Orthogonality in $\mathbb{M}_n(\mathbb{C})$ and geometry of the numerical range

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(1) We consider two types of orthogonality in a normed linear space X:

– Birkhoff–James orthogonality,

– Roberts orthogonality.

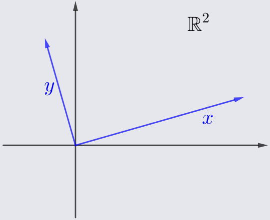
(2) In $X := \mathbb{M}_n(\mathbb{C})$ equipped with the operator norm, some geometric properties of the (generalized) numerical range of a matrix can be described in terms of the Birkhoff–James or the Roberts orthogonality.

DEFINITION

Two elements x and y of an inner product space $(X, (\cdot|\cdot))$ over a field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$) are *orthogonal* if $(x|y) = 0$. We write $x \perp y$.

EXAMPLE

Two vectors $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in $\mathbb{R}^2$ are orthogonal (in the sense of $((x_1, x_2)|(y_1, y_2)) = x_1 y_1 + x_2 y_2$) if and only if they are perpendicular in the usual sense of plane geometry.



$x \perp y$

DEFINITION

A *norm* on a linear space X is a mapping $\|\cdot\| : X \to \mathbb{R}$ such that

(1) $\|x\| \geq 0$ for $x \in X$; $\|x\| = 0$ if and only if $x = 0$;

(2) $\|\alpha x\| = |\alpha|\|x\|$ for $x \in X$, $\alpha \in \mathbb{F}$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in X$.

A *normed space* is a linear space with a norm.

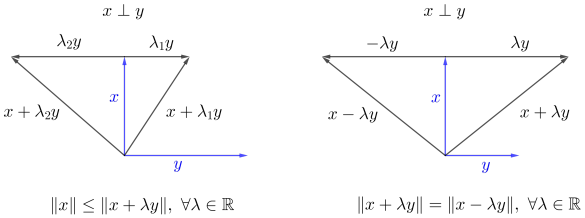
Every inner product space X is a normed space $\left(\|x\| = (x|x)^{1/2}\right)$, but the converse does not hold in general.

- Can we define orthogonality in an arbitrary normed space?

- The expression $(x|y) = 0$ makes no sense in a general normed space!

- Can we express $(x|y) = 0$ in terms of norm?

As a motivation, consider perpendicular vectors in $\mathbb{R}^2$.



- If X is an inner product space, and $x, y \in X$, then

$$(x|y) = 0 \quad \Leftrightarrow \quad \big(\|x\| \leq \|x + \lambda y\|, \ \forall \lambda \in \mathbb{F}\big);$$

$$(x|y) = 0 \quad \Leftrightarrow \quad \big(\|x + \lambda y\| = \|x - \lambda y\|, \ \forall \lambda \in \mathbb{F}\big).$$

DEFINITION

Let X be a normed space, and $x, y \in X$.

(I) We say that x is *Birkhoff–James orthogonal* to y, $x \perp_{BJ} y$, if

$$\|x\| \leq \|x + \lambda y\|, \quad \forall \lambda \in \mathbb{F}.$$

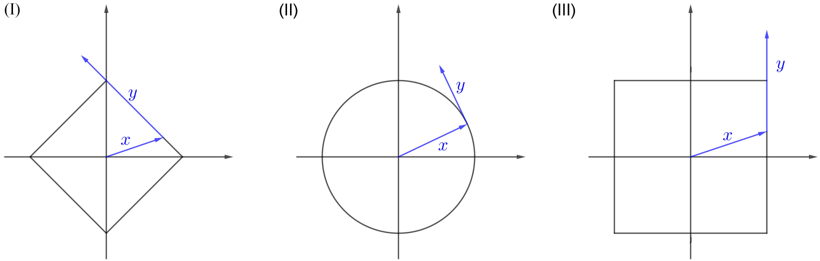
(II) We say that x is *Roberts orthogonal* to y, $x \perp_R y$, if

$$\|x + \lambda y\| = \|x - \lambda y\|, \quad \forall \lambda \in \mathbb{F}.$$

- If $(X, (\cdot|\cdot))$ is an inner product space, then $x \perp_{BJ} y \Leftrightarrow (x|y) = 0$.
- If $(X, (\cdot|\cdot))$ is an inner product space, then $x \perp_R y \Leftrightarrow (x|y) = 0$.
- In general, $x \perp_R y \Rightarrow x \perp_{BJ} y$.
- R-orthogonality is equivalent to BJ-orthogonlity if and only if X is an inner product space.

