \( 8t^2 - 42t + 37 > 0 \).
Therefore,

\[ \frac{1}{2} |E_{2^2}[E_{2^k}]| - \text{deg}(V) = \frac{1}{6}(4t - 1)(2t - 1) - 6(t - 1) = \frac{1}{6}(8t^2 - 42t + 37). \]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).

For \( k \geq 5 \) we have \( 8t^2 - 42t + 37 > 0 \).

Thus, we can not use Ore’s theorem for \( k \geq 5 \).
Therefore,
\[
\frac{1}{2} |E_2^2[E_2^k]| - \deg(V) = \frac{1}{6} (4t - 1)(2t - 1) - 6(t - 1) = \frac{1}{6} (8t^2 - 42t + 37).
\]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).

For \( k \geq 5 \) we have \( 8t^2 - 42t + 37 > 0 \).

Thus, we can not use Ore’s theorem for \( k \geq 5 \).

**Hamiltonian cycle in** \((E_2^2[E_2^5], \mathcal{E}_5)\)
Therefore,

\[
\frac{1}{2} |E_{2^2}[E_{2^k}]| - \deg(V) = \frac{1}{6} (4t - 1)(2t - 1) - 6(t - 1) = \frac{1}{6} (8t^2 - 42t + 37).
\]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).

For \( k \geq 5 \) we have \( 8t^2 - 42t + 37 > 0 \).

Thus, we can not use Ore’s theorem for \( k \geq 5 \).

**Hamiltonian cycle in** \((E_{2^2}[E_{2^5}], \mathcal{E}_5)\)

Let \( E_{2^5} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle = \langle a, b, c, d, e \rangle \),
Therefore,

\[
\frac{1}{2}|E_2^2[E_{2^k}]| - \deg(V) = \frac{1}{6}(4t-1)(2t-1) - 6(t-1) = \frac{1}{6}(8t^2 - 42t + 37).
\]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).

For \( k \geq 5 \) we have \( 8t^2 - 42t + 37 > 0 \).

Thus, we can not use Ore’s theorem for \( k \geq 5 \).

**Hamiltonian cycle in** \((E_2^2[E_{2^5}], E_5)\)

Let \( E_{2^5} = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle = \langle a, b, c, d, e \rangle \),

where \( a, b, c, d, e \) are generators of \( E_{2^5} \).
Therefore,
\[ \frac{1}{2} |E_2[E_2^k]| - \deg(V) = \frac{1}{6} (4t - 1)(2t - 1) - 6(t - 1) = \frac{1}{6} (8t^2 - 42t + 37). \]

For \( k = 3 \) and \( k = 4 \) we get \( 8t^2 - 42t + 37 < 0 \).

For \( k \geq 5 \) we have \( 8t^2 - 42t + 37 > 0 \).

Thus, we can not use Ore’s theorem for \( k \geq 5 \).

**Hamiltonian cycle in \( (E_2[E_2^5], \mathcal{E}_5) \)**

Let \( E_2^5 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle \times \langle e \rangle = \langle a, b, c, d, e \rangle \),

where \( a, b, c, d, e \) are generators of \( E_2^5 \).

Any automorphism \( \alpha \in \text{Aut}(E_2^5) \) is represented by its action on a generators like
\[ \alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix}, \]
\[ \alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix}, \]

for some \( g_i \in E_{25}^* \) such that \( \langle g_i \mid i = 1, \ldots, 5 \rangle = E_{25} \).
\[ \alpha = \left( \begin{array}{cccccc} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \right), \]

for some \( g_i \in E_{2^5}^* \) such that \( \langle g_i \mid i = 1, \ldots, 5 \rangle = E_{2^5} \).

If \( \alpha \) is of order \( n \),
\[ \alpha = \left( \begin{array}{ccccc} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{array} \right), \]

for some \( g_i \in E_{2^5}^* \) such that \( \langle g_i \mid i = 1, \ldots, 5 \rangle = E_{2^5} \)

If \( \alpha \) is of order \( n \),

then an orbit \( X^{(\alpha)} \) can be represented in a group ring \( \mathbb{Z}[E_{2^5}] \) like this:
$$\alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix},$$

for some $g_i \in E_{25}^*$ such that $\langle g_i \mid i = 1, \ldots, 5 \rangle = E_{25}$.

If $\alpha$ is of order $n$,

then an orbit $X^{\langle \alpha \rangle}$ can be represented in a group ring $\mathbb{Z}[E_{25}]$ like this:

$$X^{\langle \alpha \rangle} = X + X^\alpha + \cdots + X^{\alpha^{n-1}}.$$
\[ \alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix}, \]

for some \( g_i \in E_{25}^* \) such that \( \langle g_i \mid i = 1, \ldots, 5 \rangle = E_{25} \).

