Some nice combinatorial objects*

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University of Zagreb, Croatia

12.4.2024.

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Algorithmic Constructions of Combinatorial Objects

HRZZ Croatian Science Foundation

Algorithmic Constructions of Combinatorial Objects (ACCO)

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The topic of this research project are constructions of combinatorial objects with additional algebraic structure, such as quasi-symmetric designs, schematic designs, q-analogs of designs, difference sets, (semi)partial geometries, and generalisations. Results in algebraic combinatorics impose restrictions on the parameters and properties of such objects that can be exploited to narrow-down the search space and develop specialised algorithms for their construction and classification.

Research objectives

- Development of algorithmic methods for the construction and classification of combinatorial objects with strong algebraic structure. These methods utilise known algebraic and combinatorial properties of the objects to handle larger parameters and problems that have been out of reach with traditional construction methods.
- Widening of theoretical knowledge about combinatorial objects that are the topic of research. Interesting theorems are often discovered and proved on the basis of available examples. It is expected that the results of the project will lead to such discoveries.
- Development of a software package, implemented in GAP, for the construction and analysis of combinatorial objects.

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Rudi Mathon

Rudi Mathon (1940-2022)

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It appears to me that he regarded combinatorial designs as rare gems, and computers and algorithms as instruments used to mine for them. His ultimate goal was to find the gems, but he also paid attention to developing his craft of efficient algorithm design and effective computational methods, which he passed on to the next generation of researchers. The door of his office was always open, and as a student, I was always welcome to drop by at any time. Very often I would find him on his computer, writing programs or verifying results of his ongoing computational searches. Then, he would share some details of the particular gems he was looking for: their properties, their symmetries, their beauty. His eyes would glitter and in those moments we could catch a glimpse of his appreciation for the beauty in combinatorial structures.

— Lucia Moura

R. C. Bose, *Strongly regular graphs, partial geometries and partially balanced designs*, Pacific J. Math. **13** (1963), 389–419.

Combinatorial configurations & strongly regular graphs

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Definition.

A partial geometry $pg(s, t, \alpha)$ is an incidence structure such that:

- every line is incident with $s + 1$ points
- every point is incident with $t + 1$ lines
- every pair of points is incident with at most one line
- for every non-incident point-line pair (P, ℓ), there are exactly α points on ℓ collinear with P

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A finite semiplane is a partial linear space with parallelism of lines and non-collinearity of points being equivalence relations. It is of order n if the largest degree of a point or line is $n+1$.

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Theorem.

The set of all degrees in a semiplane is either $\{n-1, n, n+1\}$, or $\{n, n+1\}$, or $\{n+1\}$.

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All hyperbolic and parabolic, and most elliptic semiplanes are of the form $\mathcal{P} - B$, where $\mathcal{P}$ is a projective plane of order n, and B is a closed subset.

Exceptions:

R. D. Baker, *An elliptic semiplane*, J. Combin. Theory Ser. A **25** (1978), no. 2, 193–195.

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R. Mathon, *Divisible semiplanes*, in: *Handbook of combinatorial designs. Second edition* (eds. C. J. Colbourn, J. H. Dinitz), Chapman & Hall/CRC, 2007, pp. 729–731.

(135_{12}) configuration based on $(45, 12, 3)$ design.

Combinatorial configurations & strongly regular graphs

M. Abreu, M. Funk, V. Krčadinac, D. Labbate, *Strongly regular configurations*, Des. Codes Cryptogr. **90** (2022), 1881–1897.

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A strongly regular configuration with parameters $(v_k; \lambda, \mu)$ is a (v_k) configuration with the associated point graph a $SRG(v, k(k-1), \lambda, \mu)$.

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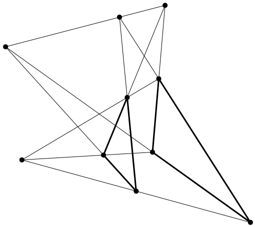

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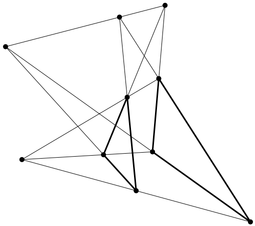

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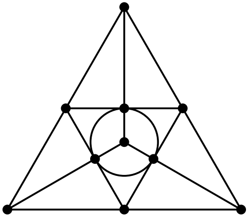

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Proper & primitive strongly regular configurations

