

Hamiltonian graphs in Abelian 2-groups

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Graphs defined on Abelian groups of exponent 2



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$$E_{2^k} = \langle x_1 \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_k \rangle, \quad x_i^2 = 1$$



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where H/T is a quotient group isomorphic to $E_{2^{m-t}}$ and $\begin{bmatrix} a \\ b \end{bmatrix}_2$ is a Gaussian coefficient.
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Ore's Theorem immediately yields that $(E_{2^2}[E_{2^3}], \mathcal{E}_3)$ and $(E_{2^2}[E_{2^4}], \mathcal{E}_4)$ are Hamiltonian.



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A graph $(E_{2^2}[E_{2^k}], \mathcal{E}_k)$ is $6(2^{k-2} - 1)$ -regular. The inequality

$$\frac{1}{2}|E_{2^2}[E_{2^k}]| - \deg(V) < 0$$

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Using the inclusion-exclusion formula, the following holds



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 \deg(V) &= \sum_{g \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1}| - \\
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 &\quad + \sum_{g \neq h \neq k \neq g, g, h, k \in V^*} |E_{2^2}[\langle g \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle h \rangle, E_{2^k}]^{-1} \cap E_{2^2}[\langle k \rangle, E_{2^k}]^{-1}| - 1 = \\
 &= \binom{3}{1} (2^{k-1} - 1) - \binom{3}{2} \cdot 1 + 1 - 1 = 6(2^{k-2} - 1).
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Put $t = 2^{k-2}$.



Therefore,

$$\frac{1}{2}|E_{2^{2^k}}[E_{2^k}]| - \deg(V) = \frac{1}{6}(4t-1)(2t-1) - 6(t-1) = \frac{1}{6}(8t^2 - 42t + 37).$$



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Any automorphism $\alpha \in \text{Aut}(E_{2^5})$ is represented by its action on a generators like



$$\alpha = \begin{pmatrix} a & b & c & d & e \\ g_1 & g_2 & g_3 & g_4 & g_5 \end{pmatrix},$$



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The following is crucial for a construction of a Hamiltonian cycle in $(E_{2^2}[E_{2^5}], \mathcal{E}_5)$.





Lemma: Let $E_{2^5} = \langle a, b, c, d, e \rangle$ and let $\alpha \in \text{Aut}(E_{2^5})$ be given by $\alpha = \begin{pmatrix} a & b & c & d & e \\ bc & cd & bcd & de & a \end{pmatrix}$, then $o(\alpha) = 31$ and $H^{\langle \alpha \rangle} = E_{2^4}[E_{2^5}]$ where $H = \langle a, b, c, d \rangle$. If $T = \langle a, b, c \rangle$ and $\Delta_i = T \cap T^{\alpha^i}$ for $i \in \mathbb{Z}_{31}$, then

$$\Delta_i = \begin{cases} \langle b, c \rangle, & \text{if } i = 1, 14 \\ \langle a, bc \rangle, & \text{if } i = 13, 30 \\ \langle ab, c \rangle, & \text{if } i = 17, 18 \\ \cong \mathbb{Z}_2 & \text{otherwise.} \end{cases}$$





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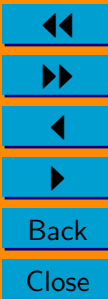
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then A is not a link, but it is a vertex in graphs $E_{2^2}[T^{\alpha^i}]_{m_i}$ and $E_{2^2}[T^{\alpha^j}]_{m_j}$.

Then, we delete a vertex A and the edges incident to it.

In this case let $E_{2^2}[T^{\alpha^i}]_{m_{i+1}} = E_{2^2}[T^{\alpha^i}]_{m_i} \setminus \{A\}$.

If such a vertex A does not exist, we leave $E_{2^2}[T^{\alpha^i}]_{m_i}$ unchanged and

denote that by $\tilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$.

Now, continue the same procedure with $E_{2^2}[T^{\alpha^{i+1}}]_{m_{i+1}}$.



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Therefore, there is always a path through each vertex of $\tilde{E}_{2^2}[T^{\alpha^i}]_{m_i}$, where endvertices belong to $L(T^{\alpha^i})$.



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