Looking for Additive Steiner 2-Designs

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Definition (2-Design)

A 2-(v, k, λ) design is a pair $(\mathcal{P}, \mathcal{B})$ such that

- $\mathcal{P}$ is a set of v points;
- $\mathcal{B}$ is a collection of k-subsets of $\mathcal{P}$ (blocks);
- each 2-subset of $\mathcal{P}$ is contained in exactly λ blocks.

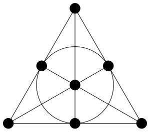

Figure: The Fano plane. 2-$(7, 3, 1)$ design.

- A 2-design is symmetric if $|\mathcal{P}| = |\mathcal{B}|$.
- A Steiner system is a design with $\lambda = 1$.

Definition (Cageggi, Falcone, Pavone, 2017)

A design $(\mathcal{P}, \mathcal{B})$ is additive under an abelian group G if

- $\mathcal{P} \subseteq G$ and
- $\sum_{x \in B} x = 0, \quad \forall B \in \mathcal{B}.$

- Examples of additive Steiner 2-designs:

Parameters	Group	Description
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$	points and lines of $AG(n, p^m)$
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	points and lines of $PG(n-1, 2)$
$(q^2 - 1, q + 1, 1)$	$\mathbb{Z}_q^{\frac{q-1}{2}}$	points and lines of $PG(2, q)$

Definition (Cameron, 1974. Delsarte, 1976.)

A 2-(v,k,λ) design over $\mathbb{F}_q$ is a pair $(\mathcal{P},\mathcal{B})$ such that

- $\mathcal{P}$ is the set of points of $\mathrm{PG}(v-1,q)$
- $\mathcal{B}$ is a collection of $(k-1)$-dimensional subspaces of $\mathrm{PG}(v-1,q)$ (blocks)
- each line is contained in exactly λ blocks.

Properties:

- 2-(v,k,λ) design over $\mathbb{F}_q$ is a classical 2-$(\frac{q^v-1}{q-1},\frac{q^k-1}{q-1},\lambda)$ design
- 2-(v,k,λ) design over $\mathbb{F}_2$ is additive under $\mathbb{Z}_2^v$

A sporadic example of an additive Steiner 2-design:

Parameters	Description	Reference
2-$(8191,7,1)$	2-$(13,3,1)$ design over $\mathbb{F}_2$	Braun, Etzion, Ostergaard, Vardy, Wassermann, 2017

Definition

$(\mathcal{P}, \mathcal{B})$ is additive under an abelian group G if $\mathcal{P} \subseteq G$ and $\sum_{x \in B} x = 0, \forall B \in \mathcal{B}$.

- strongly additive if $\mathcal{B} = \{B \in \binom{\mathcal{P}}{k} \mid \sum_{x \in B} x = 0\}$
- strictly additive if $\mathcal{P} = G$
- almost strictly additive if $\mathcal{P} = G \setminus \{0\}$

[Cageggi, Falcone, Pavone, 2017],
[Buratti, A.N., Super-regular Steiner 2-designs, 202?]

Parameters	Group	Strongly	Strictly	Almost str.	Description
$(2^n - 1, 3, 1)$	$\mathbb{Z}_2^n$	✓		✓	points and lines of $PG(n-1, 2)$
$(p^{mn}, p^m, 1)$	$\mathbb{Z}_p^{mn}$		✓		points and lines of $AG(n, p^m)$
$(p^2, p, 1)$	$\mathbb{Z}_p^{\frac{p(p-1)}{2}}$	✓			points and lines of $AG(2, p)$
(v, k, λ)	$\mathbb{Z}_k \times \mathbb{Z}_{k-\lambda}^{\frac{v-1}{2}}$	✓			symmetric design, $k - \lambda \not| k$, prime
$(8191, 7, 1)$	$\mathbb{Z}_2^{13}$			✓	$(13, 3, 1)$ design over $\mathbb{F}_2$, in $PG(12, 2)$

Super-regular 2-designs

Definition (Buratti, A.N., 202?)

$(\mathcal{P}, \mathcal{B})$ is super-regular under an abelian group G (or briefly G-super-regular) if it is

- strictly additive, i.e. $\mathcal{P} = G$,
- G-regular, i.e. $g + B \in \mathcal{B}, \quad \forall B \in \mathcal{B}, \quad \forall g \in G$.

Theorem (Buratti, A.N., 202?)

Given $k \geq 3$, there are infinitely many values of v for which there exists a super-regular 2-$(v, k, 1)$ design with the definite exceptions of $k \equiv 2 \pmod 4$ and the possible exceptions of all $k = 2^n 3 \geq 12$.

Constructing examples is computationally hard.

k	3	4	5	~~6~~	7	8
	AG$(n,3)$	AG$(n,4)$	AG$(n,5)$	$2 \pmod 4$	AG$(n,7)$	AG$(n,8)$

k	9	~~10~~	11	12	13	~~14~~	15
	AG$(n,9)$	$2 \pmod 4$	AG$(n,11)$	$2^2 \cdot 3$	AG$(n,13)$	$2 \pmod 4$	?