No.	$(v_k; \lambda, \mu)$	#Cf	No.	$(v_k; \lambda, \mu)$	#Cf
1	$(10_3; 3, 4)$	**2**	12	$(63_6; 13, 15)$	**4**
2	$(13_3; 2, 3)$	**1**	13	$(64_7; 26, 30)$	**29**
3	$(16_3; 2, 2)$	**1**	14	$(81_8; 37, 42)$	?
4	$(25_4; 5, 6)$	**0**	15	$(85_6; 11, 10)$	?
5	$(36_5; 10, 12)$	**1**	16	$(85_7; 20, 21)$	?
6	$(41_5; 9, 10)$	?	17	$(96_5; 4, 4)$	**1**
7	$(45_4; 3, 3)$	**0**	18	$(99_7; 21, 15)$	?
8	$(49_4; 5, 2)$	**0**	19	$(100_9; 50, 56)$	**1**
9	$(49_6; 17, 20)$	**1**	20	$(105_9; 51, 45)$	?
10	$(50_7; 35, 36)$	**211**	21	$(113_8; 27, 28)$	?
11	$(61_6; 14, 15)$	?	22	$(120_8; 28, 24)$	**1**

Proper & primitive strongly regular configurations

Theorem.

Let $\mathcal{P}$ be a projective plane of order $n \geq 5$ and A, B, C be three non-collinear points. By deleting all points on the lines AB, AC, BC and all lines through the points A, B, C, there remains a strongly regular $(v_k; \lambda, \mu)$ configuration with $v = (n-1)^2$, $k = n-2$, $\lambda = (n-4)^2 + 1$, and $\mu = (n-3)(n-4)$.

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Theorem.

For every prime power q, there are at least four strongly regular $(v_k; \lambda, \mu)$ configuration with parameters $v = (q^2+1)(q^4+q^3+q^2+q+1)$, $k = q^2+q+1$, $\lambda = q^3+2q^2+q-1$, and $\mu = (q+1)^2$. One of them is the semipartial geometry $LP(4, q)$, and three are not semipartial geometries.

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Theorem.

There are no strongly regular $(v_k; \lambda, \mu)$ configurations with $v = \left(\binom{k}{2}+1\right)^2$, $\lambda = \binom{k}{2} - 1$, and $\mu = 2$.

Higher-dimensional Hadamard matrices

Paul J. Shlichta, *Higher dimensional Hadamard matrices*, IEEE Trans. Inform. Theory **25** (1979), no. 5, 566–572.

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A Hadamard matrix of dimension n and order v is a function

$$H : \{1, \dots, v\}^n \to \{-1, 1\}$$

such that all parallel $(n-1)$-dimensional slices are mutually orthogonal. It is proper if all 2-dimensional slices are ordinary Hadamard matrices.

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Theorem (Y. X. Yang, 1986).

Let $h : \{1, \ldots, v\}^2 \to \{-1, 1\}$ be an ordinary Hadamard matrix of order v. Then

$$H(i_1, \ldots, i_n) = \prod_{1 \le j < k \le n} h(i_j, i_k)$$

is a proper n-dimensional Hadamard matrix of order v.

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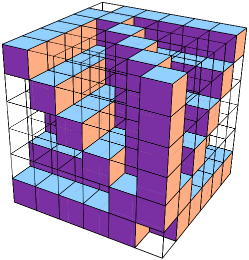

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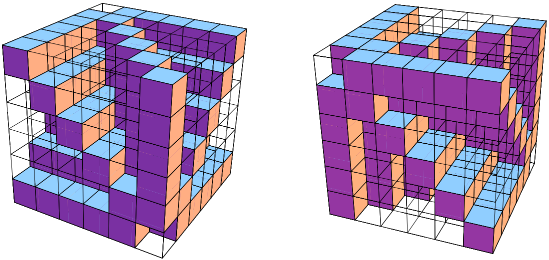

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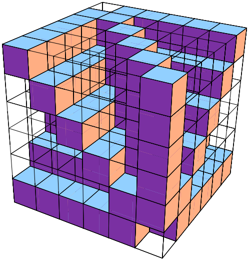

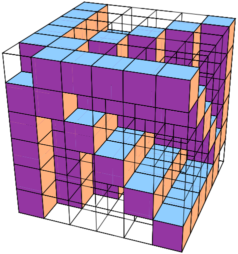

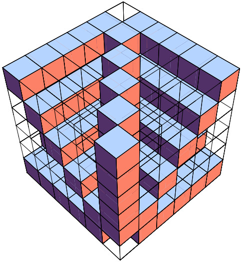

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Hadamard matrices of dimension $n = 3$ and order $v = q + 1$ exist for all odd prime powers q.