If \( \alpha \) is of order \( n \),

then an orbit \( X^{(\alpha)} \) can be represented in a group ring \( \mathbb{Z}[E_{25}] \) like this:

\[ X^{(\alpha)} = X + X^\alpha + \cdots + X^{\alpha^{n-1}}. \]

The following is crucial for a construction of a Hamiltonian cycle in \((E_{22}[E_{25}], \mathcal{E}_5)\).
Lemma: Let $E_{25} = \langle a, b, c, d, e \rangle$ and let $\alpha \in \text{Aut}(E_{25})$ be given by

\[
\alpha = \begin{pmatrix}
a & b & c & d & e \\
b c & c d & b c d & b c d & e a
\end{pmatrix},
\]

then $o(\alpha) = 31$ and $H^{(\alpha)} = E_{24}[E_{25}]$ where $H = \langle a, b, c, d \rangle$. If $T = \langle a, b, c \rangle$ and $\Delta_i = T \cap T^{\alpha^i}$ for $i \in \mathbb{Z}_{31}$, then

\[
\Delta_i = \begin{cases}
\langle b, c \rangle, & \text{if } i = 1, 14 \\
\langle a, bc \rangle, & \text{if } i = 13, 30 \\
\langle ab, c \rangle, & \text{if } i = 17, 18 \\
\cong \mathbb{Z}_2 & \text{otherwise.}
\end{cases}
\]
Lemma: Let $E_{25} = \langle a, b, c, d, e \rangle$ and let $\alpha \in \text{Aut}(E_{25})$ be given by $\alpha = 
abla \begin{pmatrix} a & b & c & d & e \\ bc & cd & bcd & de & a \end{pmatrix}$, then $o(\alpha) = 31$ and $H^{\langle \alpha \rangle} = E_{24}[E_{25}]$ where $H = \langle a, b, c, d \rangle$. If $T = \langle a, b, c \rangle$ and $\Delta_i = T \cap T^{\alpha^i}$ for $i \in \mathbb{Z}_{31}$, then

$$
\Delta_i = \begin{cases} 
\langle b, c \rangle, & \text{if } i = 1, 14 \\
\langle a, bc \rangle, & \text{if } i = 13, 30 \\
\langle ab, c \rangle, & \text{if } i = 17, 18 \\
\cong \mathbb{Z}_2 & \text{otherwise.}
\end{cases}
$$

We can rewrite an automorphism $\alpha$ in a simplified form over $\mathbb{Z}_2$. 
Lemma: Let $E_{25} = \langle a, b, c, d, e \rangle$ and let $\alpha \in \text{Aut}(E_{25})$ be given by $\alpha = \begin{pmatrix} a & b & c & d & e \\ bc & cd & bcd & de & a \end{pmatrix}$, then $o(\alpha) = 31$ and $H^{(\alpha)} = E_{24}[E_{25}]$ where $H = \langle a, b, c, d \rangle$. If $T = \langle a, b, c \rangle$ and $\Delta_i = T \cap T^{\alpha^i}$ for $i \in \mathbb{Z}_{31}$, then

$$
\Delta_i = \begin{cases} 
\langle b, c \rangle, & \text{if } i = 1, 14 \\
\langle a, bc \rangle, & \text{if } i = 13, 30 \\
\langle ab, c \rangle, & \text{if } i = 17, 18 \\
\cong \mathbb{Z}_2 & \text{otherwise.}
\end{cases}
$$

We can rewrite an automorphism $\alpha$ in a simplified form over $\mathbb{Z}_2$

$$
\alpha = \begin{pmatrix} 
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 
\end{pmatrix}.
$$
Lemma: Let $E_{2^5} = \langle a, b, c, d, e \rangle$ and let $\alpha \in \text{Aut}(E_{2^5})$ be given by $\alpha = \begin{pmatrix} a & b & c & d & e \\ bc & cd & bcd & de & a \end{pmatrix}$, then $o(\alpha) = 31$ and $H^{(\alpha)} = E_{2^4}[E_{2^5}]$ where $H = \langle a, b, c, d \rangle$. If $T = \langle a, b, c \rangle$ and $\Delta_i = T \cap T^{\alpha^i}$ for $i \in \mathbb{Z}_{31}$, then

$$\Delta_i = \begin{cases} 
\langle b, c \rangle, & \text{if } i = 1, 14 \\
\langle a, bc \rangle, & \text{if } i = 13, 30 \\
\langle ab, c \rangle, & \text{if } i = 17, 18 \\
\cong \mathbb{Z}_2 & \text{otherwise.}
\end{cases}$$

We can rewrite an automorphism $\alpha$ in a simplified form over $\mathbb{Z}_2$

$$\alpha = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}.$$
we introduce slightly different notation:
we introduce slightly different notation:

\[ \Delta_{\Omega_1} = \langle b, c \rangle, \quad \Omega_1 = \{1, 14\}, \]
\[ \Delta_{\Omega_2} = \langle a, bc \rangle, \quad \Omega_2 = \{13, 30\}, \]
\[ \Delta_{\Omega_3} = \langle ab, c \rangle, \quad \Omega_3 = \{17, 18\}. \]
we introduce slightly different notation:

\[ \Delta_{\Omega_1} = \langle b, c \rangle, \; \Omega_1 = \{1, 14\}, \]
\[ \Delta_{\Omega_2} = \langle a, bc \rangle, \; \Omega_2 = \{13, 30\}, \]
\[ \Delta_{\Omega_3} = \langle ab, c \rangle, \; \Omega_3 = \{17, 18\}. \]