- $v = 3 \cdot 5^{31}$

- Design parameters $2\text{-}(13,4,1)$
- G is the additive group of $\mathbb{F}_{27}$, $G = (\mathbb{F}_{3^3},+)$
- $\mathcal{P}$ is the subgroup of squares of $\mathbb{F}_{27}^*$

$$\begin{aligned}\mathcal{P} = \{&(0,0,1), && (0,2,0), && (0,2,1), && (0,2,2),\\ &(1,0,0), && (1,0,2), && (1,1,0), && (1,1,1),\\ &(1,2,0), && (1,2,1), && (2,0,2), && (2,1,1),\\ &(2,2,1)\}\end{aligned}$$

- Blocks $\mathcal{B}$:

$$\begin{array}{ll}
\{(0,0,1),(1,0,0),(1,1,1),(1,2,1)\}, & \{(1,0,0),(1,2,0),(2,0,2),(2,1,1)\},\\
\{(1,2,0),(1,1,1),(1,1,0),(0,2,2)\}, & \{(1,1,1),(2,0,2),(1,0,2),(2,2,1)\},\\
\{(2,0,2),(1,1,0),(0,2,0),(0,0,1)\}, & \{(1,1,0),(1,0,2),(0,2,1),(1,0,0)\},\\
\{(1,0,2),(0,2,0),(1,2,1),(1,2,0)\}, & \{(0,2,0),(0,2,1),(2,1,1),(1,1,1)\},\\
\{(0,2,1),(1,2,1),(0,2,2),(2,0,2)\}, & \{(1,2,1),(2,1,1),(2,2,1),(1,1,0)\},\\
\{(2,1,1),(0,2,2),(0,0,1),(1,0,2)\}, & \{(0,2,2),(2,2,1),(1,0,0),(0,2,0)\},\\
\{(2,2,1),(0,0,1),(1,2,0),(0,2,1)\} &
\end{array}$$

$(\mathcal{P},\mathcal{B})$ is a G-additive $2\text{-}(13,4,1)$ design

Definition

A $(v, k, 1)$ difference family in a multiplicative group $\mathcal{P}$ of order v is a set $\mathcal{F}$ of k-subsets of $\mathcal{P}$ such that

$$\Delta\mathcal{F} = \bigcup_{B\in\mathcal{F}} \{gh^{-1} \,:\, g, h \in B, g \neq h\} = \mathcal{P} \setminus \{1\}.$$

- Briefly $(\mathcal{P}, k, 1)$-DF
- The number of members of $\mathcal{F}$ (base blocks) is $\frac{v-1}{k(k-1)}$
- The development of $\mathcal{F}$ is the set dev$\mathcal{F} = \{Bp \,:\, B \in \mathcal{F}, p \in \mathcal{P}\}$ of all the translates of the base blocks
- $(\mathcal{P}$, dev$\mathcal{F})$ is a $\mathcal{P}$-regular Steiner 2-$(v, k, 1)$ design

Theorem

Let $\mathcal{F}$ be a $(v, k, 1)$-DF in $\mathcal{P} \leq \mathbb{F}_q^*$. If all the base blocks of $\mathcal{F}$ are zero-sum, then $(\mathcal{P}$, dev$\mathcal{F})$ is an $\mathbb{F}_q$-additive Steiner 2-$(v, k, 1)$ design.

Definition

Let H be a subgroup of order n of a multiplicative group $\mathcal{P}$ of order v. A $(v, n, k, 1)$ difference family, relative to H, is a set $\mathcal{F}$ of k-subsets of $\mathcal{P}$ such that $\Delta\mathcal{F} = \mathcal{P} \setminus H$.

- Briefly $(\mathcal{P}, H, k, 1)$-DF
- Ordinary $(v, k, 1)$-DF $= (v, 1, k, 1)$-DF
- Number of base blocks is $\frac{v-n}{k(k-1)}$
- Necessary condition: $v - n$ is divisible by $k(k-1)$
- If $|H| = k$, then $(\mathcal{P}, \text{dev}\mathcal{F} \bigcup \{\text{right cosets of } H \text{ in } \mathcal{P}\})$ is a $\mathcal{P}$-regular Steiner 2-$(v, k, 1)$ design

Theorem

Let $\mathcal{F}$ be a $(v, k, k, 1)$-DF in $\mathcal{P} \leq \mathbb{F}_q^*$, relative to H. If all the base blocks of $\mathcal{F}$ are zero-sum, then $(\mathcal{P}, \text{dev}\mathcal{F})$ is an $\mathbb{F}_q$-additive Steiner 2-$(v, k, 1)$ design.

