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Introduction to fractal analysis of orbits of dynamical systems

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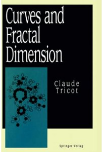
Content

Definition of box dimension

ε-neighborhood of a bounded set $A \subset \mathbb{R}^n$
$A_\varepsilon = \{y \in \mathbb{R}^n : d(y, A) < \varepsilon\}$. Box dimension is



$$d = \dim_B A = n - \lim_{\varepsilon \to 0} \frac{\log |A_\varepsilon|}{\log \varepsilon}$$

which means that

$$|A_\varepsilon| \simeq \varepsilon^{n-d}, \text{ read comparable}$$

i.e. there exist positive constants C_1, C_2

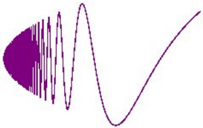
$$C_1 \varepsilon^{n-d} \leq |A_\varepsilon| \leq C_2 \varepsilon^{n-d}, \text{ for small } \varepsilon.$$

- Other names: Minkowski dimension, Bouligand dimension, limit capacity, box counting dimension
- Other definitions for box dimension e.g. $d = \dim_B A = \lim_{\varepsilon \to 0} \frac{\log N(A,\varepsilon)}{-\log \varepsilon}$, $N(A, \varepsilon)$ number of boxes in ε-grid intersecting A

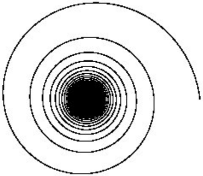
Examples from Claude Tricot book

Chirp $f(x) = x^{\alpha} \sin x^{-\beta}$ $d = \dim_B \Gamma = 2 - \frac{1+\alpha}{1+\beta}$, $0 < \alpha < \beta \leq 1$

Spiral $r = \varphi^{-\alpha}$ $d = \dim_B \Gamma = \frac{2}{1+\alpha}$, $0 < \alpha \leq 1$



$\dim_B \Gamma = \frac{7}{4}$, $\alpha = 0.5, \beta = 1$



$\dim_B \Gamma = \frac{4}{3}$, $\alpha = 0.5$

Sequence $S \ldots (\frac{1}{n^{\alpha}})$ $d = \dim_B S = \frac{1}{1+\alpha}$, $\alpha > 0$

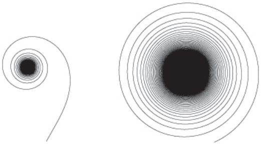
$\dim_B S = \frac{1}{2}$, $\alpha = 1$

$\dim_B S = \frac{4}{5}$, $\alpha = 0.25$

Why box dimension of smooth curves?

From the foreword of the book Claude Tricot: Curves and Fractal Dimensions, Springer-Verlag New York (1995)
"A straight line is the total negation of the plane whereas a **curved line is potentially the plane** in that it contains the essence of the plane within itself." Wassily Kandinsky: Point, Line and Plane, (1926)

Wassily Kandinsky
(Moscou 1866- Neuilly-sur-Seine 1944)

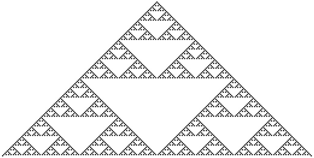


How to quantify this idea? By fractal dimension?
The bigger accumulation, the bigger fractal dimension.
The box dimension.

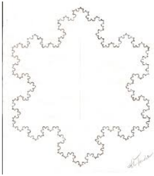
Other fractal dimensions

- Hausdorff dimension, Lyapunov dimension, Rényi dimension, correlation dimension, information dimension, Hentschel-Procaccia spectrum for dimensions, packing dimension, Assouad dimension...
- In the Tricot style, we were concentrated on curves, at the beginning.
- Nonrectifiable curve is a curve with infinite length, rectifiable curve has finite length.
- **Hausdorff dimension** do not see the difference between nonrectifiable curves, because of the countable stability.
- Every nonrectifiable smooth curve is a countable union of rectifiable smooth curves, so every nonrectifiable smooth curve always has Hausdorff dimension 1.
- Box dimension $\geq$ Hausdorff dimension.

Standard fractals-Sierpinski triangle, Koch snowflake



$\dim_B A = \frac{\log 3}{\log 2} \approx 1.585$

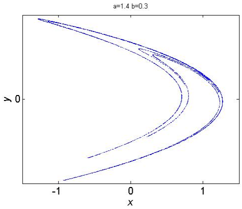


$\dim_B A = \frac{\log 4}{\log 3} \approx 1.262$

Hausdorff dimension is the same.