Lemma: Groups \( \Delta_{\alpha^k_{\Omega_i}} \) and \( \Delta_{\Omega_i} \) are distinct for all \( i \in [3] \) and \( k \in [30] \).
we introduce slightly different notation:

\[ \Delta_{\Omega_1} = \langle b, c \rangle, \ \Omega_1 = \{1, 14\}, \]
\[ \Delta_{\Omega_2} = \langle a, bc \rangle, \ \Omega_2 = \{13, 30\}, \]
\[ \Delta_{\Omega_3} = \langle ab, c \rangle, \ \Omega_3 = \{17, 18\}. \]

Lemma: Groups \( \Delta_{\Omega_i}^\alpha \) and \( \Delta_{\Omega_i} \) are distinct for all \( i \in [3] \) and \( k \in [30] \).

Corollary: If \( \Delta_{\Omega_i}^\alpha = \Delta_{\Omega_j} \), then \( \alpha^k \) is a unique element from \( \langle \alpha \rangle \).
we introduce slightly different notation:

\[ \Delta_{\Omega_1} = \langle b, c \rangle, \quad \Omega_1 = \{1, 14\}, \]
\[ \Delta_{\Omega_2} = \langle a, bc \rangle, \quad \Omega_2 = \{13, 30\}, \]
\[ \Delta_{\Omega_3} = \langle ab, c \rangle, \quad \Omega_3 = \{17, 18\}. \]

**Lemma:** Groups \( \Delta_{\Omega_i}^{\alpha^k} \) and \( \Delta_{\Omega_i} \) are distinct for all \( i \in [3] \) and \( k \in [30] \).

**Corollary:** If \( \Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_j} \), then \( \alpha^k \) is a unique element from \( \langle \alpha \rangle \).

**Lemma:** Subgroups \( \Delta_{\Omega_i} \), \( i \in [3] \) satisfy the following:

\[ \Delta_{\Omega_1}^{\alpha^30} = \Delta_{\Omega_2}, \quad \Delta_{\Omega_2}^{\alpha^18} = \Delta_{\Omega_3}, \quad \Delta_{\Omega_3}^{\alpha^14} = \Delta_{\Omega_1}. \]
we introduce slightly different notation:

\[ \Delta_{\Omega_1} = \langle b, c \rangle, \ \Omega_1 = \{1, 14\}, \]
\[ \Delta_{\Omega_2} = \langle a, bc \rangle, \ \Omega_2 = \{13, 30\}, \]
\[ \Delta_{\Omega_3} = \langle ab, c \rangle, \ \Omega_3 = \{17, 18\}. \]

**Lemma:** Groups \( \Delta_{\Omega_i}^{\alpha_k} \) and \( \Delta_{\Omega_i} \) are distinct for all \( i \in [3] \) and \( k \in [30] \).

**Corollary:** If \( \Delta_{\Omega_i}^{\alpha_k} = \Delta_{\Omega_j} \), then \( \alpha_k \) is a unique element from \( \langle \alpha \rangle \).

**Lemma:** Subgroups \( \Delta_{\Omega_i}, i \in [3] \) satisfy the following:

\[ \Delta_{\Omega_1}^{\alpha_{30}} = \Delta_{\Omega_2}, \ \Delta_{\Omega_2}^{\alpha_{18}} = \Delta_{\Omega_3}, \ \Delta_{\Omega_3}^{\alpha_{14}} = \Delta_{\Omega_1}. \]

**Theorem:** For \( T \) and \( \alpha \) the following holds

\[ \bigcup_{i=0}^{30} E_{2^2}[T^{\alpha_i}] = E_{2^2}[E_{2^5}]. \]
we introduce slightly different notation:

\[
\Delta_{\Omega_1} = \langle b, c \rangle, \ \Omega_1 = \{1, 14\},
\]
\[
\Delta_{\Omega_2} = \langle a, bc \rangle, \ \Omega_2 = \{13, 30\},
\]
\[
\Delta_{\Omega_3} = \langle ab, c \rangle, \ \Omega_3 = \{17, 18\}.
\]

**Lemma:** Groups \(\Delta_{\Omega_i}^{\alpha^k}\) and \(\Delta_{\Omega_i}\) are distinct for all \(i \in [3]\) and \(k \in [30]\).

**Corollary:** If \(\Delta_{\Omega_i}^{\alpha^k} = \Delta_{\Omega_j}\), then \(\alpha^k\) is a unique element from \(\langle \alpha \rangle\).