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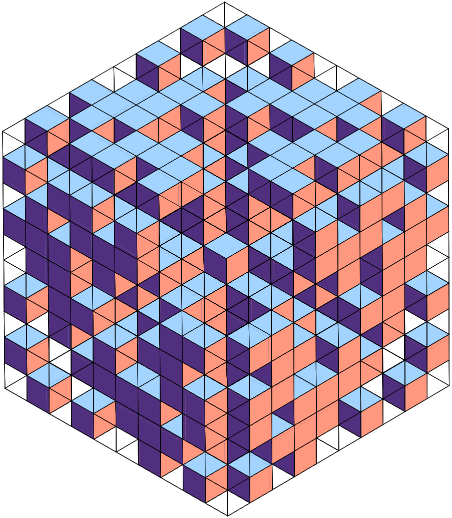

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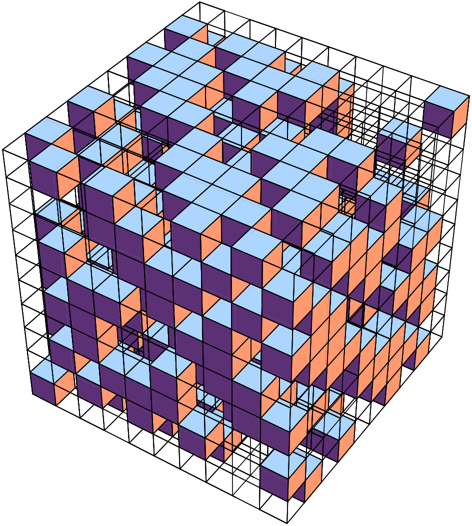

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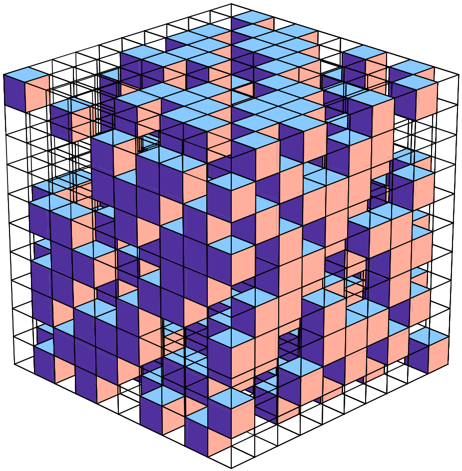

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$H : PG(1,q)^3 \to \{1,-1\}$, $q \equiv 1 \text{ or } 3 \pmod 4$,

$$H(x,y,z) = \begin{cases} -1, & \text{if } x = y = z, \\ 1, & \text{if } x = y \neq z \\ & \quad \text{or } x = z \neq y \\ & \quad \text{or } y = z \neq x, \\ \chi(z-y), & \text{if } x = \infty, \\ \chi(x-z), & \text{if } y = \infty, \\ \chi(y-x), & \text{if } z = \infty, \\ \chi((x-y)(y-z)(z-x)), & \text{otherwise.} \end{cases}$$

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W. de Launey, *On the construction of n-dimensional designs from 2-dimensional designs*, Australas. J. Combin. **1** (1990), 67–81.

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Definition.

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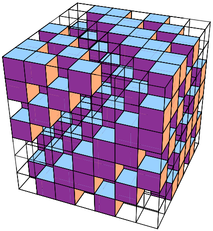

$(7, 3, 1)$

Cubes of symmetric designs

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Let D be a (v, k, λ) difference set in the group G. Order the group elements as $g_1, \ldots, g_v$. Then the function $C(i_1, \ldots, i_n) = [g_{i_1} \cdots g_{i_n} \in D]$ is an n-dimensional cube of (v, k, λ) designs.