- Design parameters $2\text{-}(40, 4, 1)$
- G is the additive group of $\mathbb{F}_{81}$
- $\mathcal{P}$ is the subgroup of squares of $\mathbb{F}_{81}^* = \langle r \rangle$

$$r^4 - r^3 - 1 = 0$$

- $H = \{r^0, r^{20}, r^{40}, r^{60}\}$
- The following is a $(\mathcal{P}, H, 4, 1)$-DF and its base blocks are zero-sum

$$\mathcal{F} = \{\{r^0, r^2, r^{14}, r^{44}\}, \quad \{r^0, r^4, r^{10}, r^{32}\}, \quad \{r^0, r^8, r^{18}, r^{64}\}\}$$

- Set $\mathcal{B} = \text{dev}\mathcal{P} \bigcup \{\text{cosets of } H \text{ in } \mathcal{P}\}$

$(\mathcal{P}, \mathcal{B})$ is a $(\mathbb{F}_{3^4}, +)$-additive $2\text{-}(40, 4, 1)$ design

- Design parameters $2\text{-}(85, 5, 1)$
- G is the additive group of $\mathbb{F}_{256}$
- $\mathcal{P}$ is the subgroup of cubes of $\mathbb{F}_{256}^* = \langle r \rangle$

$$r^8 + r^4 + r^3 + r^2 + 1 = 0$$

- $H = \{r^0, r^{51}, r^{102}, r^{153}, r^{204}\}$
- The following is a $(\mathcal{P}, H, 5, 1)$-DF and its base blocks are zero-sum

$$\mathcal{F} = \{\{r^0, r^3, r^{75}, r^{123}, r^{216}\}, \quad \{r^0, r^6, r^{150}, r^{177}, r^{246}\},$$
$$\{r^0, r^{12}, r^{45}, r^{69}, r^{237}\}, \quad \{r^0, r^{21}, r^{57}, r^{81}, r^{147}\}\}$$

- Set $\mathcal{B} = \text{dev}\mathcal{P} \bigcup \{\text{cosets of } H \text{ in } \mathcal{P}\}$

$(\mathcal{P}, \mathcal{B})$ is a $(\mathbb{F}_{4^4}, +)$-additive $2\text{-}(85, 5, 1)$ design

We found:

- $(\mathbb{F}_{3^3},+)$-additive 2-$(13,4,1)$ design isomorphic to point-line design of $\mathsf{PG}(2,3)$
- $(\mathbb{F}_{3^4},+)$-additive 2-$(40,4,1)$ design isomorphic to point-line design of $\mathsf{PG}(3,3)$
- $(\mathbb{F}_{4^4},+)$-additive 2-$(85,5,1)$ design isomorphic to point-line design of $\mathsf{PG}(3,4)$
- $(\mathbb{F}_{5^4},+)$-additive 2-$(156,6,1)$ design isomorphic to point-line design $\mathsf{PG}(3,5)$
- $(\mathbb{F}_{4^3},+)$-additive 2-$(21,5,1)$ design isomorphic to point-line design of $\mathsf{PG}(2,4)$
- $(\mathbb{F}_{7^3},+)$-additive 2-$(57,8,1)$ design isomorphic to point-line design of $\mathsf{PG}(2,7)$
- $(\mathbb{F}_{5^3},+)$-additive 2-$(31,6,1)$ design isomorphic to point-line design of $\mathsf{PG}(2,5)$

We checked for $q \leq 19$ that every point-line design of $\mathsf{PG}(2,q)$ is additive under $(\mathbb{F}_{q^3},+)$.

Conjecture

The point-line design of $\mathsf{PG}(d,q)$ is additive under $(\mathbb{F}_{q^{d+1}},+)$.

Theorem (Cageggi, Falcone, Pavone, 2017)

Every symmetric design, so in particular the point-line design of $\mathsf{PG}(2,q)$, is additive under a suitable (big) group. For instance:

For q prime, the point-line design of $\mathsf{PG}(2,q)$ is strongly additive under $(\mathbb{F}_{q^{(q-1)/2}},+)$.

- Design parameters 2-$(124, 4, 1)$
- G is the additive group of $\mathbb{F}_{125}$
- $\mathcal{P} = \mathbb{F}_{125}^{*} = \langle r \rangle$
- $H = \{r^0, r^{31}, r^{62}, r^{93}\}$
- The following is a $(\mathcal{P}, H, 4, 1)$-DF and its base blocks are zero-sum

$$\mathcal{F} = \{\{r^0, r, r^{21}, r^{55}\}, \{r^0, r^2, r^{59}, r^{112}\}, \{r^0, r^3, r^{44}, r^{63}\},$$
$$\{r^0, r^4, r^{79}, r^{95}\}, \{r^0, r^5, r^{17}, r^{48}\}, \{r^0, r^6, r^{56}, r^{94}\},$$
$$\{r^0, r^7, r^{81}, r^{99}\}, \{r^0, r^8, r^{36}, r^{106}\}, \{r^0, r^{10}, r^{35}, r^{49}\},$$
$$\{r^0, r^{13}, r^{37}, r^{89}\}\}$$

- Set $\mathcal{B} = \text{dev}\mathcal{P} \bigcup \{\text{cosets of } H \text{ in } \mathcal{P}\}$

$(\mathcal{P}, \mathcal{B})$ is a $(\mathbb{F}_{5^3}, +)$-additive 2-$(124, 4, 1)$ design

Thank you for your attention!