**Lemma:** Subgroups \(\Delta_{\Omega_i}, \ i \in [3]\) satisfy the following:

\[
\Delta_{\Omega_1}^{\alpha^{30}} = \Delta_{\Omega_2}, \ \Delta_{\Omega_2}^{\alpha^{18}} = \Delta_{\Omega_3}, \ \Delta_{\Omega_3}^{\alpha^{14}} = \Delta_{\Omega_1}.
\]

**Theorem:** For \(T\) and \(\alpha\) the following holds

\[
\bigcup_{i=0}^{30} E_2[T^{\alpha^i}] = E_2[E_5].
\]
Sketch of a proof:
Sketch of a proof:

The total number of all $E_2^2$ subgroups of $E_2^5$ is
Sketch of a proof:

The total number of all $E_2^2$ subgroups of $E_2^5$ is

$$|E_2^2[E_2^5]| = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_2 = 31 \cdot 5.$$
Sketch of a proof:

The total number of all $E_2^2$ subgroups of $E_2^5$ is

$$|E_2^2[E_2^5]| = \left[ \begin{array}{c} 5 \\ 2 \end{array} \right]_2 = 31 \cdot 5.$$ 

Using the inclusion-exclusion formula
Sketch of a proof:

The total number of all $E_2^2$ subgroups of $E_2^5$ is

$$|E_2^2[E_2^5]| = \binom{5}{2}_2 = 31 \cdot 5.$$  

Using the inclusion-exclusion formula

$$\bigg| \bigcup_{i=0}^{30} E_2^2[T^{\alpha_i}] \bigg| = \sum_{i=0}^{30} |E_2^2[T^{\alpha_i}]| - \sum_{0 \leq i < j \leq 30} |E_2^2[T^{\alpha_i}] \cap E_2^2[T^{\alpha_j}]| + \cdots +$$

$$\cdots + \sum_{0 \leq i < j < k \leq 30} |E_2^2[T^{\alpha_i}] \cap E_2^2[T^{\alpha_j}] \cap E_2^2[T^{\alpha_k}]| =$$

$$= 31 \cdot 7 - 31 \cdot 3 + 31 - 0 + 0 - \cdots = 31 \cdot 5.$$
Sketch of a proof:

The total number of all $E_{2^2}$ subgroups of $E_{2^5}$ is

$$|E_{2^2}[E_{2^5}]| = \binom{5}{2} = 31 \cdot 5.$$  

Using the inclusion-exclusion formula

$$\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha_i}] = \sum_{i=0}^{30} |E_{2^2}[T^{\alpha_i}]| - \sum_{0 \leq i < j \leq 30} |E_{2^2}[T^{\alpha_i}] \cap E_{2^2}[T^{\alpha_j}]| + \cdots +$$

$$\cdots + \sum_{0 \leq i < j < k \leq 30} |E_{2^2}[T^{\alpha_i}] \cap E_{2^2}[T^{\alpha_j}] \cap E_{2^2}[T^{\alpha_k}]| =$$

$$= 31 \cdot 7 - 31 \cdot 3 + 31 - 0 + 0 - \cdots = 31 \cdot 5.$$
Theorem: A graph \((E_2^2[E_2^5], \mathcal{E}_5)\) is Hamiltonian.
Theorem: A graph \((E_{2^2}[E_{2^5}], E_5)\) is Hamiltonian.

Sketch of a proof:
Theorem: A graph \((E_2^2[E_5], E_5)\) is Hamiltonian.

Sketch of a proof:

Since \(T \cong E_{23}\) and \(AB = T\), where
Theorem: A graph $(E_2^2[E_2^5], E_5)$ is Hamiltonian.

Sketch of a proof:

Since $T \cong E_2^3$ and $AB = T$, where

$$A, B \in E_2^2[T^\alpha],$$
Theorem: A graph \((E_{22}[E_{25}], \mathcal{E}_5)\) is Hamiltonian.

Sketch of a proof:

Since \(T \cong E_{23}\) and \(AB = T\), where

\[A, B \in E_{22}[T^{\alpha^i}],\]

it follows that \(|A \cap B| = \frac{|A| \cdot |B|}{|E_{23}|} = 2.\]
Theorem: A graph \((E_2 [E_5], \mathcal{E}_5)\) is Hamiltonian.

Sketch of a proof:

Since \(T \cong E_2^3\) and \(AB = T\), where \(A, B \in E_2 [T^{\alpha^i}]\),

it follows that \(|A \cap B| = \frac{|A| \cdot |B|}{|E_2^3|} = 2\).

Hence \(A\) and \(B\) are adjacent.
Theorem: A graph \((E_2[E_5], \mathcal{E}_5)\) is Hamiltonian.

Sketch of a proof:

Since \(T \cong E_{23}\) and \(AB = T\), where \(A, B \in E_2[T^{\alpha_i}]\),

it follows that \(|A \cap B| = \frac{|A| \cdot |B|}{|E_{23}|} = 2\).

Hence \(A\) and \(B\) are adjacent.

Vertices in \(E_2[T^{\alpha_i}] \cong K_7\) induce a complete graph on 7.
Theorem: A graph \((E_{22}[E_{25}], E_5)\) is Hamiltonian.

Sketch of a proof:

Since \(T \cong E_{23}\) and \(AB = T\), where

\(A, B \in E_{22}[T^{\alpha_i}]\),

it follows that \(|A \cap B| = \frac{|A| \cdot |B|}{|E_{23}|} = 2\).

Hence \(A\) and \(B\) are adjacent.

Vertices in \(E_{22}[T^{\alpha_i}] \cong K_7\) induce a complete graph on 7

if we delete some vertices together with the edges incident to them from \(E_{22}[T^{\alpha_i}]\),
Theorem: A graph \( (E_{2^2}[E_{2^5}], \mathcal{E}_5) \) is Hamiltonian.

