Levinson's operator inequality

Jadranka Mićić
University of Zagreb, Faculty of Mechanical Engineering and Naval Architecture, Ivana Lučića 5, 10000 Zagreb, Croatia
jmicic@fsb.hr

Joint work with Josip Pečarić and Marjan Praljak

In 1964, Norman Levinson proved that the inequality

$$\sum_{i=1}^{n} p_i f(x_i) - f(\sum_{i=1}^{n} p_i x_i) \leq \sum_{i=1}^{n} p_i f(y_i) - f(\sum_{i=1}^{n} p_i y_i)$$

holds, *for every* $f : (0, 2c) \to \mathbb{R}$, $f''' \geq 0$ *and* p_i, x_i, y_i, $i = 1, 2, \ldots, n$, *such that* $p_i > 0$, $\sum_{i=1}^{n} p_i = 1$, $0 \leq x_i \leq c$ *and* $x_1 + y_1 = x_2 + y_2 = \ldots = x_n + y_n = 2c$.

We consider the assumptions on the function f such that Levinson's inequality holds for Hilbert space operators.

We give our main result:

Let $(X_1, \ldots, X_n)$ *be an* $n-$*tuple and* $(Y_1, \ldots, Y_k)$ *be a* $k-$*tuple of self-adjoint operators* $X_i, Y_j \in B_h(H)$ *with spectra contained in* $[m_x, M_x]$ *and* $[m_y, M_y]$, *respectively, such that* $m_x < M_x \leq c \leq m_y < M_y$. *Let* $(\Phi_1, \ldots, \Phi_n)$ *be an* $n-$*tuple and* $(\Psi_1, \ldots, \Psi_k)$ *be a* $k-$*tuple of positive linear mappings* $\Phi_i, \Psi_j : B(H) \to B(K)$, *such that* $\sum_{i=1}^{n} \Phi_i(1_H) = 1_K$ *and* $\sum_{i=1}^{k} \Psi_i(1_H) = 1_K$. *Let* $f \in \mathcal{C}([m_x, M_y])$ *and* $c \in (m_x, M_y)$ *such that there exists a constant* A *for which the function* $F(x) = f(x) - \frac{A}{2}x^2$ *is operator concave on* $[m_x, c]$ *and operator convex on* $[c, M_y]$. *If*

$$C_1 := \frac{A}{2}\left[\sum_{i=1}^{n} \Phi_i\big(X_i^2\big) - \Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big)^2\right] \leq C_2 := \frac{A}{2}\left[\sum_{i=1}^{k} \Psi_i\big(Y_i^2\big) - \Big(\sum_{i=1}^{k} \Psi_i(Y_i)\Big)^2\right]$$

then

$$\sum_{i=1}^{n} \Phi_i\big(f(X_i)\big) - f\Big(\sum_{i=1}^{n} \Phi_i(X_i)\Big) \leq C_1 \leq C_2 \leq \sum_{i=1}^{k} \Psi_i\big(f(Y_i)\big) - f\Big(\sum_{i=1}^{k} \Psi_i(Y_i)\Big). \tag{1}$$

Further, using conditions on the spectra of operators we obtain that Levinson's operator inequality (1) holds without operator concavity and operator convexity of f.
We also study order among quasi-arithmetic means under the same conditions.