Cubes of Symmetric Designs [1]

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Incidence vector

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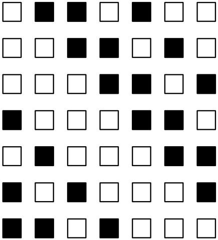
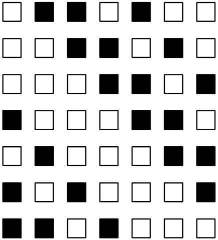
What will happen if I cyclically shift this incidence vector?

Incidence matrix

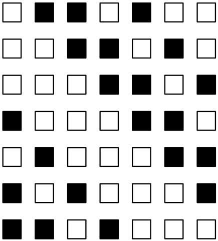


Incidence matrix



- Development of a difference set

Incidence matrix



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- A matrix indicating incidences between two kind of objects.

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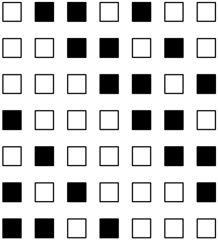

- Development of a difference set
- A matrix indicating incidences between two kind of objects.
- Symmetric design with parameters (7,3,1) - the Fano plane

How to continue?

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Construct the second layer of a cube by permuting the rows cyclically, starting with the second row of the incidence matrix:

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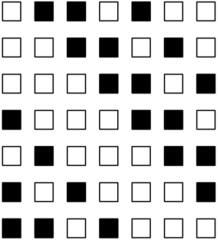
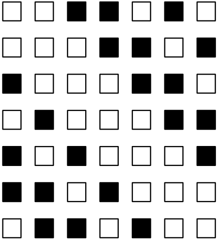
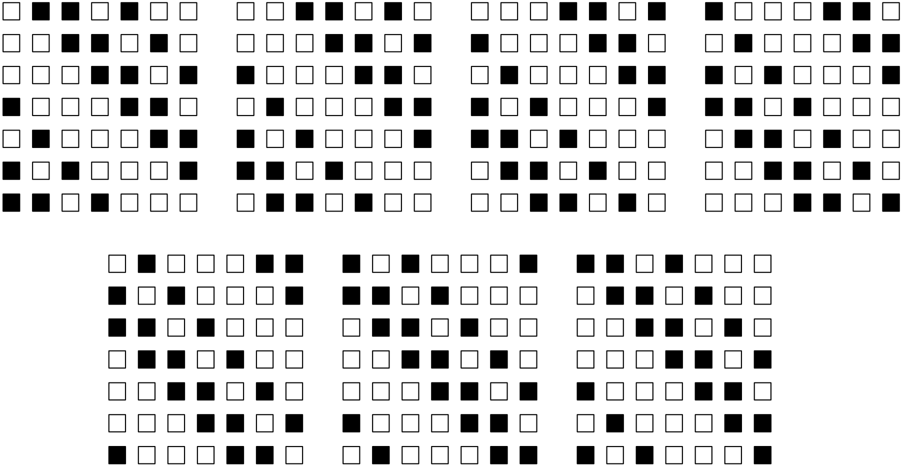
1. layer



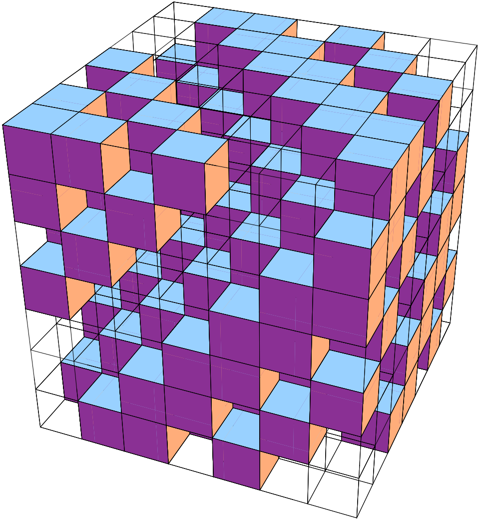
2. layer



Incidence cubes



Wow! What a picture!



The Fano cube

Conclusions:

We have constructed a 3-dimensional 0-1 matrix such that its every 2-dimensional layer (slice) is the incidence matrix of the Fano plane.

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- Are we the first to who this idea came to mind?
- ...

