On higher-dimensional Hadamard matrices and designs*

Vedran Krčadinac

University of Zagreb, Croatia

11.6.2024.

* This work was fully supported by the Croatian Science Foundation under the project 9752.

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Smallest unknown order: $v = 668$

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- is Hadamard if all $(n-1)$-dimensional parallel slices are orthogonal:

$$\sum_{1\le i_1,\ldots,\widehat{i_j},\ldots,i_n\le v} H(i_1,\ldots,a,\ldots,i_n)H(i_1,\ldots,b,\ldots,i_n) = v^{n-1}\delta_{ab}$$

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- is proper Hadamard if all 2-dimensional slices are Hadamard matrices.

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Theorem (Y. X. Yang, 1986). "Product construction"

Let $h : \{1, \ldots, v\}^2 \to \{-1, 1\}$ be an ordinary Hadamard matrix of order v. Then

$$H(i_1, \ldots, i_n) = \prod_{1 \le j < k \le n} h(i_j, i_k)$$

is an n-dimensional proper Hadamard matrix of order v.

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For dimensions $n \geq 3$, the order $v > 2$ of "improper" Hadamard matrices must be even. They can exist for $v \equiv 2 \pmod 4$!

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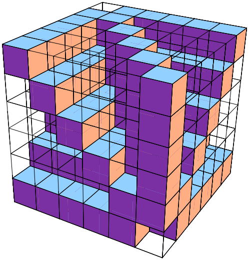

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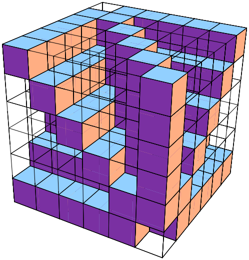

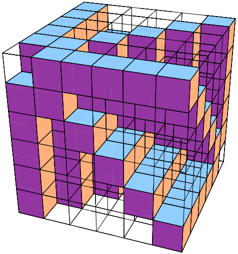

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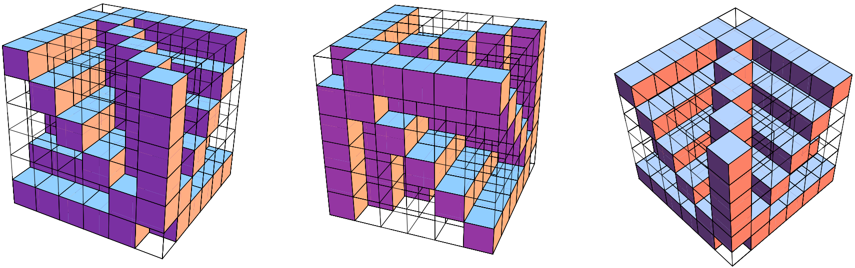

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Concluding questions: (in book from 2010)

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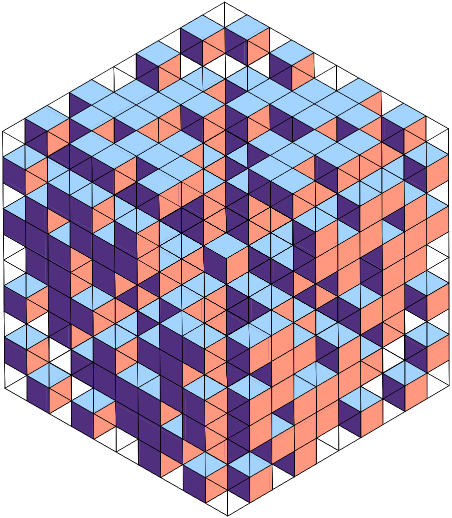

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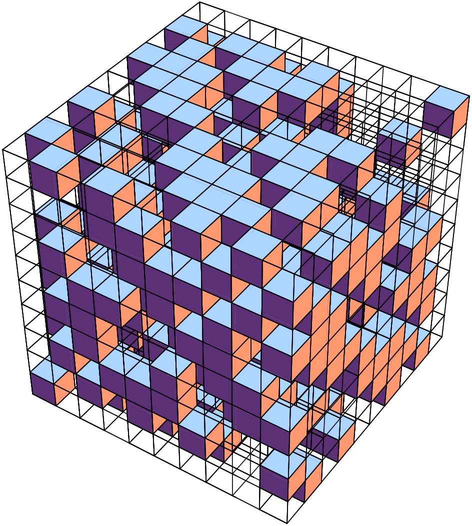