(I) $\mathbb{R}^2$ with respect to $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$,
(II) $\mathbb{R}^2$ with respect to $\|(x_1, x_2)\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$,
(III) $\mathbb{R}^2$ with respect to $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$.

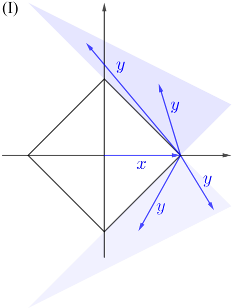
BJ-orthogonality in $\mathbb{R}^2$ with $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$:



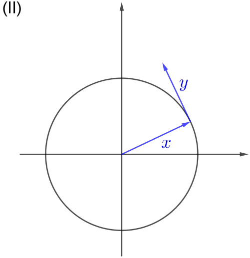
$x \perp_{BJ} y$ $x \perp_{BJ} y$ $x \perp_{BJ} y$

perpendicular vectors

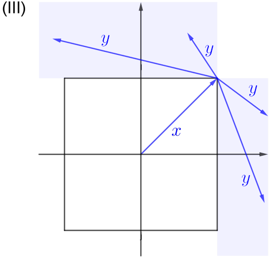
Birkhoff–James orthogonality in $\mathbb{R}^2$



$x \perp_{BJ} y$



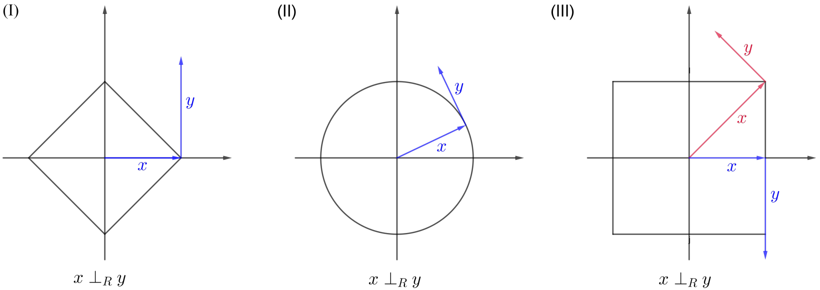
$x \perp_{BJ} y$
perpendicular vectors



$x \perp_{BJ} y$

(I) $\mathbb{R}^2$ with respect to $\|(x_1, x_2)\|_1 = |x_1| + |x_2|$,
(II) $\mathbb{R}^2$ with respect to $\|(x_1, x_2)\|_2 = \sqrt{|x_1|^2 + |x_2|^2}$,
(III) $\mathbb{R}^2$ with respect to $\|(x_1, x_2)\|_\infty = \max\{|x_1|, |x_2|\}$.

R-orthogonality in $\mathbb{R}^2$ with $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$:



perpendicular vectors

$\mathbb{C}^n$ – the linear space with the inner product of $x = (x_1 \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ defined by $(x|y) = \sum_{i=1}^n x_i \overline{y_i}$;

$$\|x\| = (x|x)^{1/2} = (\sum_{i=1}^{n} |x_i|^2)^{1/2}.$$

$\mathbb{M}_n(\mathbb{C})$ – the space of all $n \times n$ complex matrices with the norm defined by $\|A\| = \max\{\|Ax\| : x \in \mathbb{C}^n, \|x\| = 1\}$ $(A \in \mathbb{M}_n(\mathbb{C}))$

Definition

The *numerical range* of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W(A) = \{(Ax|x) : x \in \mathbb{C}^n, \|x\| = 1\}.$$

For $A \in \mathbb{M}_n(\mathbb{C})$ the following statements hold.