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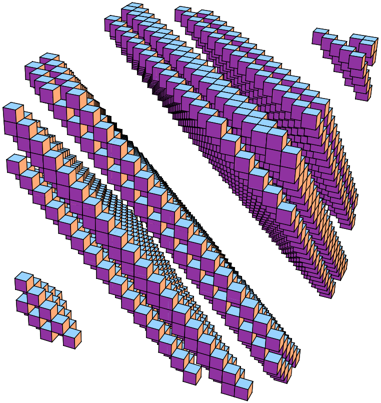

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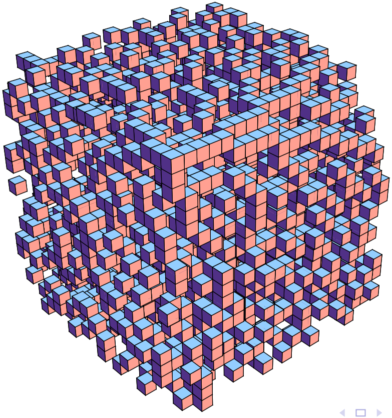

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Cubes of symmetric designs

All $(16, 6, 2)$ group 3-cubes:

ID	Structure	#Dc	#Gc	#Gc−#Dc
1	Z_{16}	0	0	0
2	Z_4^2	3	58	55
3	$(\mathbb{Z}_4 \times \mathbb{Z}_2) : \mathbb{Z}_2$	4	87	83
4	$\mathbb{Z}_4 : \mathbb{Z}_4$	3	84	81
5	$\mathbb{Z}_8 \times \mathbb{Z}_2$	2	108	106
6	$\mathbb{Z}_8 : \mathbb{Z}_2$	2	36	34
7	D_{16}	0	0	0
8	QD_{16}	2	52	50
9	Q_{16}	2	73	71
10	$\mathbb{Z}_4 \times \mathbb{Z}_2^2$	2	133	131
11	$\mathbb{Z}_2 \times D_8$	2	54	52
12	$\mathbb{Z}_2 \times Q_8$	2	199	197
13	$(\mathbb{Z}_4 \times \mathbb{Z}_2) : \mathbb{Z}_2$	2	79	77
14	$\mathbb{Z}_2^4$	1	10	9
		27	**973**	**946**

Cubes of symmetric designs

There are three $(16,6,2)$ designs:

$$|\operatorname{Aut}(\mathcal{D}_1)| = 11520, \quad |\operatorname{Aut}(\mathcal{D}_2)| = 768, \quad |\operatorname{Aut}(\mathcal{D}_3)| = 384$$

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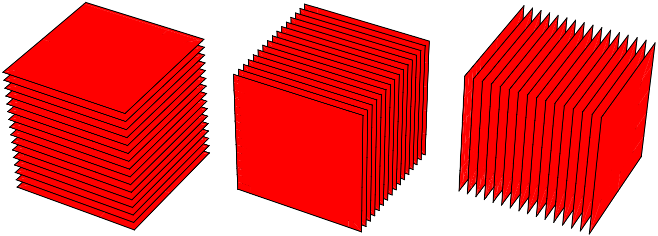
Group $\mathbb{Z}_2^4$: $\mathcal{D}_1 = \{B_1, \ldots, B_{16}\}$



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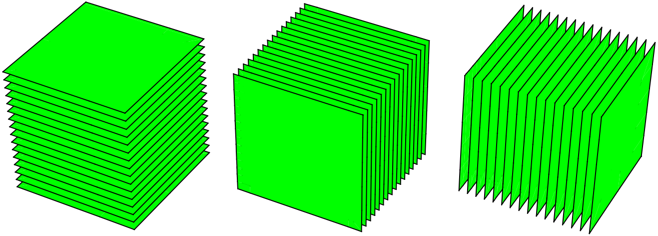

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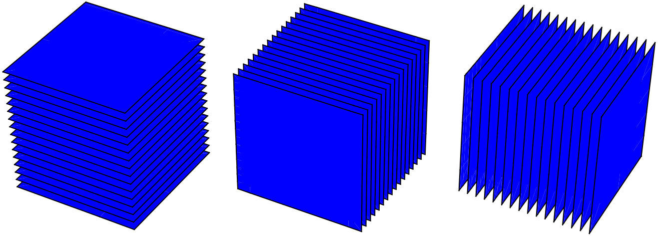

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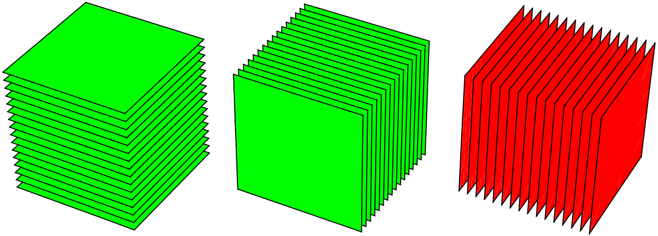