Sketch of a proof:

Since \( T \cong E_{2^3} \) and \( AB = T \), where 
\[ A, B \in E_{2^2}[T^\alpha] \],

it follows that \( |A \cap B| = \frac{|A| \cdot |B|}{|E_{2^3}|} = 2 \).

Hence \( A \) and \( B \) are adjacent.

Vertices in \( E_{2^2}[T^\alpha] \cong K_7 \) induce a complete graph on 7

if we delete some vertices together with the edges incident to them from 
\( E_{2^2}[T^\alpha] \),

there will be a path in a remaining graph that visits each remaining vertex.
The subgraphs $E_{2^2}[T^{\alpha^{i-1}}]$, $E_{2^2}[T^{\alpha^i}]$ and $E_{2^2}[T^{\alpha^{i+1}}]$ have common vertices
The subgraphs $E_{2^2}[T^{\alpha_i-1}]$, $E_{2^2}[T^{\alpha_i}]$ and $E_{2^2}[T^{\alpha_i+1}]$ have common vertices $T^{\alpha_i} \cap T^{\alpha_i-1}$ and $T^{\alpha_i} \cap T^{\alpha_i+1}$. 
The subgraphs $E_2[T^{\alpha i-1}]$, $E_2[T^{\alpha i}]$ and $E_2[T^{\alpha i+1}]$ have common vertices $T^{\alpha i} \cap T^{\alpha i-1}$ and $T^{\alpha i} \cap T^{\alpha i+1}$.

Let $L(T^{\alpha i}) = \{ T^{\alpha i} \cap T^{\alpha i-1}, T^{\alpha i} \cap T^{\alpha i+1} \}$. 
The subgraphs $E_{22}[T^{\alpha_{i-1}}]$, $E_{22}[T^{\alpha_i}]$ and $E_{22}[T^{\alpha_{i+1}}]$ have common vertices

$T^{\alpha_i} \cap T^{\alpha_{i-1}}$ and $T^{\alpha_i} \cap T^{\alpha_{i+1}}$.

Let $L(T^{\alpha_i}) = \{T^{\alpha_i} \cap T^{\alpha_{i-1}}, T^{\alpha_i} \cap T^{\alpha_{i+1}}\}$.

Notice that $L(T^{\alpha_i}) = \{\Delta^{\alpha_{i-1}}, \Delta^{\alpha_i}\}$ (since $T \cap T^{\alpha} = \Delta_1$).
The subgraphs $E_2[T^{\alpha_i-1}]$, $E_2[T^{\alpha_i}]$ and $E_2[T^{\alpha_i+1}]$ have common vertices

$T^{\alpha_i} \cap T^{\alpha_i-1}$ and $T^{\alpha_i} \cap T^{\alpha_i+1}$.

Let $L(T^{\alpha_i}) = \{ T^{\alpha_i} \cap T^{\alpha_i-1}, T^{\alpha_i} \cap T^{\alpha_i+1} \}$.

Notice that $L(T^{\alpha_i}) = \{ \Delta_1^{\alpha_i-1}, \Delta_1^{\alpha_i} \}$ (since $T \cap T^{\alpha} = \Delta_1$).

Vertices $L(T^{\alpha_i})$ are links between neighboring graphs $E_2[T^{\alpha_i-1}]$, $E_2[T^{\alpha_i}]$ and $E_2[T^{\alpha_i+1}]$. 
The subgraphs $E_2[T^{\alpha^i-1}]$, $E_2[T^{\alpha^i}]$ and $E_2[T^{\alpha^i+1}]$ have common vertices $T^{\alpha^i} \cap T^{\alpha^i-1}$ and $T^{\alpha^i} \cap T^{\alpha^i+1}$.

Let $L(T^{\alpha^i}) = \{T^{\alpha^i} \cap T^{\alpha^i-1}, T^{\alpha^i} \cap T^{\alpha^i+1}\}$.

Notice that $L(T^{\alpha^i}) = \{\Delta_1^{\alpha^i-1}, \Delta_1^{\alpha^i}\}$ (since $T \cap T^{\alpha} = \Delta_1$).

Vertices $L(T^{\alpha^i})$ are links between neighboring graphs $E_2[T^{\alpha^i-1}]$, $E_2[T^{\alpha^i}]$ and $E_2[T^{\alpha^i+1}]$.

All vertices in $\bigcup_{i=0}^{30} L(T^{\alpha^i})$ are mutually different (Lemmas above).
The subgraphs $E_{2^2}[T^{\alpha_{i-1}}]$, $E_{2^2}[T^{\alpha_i}]$ and $E_{2^2}[T^{\alpha_{i+1}}]$ have common vertices

$T^{\alpha_i} \cap T^{\alpha_{i-1}}$ and $T^{\alpha_i} \cap T^{\alpha_{i+1}}$.

Let $L(T^{\alpha_i}) = \{T^{\alpha_i} \cap T^{\alpha_{i-1}}, T^{\alpha_i} \cap T^{\alpha_{i+1}}\}$.

Notice that $L(T^{\alpha_i}) = \{\Delta_1^{\alpha_{i-1}}, \Delta_1^{\alpha_i}\}$ (since $T \cap T^{\alpha} = \Delta_1$).