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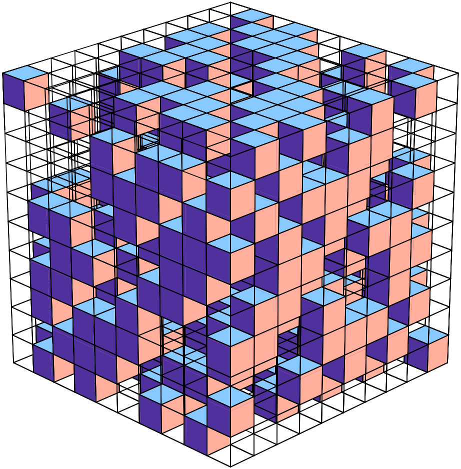


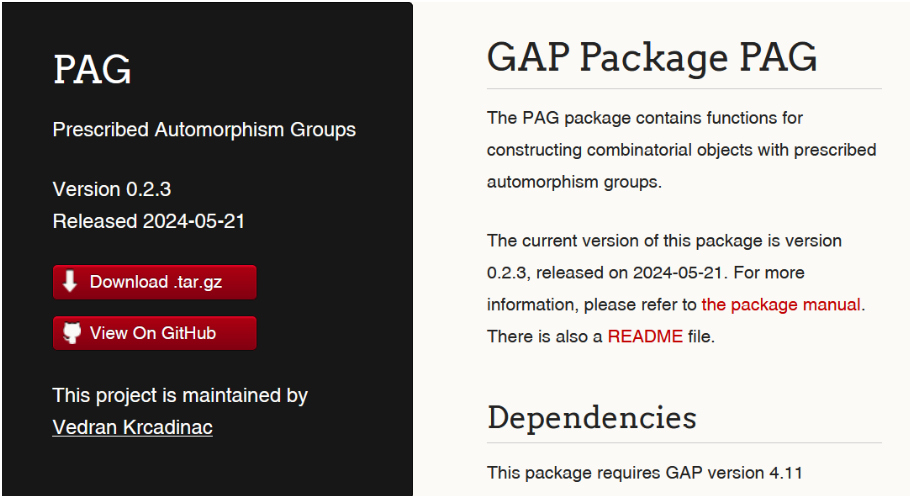
Higher-dimensional Hadamard matrices

$H : PG(1,q)^3 \to \{1,-1\}$, $q \equiv 1 \text{ or } 3 \pmod 4$,

$$H(x,y,z) = \begin{cases} -1, & \text{if } x = y = z, \\ 1, & \text{if } x = y \neq z \\ & \quad \text{or } x = z \neq y \\ & \quad \text{or } y = z \neq x, \\ \chi(z-y), & \text{if } x = \infty, \\ \chi(x-z), & \text{if } y = \infty, \\ \chi(y-x), & \text{if } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{otherwise.} \end{cases}$$

$PG(1,q) = \mathbb{F}_q \cup \{\infty\}$

Prescribed Automorphism Groups



PAG

Prescribed Automorphism Groups

Version 0.2.3
Released 2024-05-21

Download .tar.gz

View On GitHub

This project is maintained by Vedran Krcadinac

GAP Package PAG

The PAG package contains functions for constructing combinatorial objects with prescribed automorphism groups.

The current version of this package is version 0.2.3, released on 2024-05-21. For more information, please refer to the package manual. There is also a README file.

Dependencies

This package requires GAP version 4.11

https://vkrcadinac.github.io/PAG/

For the Open problems session

Question: is there a 3-dimensional Hadamard matrix of order $v = 22$?

Symmetric designs

A symmetric (v, k, λ) design is a $v \times v$ matrix with $\{0, 1\}$-entries such that $A \cdot A^\tau = (k - \lambda)\, I + \lambda\, J$ holds. The order of the design is $n = k - \lambda$.