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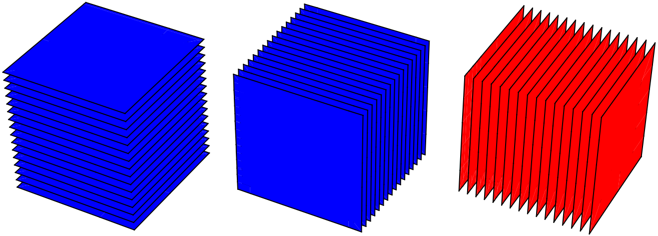

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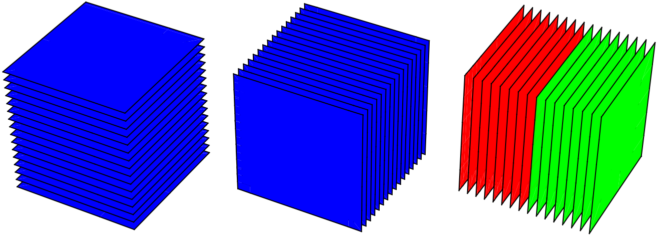

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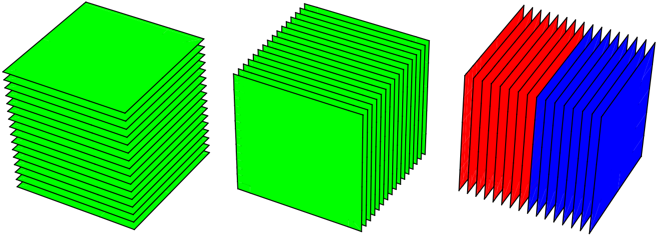


Cubes of symmetric designs

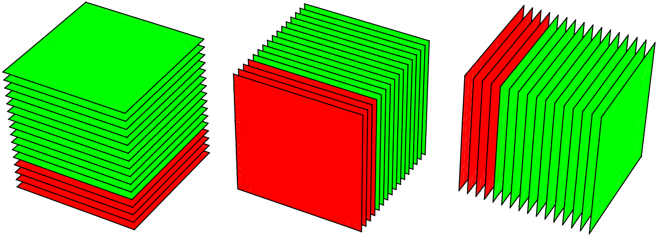
Theorem.

For every $m \geq 2$ and $n \geq 3$, there are at least two inequivalent

$$(4^m,\ 2^{m-1}(2^m-1),\ 2^{m-1}(2^{m-1}-1))$$

group n-cubes that are not difference cubes.

Non-group cube:



Cubes of symmetric designs

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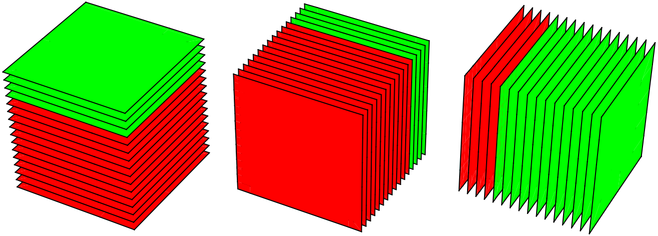
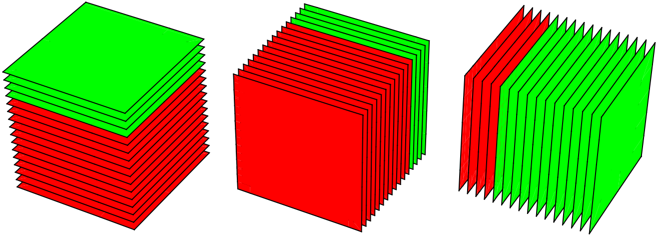


Cubes of symmetric designs

Non-group cube:



Proposition.

There are at least 1423 inequivalent $(16, 6, 2)$ non-group 3-cubes.

Mosaics of combinatorial designs

O. W. Gnilke, M. Greferath, M. O. Pavčević, *Mosaics of combinatorial designs*, Des. Codes Cryptogr. **86** (2018), no. 1, 85–95.

Definition.

A mosaic of combinatorial designs is a $v \times b$ matrix with entries from $\{1, \ldots, c\}$ such that for each i, the entries containing i are incidences of a combinatorial t_i-(v, k_i, λ_i) design.