- $W(A)$ is a compact convex set in $\mathbb{C}$. (Toeplitz–Hausdorff)
- $W(U^*AU) = W(A)$ for any unitary $U \in \mathbb{M}_n(\mathbb{C})$.
- $W(A)$ contains the spectrum $\sigma(A)$ of A.

Theorem

Suppose $A \in \mathbb{M}_2(\mathbb{C})$ has eigenvalues α and β. Then $W(A)$ is an elliptical disc (possibly degenerate) centered at $\frac{1}{2}\operatorname{tr} A$ with foci α and β.

Theorem

Let $A \in \mathbb{M}_n(\mathbb{C})$.

(a) $A = \lambda I$ if and only if $W(A) = \{\lambda\}$.

(b) A is self-adjoint (i.e., $A^* = A$) if and only if $W(A) \subseteq \mathbb{R}$.

(c) If A is normal (i.e., $A^*A = AA^*$), then $W(A) = \operatorname{conv} \sigma(A)$ is a convex polygon.

Definition

The *maximal numerical range* of $A \in \mathbb{M}_n(\mathbb{C})$ is defined by

$$W_{\max}(A) = \{(Ax|x) : x \in \mathbb{C}^n, \|x\| = 1, \|Ax\| = \|A\|\}.$$

- $W_{\max}(A)$ ia a convex compact set contained in $W(A)$.

Theorem (J.G. Stampfli, J.P. Williams)

Let $A \in \mathbb{M}_n(\mathbb{C})$.

(i) $0 \in W(A)$ if and only if $I \perp_{BJ} A$.

(ii) $0 \in W_{\max}(A)$ if and only if $A \perp_{BJ} I$.

Roberts orthogonality in $\mathbb{M}_n(\mathbb{C})$

Theorem (Lj. Arambašić, T. Berić, R.R.)

Let $A \in \mathbb{M}_2(\mathbb{C})$. Then

$$W(A) = -W(A) \Leftrightarrow A \perp_R I.$$

Theorem (Lj. Arambašić, T. Berić, R.R.)

Let $A \in \mathbb{M}_n(\mathbb{C})$.

(i) If A is normal, then $W(A) = -W(A) \Leftrightarrow A \perp_R I$.

(ii) If $A \perp_R I$, then $W(A) = -W(A)$.

Definition

The *Davis–Wielandt shell* of $A \in \mathbb{M}_n(\mathbb{C})$ is defined as the set

$$DW(A) = \{((Ax|x), (A^*Ax|x)) : x \in H, \|x\| = 1\}.$$

- $DW(A)$ is a compact set in $\mathbb{C} \times \mathbb{R}$.
- $DW(A)$ is convex if $n \geq 3$.
- If $n = 2$, then the Davis–Wielandt shell $DW(A)$ is an ellipsoid without the interior centered at $\left(\frac{\mathrm{tr}(A)}{2}, \frac{\mathrm{tr}(A^*A)}{2}\right)$.
- A is normal if and only if $DW(A) = \mathrm{conv}\{(\lambda, |\lambda|^2) : \lambda \in \sigma(A)\}$ if and only if $DW(A)$ is a polyhedron.

Definition

Let $A \in \mathbb{M}_n(\mathbb{C})$. The *upper boundary* of $DW(A)$ is the set

$$DW_{ub}(A) = \{(\mu, r) \in DW(A) : \ r = \max \mathcal{L}_\mu(A)\},$$

where

$$\mathcal{L}_\mu(A) = \{(A^* Ax|x) : \ x \in H, \|x\| = 1, (Ax|x) = \mu\}.$$

Theorem (Lj. Arambašić, T. Berić, R.R.)

Let $A \in \mathbb{M}_n(\mathbb{C})$, $n \geq 3$. Then

$$DW_{ub}(A) = DW_{ub}(-A) \Leftrightarrow A \perp_R I.$$

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Thank you for your attention!