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Example:

$(7, 3, 1)$

$n = 2$

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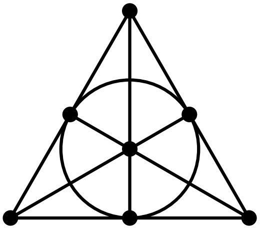

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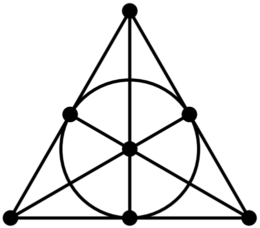

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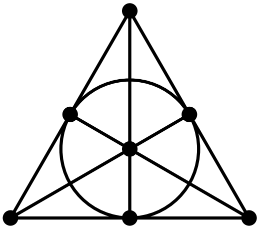

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Proposition.

If a symmetric (v, k, λ) design exists, then $\lambda(v - 1) = k(k - 1)$.

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If D is a (v,k,λ) difference set in $G = \{g_1, \ldots, g_v\}$, then

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Example:

$D = \{0, 1, 3\}$ is a $(7, 3, 1)$ difference set in $G = \mathbb{Z}_7 = \{0, \ldots, 6\}$

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Symmetric $(25, 9, 3)$ designs exist, but there are no $(25, 9, 3)$ difference sets in any group of order 25.

Cubes of symmetric designs

V. Krčadinac, M. O. Pavčević, K. Tabak, *Cubes of symmetric designs*, Ars Math. Contemp. (to appear). https://arxiv.org/abs/2304.05446

An n-dimensional cube of symmetric (v, k, λ) designs is a function

$$A : \{1, \ldots, v\}^n \to \{0, 1\}$$

such that all 2-dimensional slices are symmetric (v, k, λ) designs.

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Warwick de Launey, *On the construction of n-dimensional designs from 2-dimensional designs*, Australas. J. Combin. **1** (1990), 67–81.

W. de Launey, D. Flannery, *Algebraic design theory*, American Mathematical Society, 2011.

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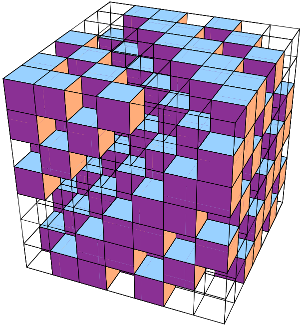
A 3-cube of symmetric $(7, 3, 1)$ designs:



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Usually: $D_i = g_i \cdot D$, i.e. the family is the development of a single D

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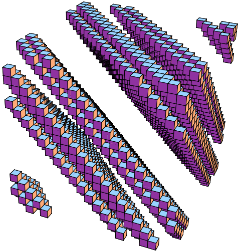
A 3-cube of $(21, 5, 1)$ designs
(projective planes of order 4)



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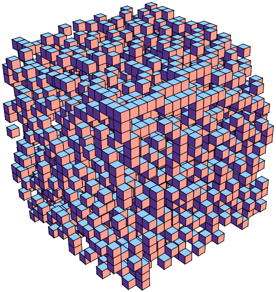

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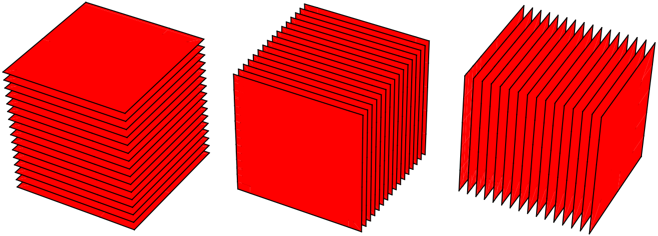

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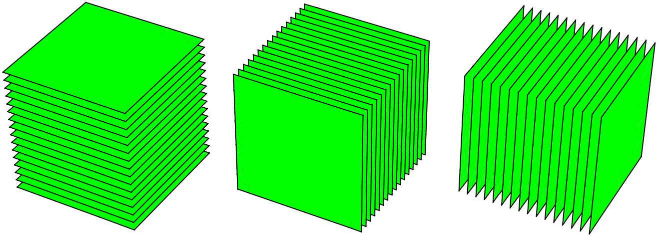

