Local modifications of 2-designs: theory and implementation

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We call the triple $(\mathcal{P}, \mathcal{B}, I)$ an incidence structure, provided $\mathcal{P}$, $\mathcal{B}$ are disjoint sets and $I \subseteq \mathcal{P} \times \mathcal{B}$. We use geometric language and call the elements of $\mathcal{P}$ points, the elements of $\mathcal{B}$ blocks, and write $P\ I\ b$ instead of $(P, b) \in I$. The incidence structure is called simple, if each block can be identified with the set of points with which it is incident. The incidence graph (also called Levi graph) $\Gamma(\mathcal{X})$ of an incidence structure $\mathcal{X} = (\mathcal{P}, \mathcal{B}, I)$ has vertex set $V = \mathcal{P} \cup \mathcal{B}$ and edge set $\{\{P, b\} \mid PIb\}$. $\Gamma(\mathcal{X})$ is a bipartite graph with vertex color classes $\mathcal{P}$, $\mathcal{B}$. Automorphisms or isomorphism of incidence structures induce automorphisms of isomorphisms of the associated incidence graphs. The converse is also true when we require that vertex colors to be preserved. Hence, when classifying combinatorial incidence structures, many problems can be reduced to the isomorphism problem of simple graphs (GI).

We first give a brief survey on the theoretical aspects of the computational complexity of the graph isomorphism problem, including Babai's seminal result [1]. Then, we focus on the implementation issues of the problem. One of the most powerful and best known of these algorithms is due to Brendan McKay [3]. It is known that his algorithm has exponential running time on some inputs, but in general it performs exceptionally well. The algorithms of `nauty` and `BLISS` [2] are based on iterative refinement techniques, on equitable partitions, and on canonical labeling.

The second lecture will present methods that locally modify 2-designs and let the main parameters invariant. The switching method allows the modification of two columns of the incidence matrix. This turns out to be a special case of the paramodification method, which affects the columns of the incidence matrix that correspond to the points of a fixed block. We will present these methods in detail, and study them for specific classes of 2-designs: affine and projective planes, Steiner triples systems, and unitals [4]. Since we can construct an enormous amount of new unitals of order 4 in this way, we will also address the issues of classifying and storing combinatorial objects on the computer.

References

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