Vertices $L(T^{\alpha_i})$ are links between neighboring graphs $E_{2^2}[T^{\alpha_{i-1}}]$, $E_{2^2}[T^{\alpha_i}]$ and $E_{2^2}[T^{\alpha_{i+1}}]$.

all vertices in $\bigcup_{i=0}^{30} L(T^{\alpha_i})$ are mutually different (Lemmas above).

As the initial step of a recursive construction of a Hamiltonian cycle, we define
The subgraphs $E_{22}[T^{\alpha i-1}]$, $E_{22}[T^{\alpha i}]$ and $E_{22}[T^{\alpha i+1}]$ have common vertices $T^{\alpha i} \cap T^{\alpha i-1}$ and $T^{\alpha i} \cap T^{\alpha i+1}$.

Let $L(T^{\alpha i}) = \{ T^{\alpha i} \cap T^{\alpha i-1}, T^{\alpha i} \cap T^{\alpha i+1} \}$.

Notice that $L(T^{\alpha i}) = \{ \Delta^{\alpha i-1}, \Delta^{\alpha i} \}$ (since $T \cap T^{\alpha} = \Delta_1$).

Vertices $L(T^{\alpha i})$ are links between neighboring graphs $E_{22}[T^{\alpha i-1}]$, $E_{22}[T^{\alpha i}]$ and $E_{22}[T^{\alpha i+1}]$.

All vertices in $\bigcup_{i=0}^{30} L(T^{\alpha i})$ are mutually different (Lemmas above).

As the initial step of a recursive construction of a Hamiltonian cycle, we define

$E_{22}[T^{\alpha i}]_0 = E_{22}[T^{\alpha i}]$ for all $i \in \mathbb{Z}_{31}$. 
The subgraphs $E_{22}[T^\alpha_{i-1}]$, $E_{22}[T^\alpha_i]$ and $E_{22}[T^\alpha_{i+1}]$ have common vertices

$T^\alpha_i \cap T^\alpha_{i-1}$ and $T^\alpha_i \cap T^\alpha_{i+1}$.

Let $L(T^\alpha_i) = \{T^\alpha_i \cap T^\alpha_{i-1}, T^\alpha_i \cap T^\alpha_{i+1}\}$.

Notice that $L(T^\alpha_i) = \{\Delta_1^\alpha_{i-1}, \Delta_1^\alpha_i\}$ (since $T \cap T^\alpha = \Delta_1$).

Vertices $L(T^\alpha_i)$ are links between neighboring graphs $E_{22}[T^\alpha_{i-1}]$, $E_{22}[T^\alpha_i]$ and $E_{22}[T^\alpha_{i+1}]$.

all vertices in $\bigcup_{i=0}^{30} L(T^\alpha_i)$ are mutually different (Lemmas above).

As the initial step of a recursive construction of a Hamiltonian cycle, we define

$E_{22}[T^\alpha_i]_0 = E_{22}[T^\alpha_i]$ for all $i \in \mathbb{Z}_{31}$. 
Assume that we have formed a sequence \( \left( E_{2^2}[T^{\alpha_i}]_{m_i} \right)_{i \in \mathbb{Z}_{31}} \).
Assume that we have formed a sequence \( \left( E_{2^2}[T^{\alpha_i}]_{m_i} \right)_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2}[T^{\alpha_i}] \).
Assume that we have formed a sequence \( \left( E_{2^2}[T^{\alpha^i}]_{m_i} \right)_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2}[T^{\alpha^i}] \).

If there is a vertex \( A \) and \( j \neq i \) such that
Assume that we have formed a sequence \( \left( E_{22}[T^{\alpha^i}]_{m_i} \right)_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{22}[T^{\alpha^i}] \).

If there is a vertex \( A \) and \( j \neq i \) such that

\[
A \in (E_{22}[T^{\alpha^i}]_{m_i} \setminus L(T^{\alpha^i})) \cap E_{22}[T^{\alpha^j}]_{m_j},
\]
Assume that we have formed a sequence \( (E_{2^2}[T^{\alpha_i}]_{m_i})_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2}[T^{\alpha_i}] \).

If there is a vertex \( A \) and \( j \neq i \) such that

\[ A \in (E_{2^2}[T^{\alpha_i}]_{m_i} \setminus L(T^{\alpha_i})) \cap E_{2^2}[T^{\alpha_j}]_{m_j}, \]

then \( A \) is not a link, but it is a vertex in graphs \( E_{2^2}[T^{\alpha_i}]_{m_i} \) and \( E_{2^2}[T^{\alpha_j}]_{m_j} \).
Assume that we have formed a sequence \( \left( E_{2^2[T^{\alpha_i}]} m_i \right)_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2[T^{\alpha_i}]} \).

If there is a vertex \( A \) and \( j \neq i \) such that

\[
A \in (E_{2^2[T^{\alpha_i}]} m_i \setminus L(T^{\alpha_i})) \cap E_{2^2[T^{\alpha_j}]} m_j,
\]

then \( A \) is not a link, but it is a vertex in graphs \( E_{2^2[T^{\alpha_i}]} m_i \) and \( E_{2^2[T^{\alpha_j}]} m_j \).