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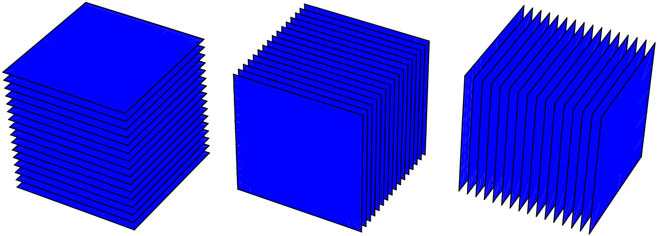

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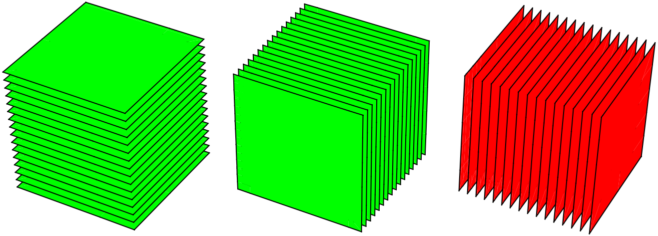

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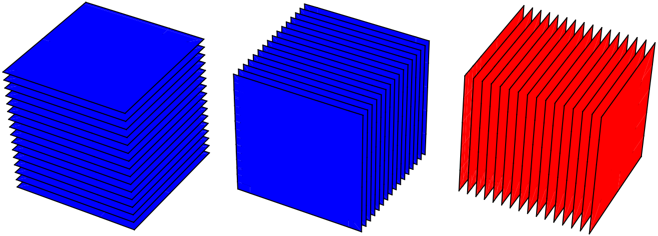

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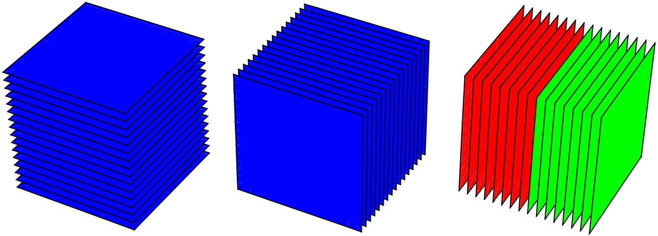

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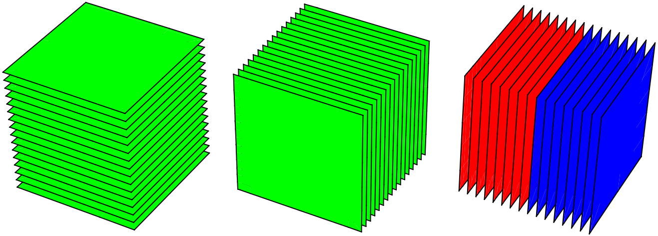

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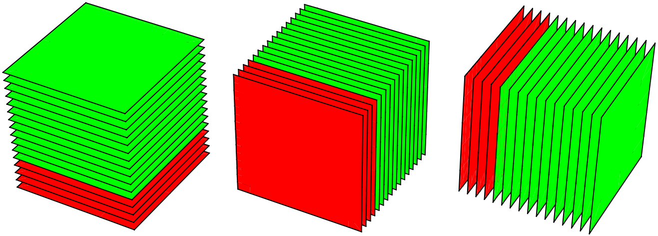

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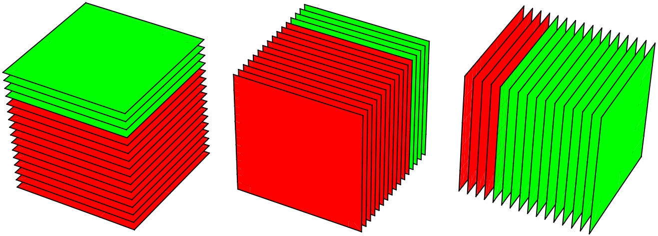

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Proposition.

There are at least 1423 inequivalent non-group 3-cubes of symmetric $(16, 6, 2)$ designs.

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Thanks for your attention!