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$$t_1\text{-}(v, k_1, \lambda_1) \oplus t_2\text{-}(v, k_2, \lambda_2) \oplus \cdots \oplus t_c\text{-}(v, k_c, \lambda_c)$$

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$$2\text{-}(9,3,1) \oplus 2\text{-}(9,3,1) \oplus 2\text{-}(9,3,1)$$

$$\begin{bmatrix}
1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 \\
1 & 2 & 3 & 2 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 \\
1 & 2 & 3 & 3 & 1 & 2 & 3 & 1 & 2 & 3 & 1 & 2 \\
2 & 3 & 1 & 1 & 2 & 3 & 2 & 3 & 1 & 3 & 1 & 2 \\
2 & 3 & 1 & 2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 \\
2 & 3 & 1 & 3 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 \\
3 & 1 & 2 & 1 & 2 & 3 & 3 & 1 & 2 & 2 & 3 & 1 \\
3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 3 & 1 & 2 \\
3 & 1 & 2 & 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3
\end{bmatrix}$$

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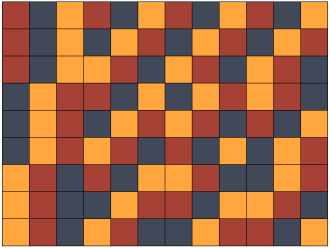

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Theorem.

If there exists a resolvable t-(v, k, λ) design, then there exists a c-mosaic

$$t\text{-}(v, k, \lambda) \oplus \cdots \oplus t\text{-}(v, k, \lambda)$$

for $c = v/k$.

Mosaics of combinatorial designs

A. Ćustić, V. Krčadinac, Y. Zhou, *Tiling groups with difference sets*, Electron. J. Combin. **22** (2015), no. 2, Paper 2.56, 13 pp.

Definition.

Let G be a finite group of order v with identity element 0. A (v, k, λ) tiling of G is a collection $\{D_1, \ldots, D_t\}$ of mutually disjoint (v, k, λ) difference sets such that $D_1 \cup \cdots \cup D_t = G \setminus \{0\}$.

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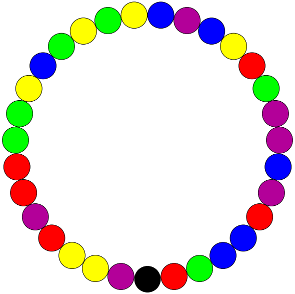


$(31, 6, 1)$ tiling of $\mathbb{Z}_{31}$

Mosaics of combinatorial designs

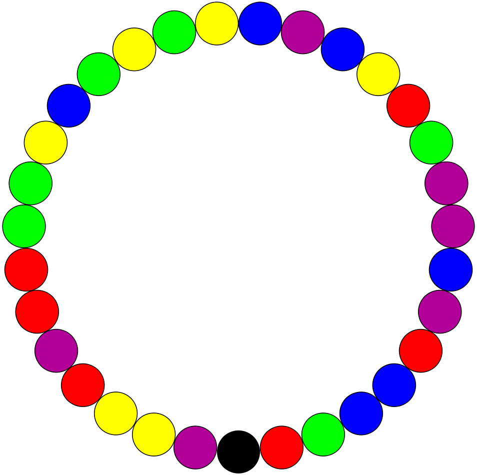
https://www.imaginary.org/gallery/difference-bracelets

A model of the (31,6,1) bracelet

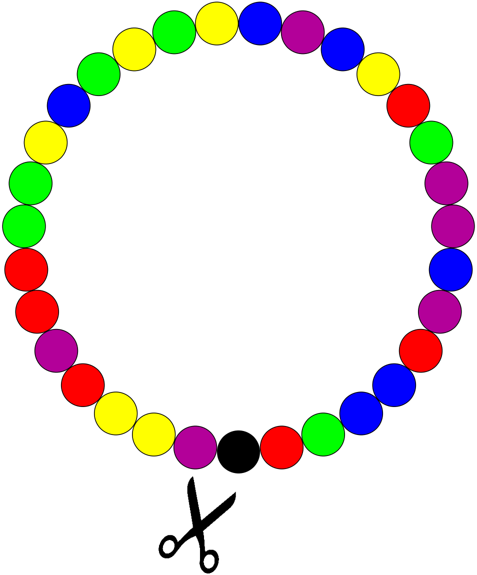
Difference bracelets cannot be built without the black bead, representing the identity element of the group. A slightly larger faceted bead was used here.