Then, we delete a vertex \( A \) and the edges incident to it.
Assume that we have formed a sequence \( \left( E_{2^2[T^\alpha]^i} \right)_{m_i} \in \mathbb{Z}_{31} \), where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2[T^\alpha]^i} \).

If there is a vertex \( A \) and \( j \neq i \) such that
\[
A \in \left( E_{2^2[T^\alpha]^i} \setminus L(T^\alpha) \right) \cap E_{2^2[T^\alpha]^j} \cap E_{2^2[T^\alpha]^m},
\]
then \( A \) is not a link, but it is a vertex in graphs \( E_{2^2[T^\alpha]^i} \) and \( E_{2^2[T^\alpha]^j} \).

Then, we delete a vertex \( A \) and the edges incident to it.

In this case let \( E_{2^2[T^\alpha]^i} = E_{2^2[T^\alpha]^i} \setminus \{ A \} \).
Assume that we have formed a sequence \( \left( E_{2^2}[T^{\alpha^i}]_{m_i} \right)_{i \in \mathbb{Z}_{31}}, \)

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2}[T^{\alpha^i}] \).

If there is a vertex \( A \) and \( j \neq i \) such that

\[ A \in \left( E_{2^2}[T^{\alpha^i}]_{m_i} \setminus L(T^{\alpha^i}) \right) \cap E_{2^2}[T^{\alpha^j}]_{m_j}, \]

then \( A \) is not a link, but it is a vertex in graphs \( E_{2^2}[T^{\alpha^i}]_{m_i} \) and \( E_{2^2}[T^{\alpha^j}]_{m_j} \).

Then, we delete a vertex \( A \) and the edges incident to it.

In this case let \( E_{2^2}[T^{\alpha^i}]_{m_i+1} = E_{2^2}[T^{\alpha^i}]_{m_i} \setminus \{ A \}. \)

If such a vertex \( A \) does not exist, we leave \( E_{2^2}[T^{\alpha^i}]_{m_i} \) unchanged and
Assume that we have formed a sequence \( \left( E_{2^2[T^\alpha]}(m_i) \right)_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2[T^\alpha]} \).

If there is a vertex \( A \) and \( j \neq i \) such that
\[
A \in \left( E_{2^2[T^\alpha]}(m_i) \setminus L(T^\alpha) \right) \cap E_{2^2[T^\alpha]}(m_j),
\]
then \( A \) is not a link, but it is a vertex in graphs \( E_{2^2[T^\alpha]}(m_i) \) and \( E_{2^2[T^\alpha]}(m_j) \).

Then, we delete a vertex \( A \) and the edges incident to it.

In this case let \( E_{2^2[T^\alpha]}(m_i+1) = E_{2^2[T^\alpha]}(m_i) \setminus \{ A \} \).

If such a vertex \( A \) does not exist, we leave \( E_{2^2[T^\alpha]}(m_i) \) unchanged and denote that by \( \tilde{E}_{2^2[T^\alpha]}(m_i) \).
Assume that we have formed a sequence \( \left( E_{2^2}[T^{\alpha^i}]_{m_i} \right)_{i \in \mathbb{Z}_{31}} \),

where \( m_i \) is a sequence of integers that count number of steps (deletions) that we have done in the recursive procedure within \( E_{2^2}[T^{\alpha^i}] \).

If there is a vertex \( A \) and \( j \neq i \) such that
\[
A \in \left( E_{2^2}[T^{\alpha^i}]_{m_i} \setminus L(T^{\alpha^i}) \right) \cap E_{2^2}[T^{\alpha^j}]_{m_j},
\]
then \( A \) is not a link, but it is a vertex in graphs \( E_{2^2}[T^{\alpha^i}]_{m_i} \) and \( E_{2^2}[T^{\alpha^j}]_{m_j} \).

Then, we delete a vertex \( A \) and the edges incident to it.

In this case let \( E_{2^2}[T^{\alpha^i}]_{m_i+1} = E_{2^2}[T^{\alpha^i}]_{m_i} \setminus \{ A \} \).

If such a vertex \( A \) does not exist, we leave \( E_{2^2}[T^{\alpha^i}]_{m_i} \) unchanged and denote that by \( \tilde{E}_{2^2}[T^{\alpha^i}]_{m_i} \).

Now, continue the same procedure with \( E_{2^2}[T^{\alpha^i+1}]_{m_i+1} \).
Following this process, after finite number of steps, we will construct a sequence
Following this process, after finite number of steps, we will construct a sequence

$$( \tilde{E}^i_{22} [T^{\alpha^i}]_{m_i} )_{i \in \mathbb{Z}_{31}}.$$
Following this process, after finite number of steps, we will construct a sequence

\((\tilde{E}_{2^2}[T^{\alpha_i}]m_i)_{i \in \mathbb{Z}_{31}}\).