Are we the first? Similar ideas found:

P. J. Shlichta, *Higher dimensional Hadamard matrices*, IEEE Trans. Inform. Theory **25** (1979), no. 5, 566–572.

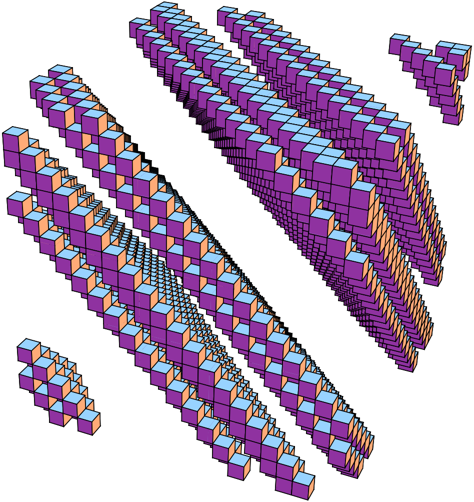
J. Seberry, *Higher-dimensional orthogonal designs and Hadamard matrices*, Combinatorial mathematics VII (Proc. Seventh Australian Conf., Univ. Newcastle, Newcastle, 1979), pp. 220–223, Lecture Notes in Math. **829**, Springer, Berlin, 1980.

W. de Launey, *On the construction of n-dimensional designs from 2-dimensional designs*, Combin. mathematics and combin. computing, Vol. 1 (Brisbane, 1989), Australas. J. Combin. **1** (1990), 67–81.

W. de Launey, K. J. Horadam, *A weak difference set construction for higher-dimensional designs*, Des. Codes Cryptogr. **3** (1993), no. 1, 75–87.

Difference set $\Rightarrow$ cube of designs, YES

Difference set ⇒ cube of designs, YES



Always cyclic? NO!

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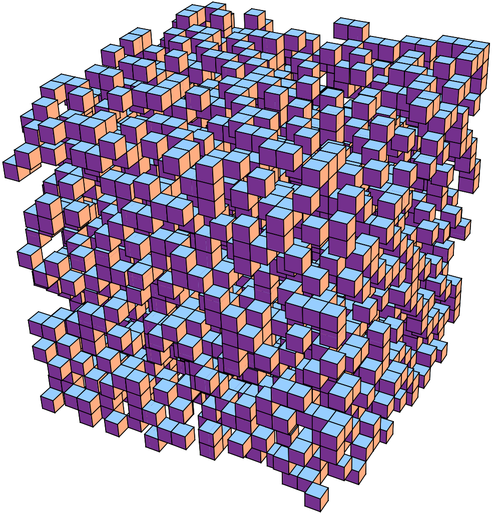

Cubes always from a difference set? NO!

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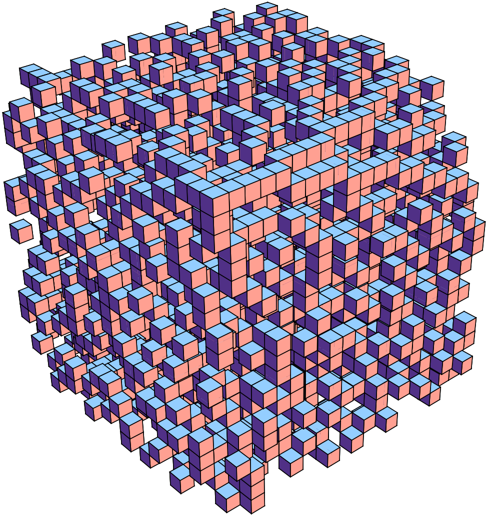

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There are many such cubes the slices of which are different 2-$(16, 6, 2)$ designs. They don't come from a proposed group action at all.

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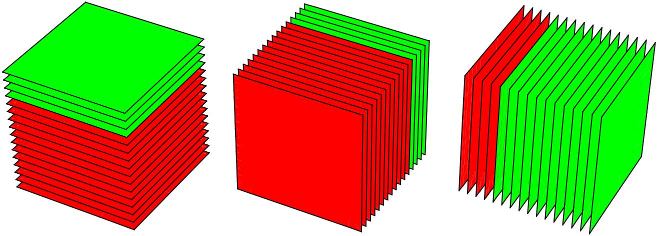

Preprints and Info:

V. Krčadinac, M. O. Pavčević, K. Tabak,
Cubes of symmetric designs,
to appear in Ars Math. Contemp.
http://arxiv.org/abs/2304.05446

V. Krčadinac, The PAG manual, 2023.
https://web.math.pmf.unizg.hr/acco/PAGmanual.pdf

https://web.math.pmf.unizg.hr/ krcko/results/cubes.html

Talk almost done, without a definition?!

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Definition

An *n-dimensional cube of symmetric (v, k, λ) designs* is a function

$$A : \{1, \dots, v\}^n \to \{0, 1\}$$

such that all 2-dimensional slices are symmetric (v, k, λ) designs.

What's next?

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Thanks for your attention!