More Galleries

Mosaics of combinatorial designs



Mosaics of combinatorial designs



Mosaics of combinatorial designs

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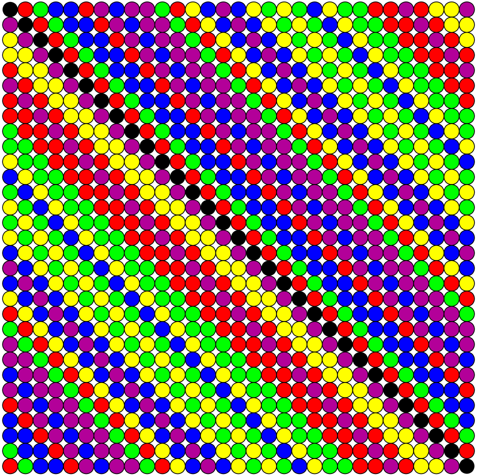
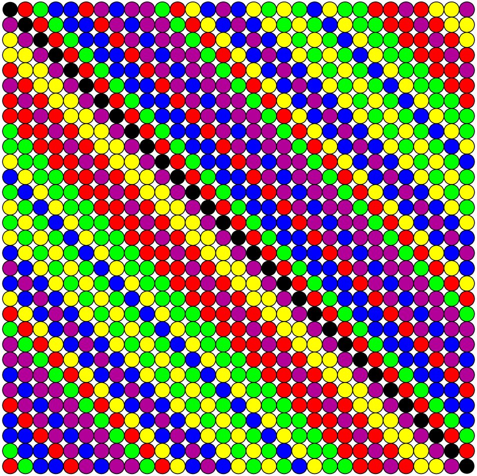


Mosaics of combinatorial designs



$$(31,6,1) \oplus (31,6,1) \oplus (31,6,1) \oplus (31,6,1) \oplus (31,6,1) \oplus (31,1,0)$$

Mosaics of combinatorial designs

Theorem.

If there exists a (v, k, λ) tiling of a group, then exists a $(c+1)$-mosaic

$$(v, k, \lambda) \oplus \cdots \oplus (v, k, \lambda) \oplus (v, 1, 0)$$

for $c = (v-1)/k = (k-1)/\lambda$.

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$$(q^2+q+1, q+1, 1) \oplus \cdots \oplus (q^2+q+1, q+1, 1) \oplus (q^2+q+1, 1, 0)$$

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q	2	3	4	5	7	8	9	$\cdots$
Tiling	✓	✗	✗	✓	✓	✓	?	$\cdots$
Mosaic								

Mosaics of combinatorial designs

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q	2	3	4	5	7	8	9	…
Tiling	✓	✗	✗	✓	✓	✓	?	…
Mosaic	✓	?	?	✓	✓	✓	?	…

Mosaics of combinatorial designs

Theorem.

If there exists a (v,k,λ) tiling of a group, then exists a $(c+1)$-mosaic

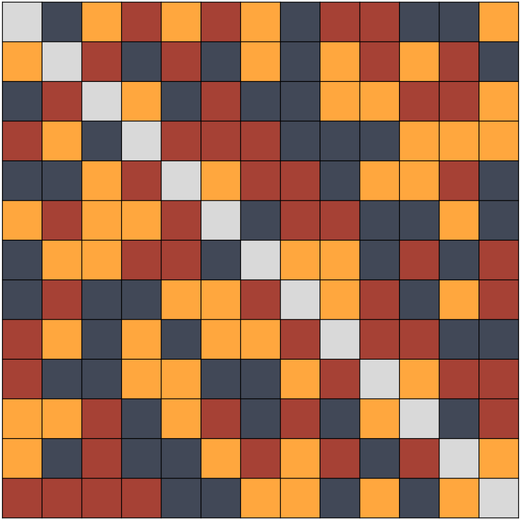
$$(v,k,\lambda)\oplus\cdots\oplus(v,k,\lambda)\oplus(v,1,0)$$

for $c=(v-1)/k=(k-1)/\lambda$.

$$(q^2+q+1,q+1,1)\oplus\cdots\oplus(q^2+q+1,q+1,1)\oplus(q^2+q+1,1,0)$$

q	2	3	4	5	7	8	9	…
Tiling	✓	✗	✗	✓	✓	✓	?	…
Mosaic	✓	✓	?	✓	✓	✓	?	…

The End



Combinatorial Constructions Conference

April 7-13, 2024, Dubrovnik, Croatia

Closing