Using a notation in a group ring \(\mathbb{Z}[E_{2^2}[E_{2^5}]]\), we have the following:
Following this process, after finite number of steps, we will construct a sequence
\[ \left( \tilde{E}_{2^2}[T^{\alpha^i}_m] \right)_{i \in \mathbb{Z}_{31}}. \]

Using a notation in a group ring \( \mathbb{Z}[E_{2^2}E_{2^5}] \), we have the following:
\[
\bigcup_{i \in \mathbb{Z}_{31}} \bigcup_{A \in \tilde{E}_{2^2}[T^{\alpha^i}_m]} A = E_{2^2}[E_{2^5}].
\]
Following this process, after finite number of steps, we will construct a sequence
\[
(\tilde{E}_2^2[T^{\alpha^i}]_{m_i})_{i \in \mathbb{Z}_{31}}.
\]

Using a notation in a group ring \( \mathbb{Z}[E_2^2[E_2^5]] \), we have the following:
\[
\bigcup_{i \in \mathbb{Z}_{31}} \bigcup_{A \in \tilde{E}_2^2[T^{\alpha^i}]_{m_i}} A = E_2^2[E_2^5].
\]

\[
\bigcup_{i=0}^{30} E_2^2[T^{\alpha^i}] \text{ contains all edges in } E_2^5.
\]
Following this process, after finite number of steps, we will construct a sequence
\[(\tilde{E}_{2^2}[T^{\alpha^i}]_{m_i})_{i \in \mathbb{Z}_{31}}.\]

Using a notation in a group ring \(\mathbb{Z}[E_{2^2}[E_{2^5}]]\), we have the following:

\[
\bigcup_{i \in \mathbb{Z}_{31}} \bigcup_{A \in \tilde{E}_{2^2}[T^{\alpha^i}]_{m_i}} A = E_{2^2}[E_{2^5}].
\]

\[\bigcup_{i=0}^{30} E_{2^2}[T^{\alpha^i}] \text{ contains all edges in } E_{2^5}.\]

From \(|E_{2^2}[T^{\alpha^i}]| = 7\) and the fact that we do not delete links in this procedure,
Following this process, after finite number of steps, we will construct a sequence

\((\tilde{E}_2^2[T^{\alpha^i}]m_i)_{i \in \mathbb{Z}_{31}}\).

Using a notation in a group ring \(\mathbb{Z}[E_2^2[E_2^5]]\), we have the following:

\[
\bigcup_{i \in \mathbb{Z}_{31}} \bigcup_{A \in \tilde{E}_2^2[T^{\alpha^i}]m_i} A = E_2^2[E_2^5].
\]

\[\bigcup_{i=0}^{30} E_2^2[T^{\alpha^i}]\] contains all edges in \(E_2^5\).

From \(|E_2^2[T^{\alpha^i}]| = 7\) and the fact that we do not delete links in this procedure,

we get \(m_i \leq 5\) and \(\tilde{E}_2^2[T^{\alpha^i}]m_i \cong K_{7-m_i}\).
Following this process, after finite number of steps, we will construct a sequence
\[
(\tilde{E}_2^2[T^{\alpha_i}]m_i)_{i \in \mathbb{Z}_{31}}.
\]

Using a notation in a group ring \( \mathbb{Z}[E_2^2[E_2^5]] \), we have the following:
\[
\bigcup_{i \in \mathbb{Z}_{31}} \bigcup_{A \in \tilde{E}_2^2[T^{\alpha_i}]m_i} A = E_2^2[E_2^5].
\]

\[
\bigcup_{i=0}^{30} E_2^2[T^{\alpha_i}]
\]
contains all edges in \( E_2^5 \).

From \( |E_2^2[T^{\alpha_i}]| = 7 \) and the fact that we do not delete links in this procedure,
we get \( m_i \leq 5 \) and \( \tilde{E}_2^2[T^{\alpha_i}]m_i \cong K_{7-m_i} \).
Therefore, there is always a path through each vertex of $\tilde{E}_{2}^{2}[T^{\alpha i}]_{m_{i}}$, where endvertices belong to $L(T^{\alpha i})$. 
Therefore, there is always a path through each vertex of $\tilde{E}_{2}^{2}[T^{\alpha^{i}}]_{m_{i}}$, where endvertices belong to $L(T^{\alpha^{i}})$.

Since all links are preserved, the mentioned paths, after being joined together, make a Hamiltonian cycle in $(E_{2}^{2}[E_{2}^{5}], E_{5})$. 
Therefore, there is always a path through each vertex of $\tilde{E}_2^2[T^{\alpha^i}]_{m_i}$, where endvertices belong to $L(T^{\alpha^i})$.

Since all links are preserved, the mentioned paths, after being joined together, make a Hamiltonian cycle in $(E_2^2[E_5^5], \mathcal{E}_5)$.

Thank you for your attention
Therefore, there is always a path through each vertex of $\tilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$, where endvertices belong to $L(T^{\alpha^i})$.

Since all links are preserved, the mentioned paths, after being joined together, make a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$.

Thank you for your attention

Questions?
Therefore, there is always a path through each vertex of $\tilde{E}_2^2[T^{\alpha^i}]_{m_i}$, where endvertices belong to $L(T^{\alpha^i})$.

Since all links are preserved, the mentioned paths, after being joined together, make a Hamiltonian cycle in $(E_2^2[E_2^5], E_5)$.

Thank you for your attention

Questions?