Fractal dimensions in dynamics from 1970s

- Since 1970 thermodynamics formalism, developed by Sinai, Ruelle, and Bowen, resulted in Hausdorff dimension of the Smale horseshoe and a results about Hausdorff dimension of Julia and Mandelbrot sets.
- Since 1980 physicists and mathematicians started to estimate and compute fractal dimensions of strange attractors (Lorenz, Henon,...). Fractal dimensions are estimated also for attractors of infinite-dimensional dynamical systems.



Different approach

- A natural idea is that "density" of orbit is related to quantity and quality of objects which could be produced by perturbation of the system (fixed points, periodic orbits, polycycles).
- We are interested in connection between the change of box dimension and the bifurcation of dynamical system.
- The box dimension is read from the exponent of the first term of area (length) of ε-neighborhood of an orbit. Coefficient of the first term is called Minkowski content.
- Bounded set $A \subset \mathbb{R}^n$,

$$\mathcal{M}^s(A) := \lim_{\varepsilon \to 0} \frac{|A_\epsilon|}{\varepsilon^{n-s}}$$

- What about other terms?
- In general- we are interested in reading properties of dynamical system form ε-neighborhood of orbit.

Weak focus

D. Žubrinić, V. Županović: Fractal analysis of spiral trajectories of some planar vector fields, Bull. Sci. Math., 129/6 (2005), 457-485.

$$\begin{cases} \dot{r} = r(r^{2l} + \sum_{i=0}^{l-1} a_i r^{2i}), \\ \dot{\varphi} = 1. \end{cases} \tag{1}$$

Theorem

Γ a part of a trajectory of (1) near the origin.
(a) $a_0 \neq 0$, then the spiral Γ is of exponential type, that is, comparable with $r = e^{a_0 \varphi}$, and hence $\dim_B \Gamma = 1$.
(b) k is fixed, $1 \leq k \leq l$, $a_l = 1$ and $a_0 = \cdots = a_{k-1} = 0$, $a_k \neq 0$. Then Γ is comparable with the spiral $r = \varphi^{-1/2k}$, and

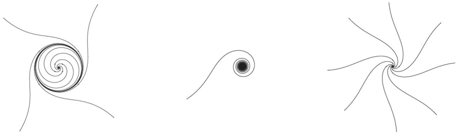
$$d := \dim_B \Gamma = \frac{4k}{2k+1}.$$

Γ is Minkowski measurable with the value equal to explicit constant.

Box dimension for Hopf bifurcation $4/3$

Hopf bifurcation occurs for $l = 1$ if $a_0 = 0$, then

$$\dim_B \Gamma = \frac{4}{3}$$



Hopf-Takens bifurcation occurs for $l > 1$, producing l limit cycles in the system

About proof

- Strong hyperbolic focus is given by exponential spiral-rectifiable case, box dimension=topological dimension.
- Weak focus with pure imaginary eigenvalues-power spiral.
- From Tricot book $r = \varphi^{-\alpha}$ $d = \dim_B \Gamma = \frac{2}{1+\alpha}$, $0 < \alpha \leq 1$, near accumulation point.
- We obtained new criteria for $r = f(\varphi)$ in polar coordinates to have the same box dimension as $r = \varphi^{-\alpha}$.
- We checked that spiral obtained by the given system in normal form satisfy the conditions for $\alpha = \frac{1}{2k}$.

Radial box dimension of a spiral

- We introduced radial box dimension obtained from radial ε-neighborhood $A_{\varepsilon,rad}$.
- The radial box dimension of spiral $r = f(\varphi)$ is $\frac{2}{1+\alpha}$ under conditions

$$\underline{m}\,\varphi^{-\alpha} \leq f(\varphi) \leq \overline{m}\,\varphi^{-\alpha}$$

-

$$\underline{a}\,\varphi^{-\alpha-1} \leq f(\varphi) - f(\varphi + 2\pi) \leq \overline{a}\,\varphi^{-\alpha-1}.$$

where f is radially decreasing and there exist positive constants $\underline{m}$, $\underline{a}$ and $\overline{m}$, $\overline{a}$, and $\varphi \geq \varphi_1$.

Box dimension of a spiral

- We obtained conditions for $\dim_B A = \dim_B(A, rad)$ by inequality involving f
- We prove an Excision lemma, saying that φ_1 from $\varphi \geq \varphi_1$ does not change the box dimension.
- We obtained conditions with derivatives, which are easy to check:
- f is decreasing, C^2,
- $M_1\,\varphi^{-\alpha-1} \leq\mid f'(\varphi)\mid\leq \overline{M}_2\,\varphi^{-\alpha-1}$,for $\varphi \geq \varphi_1$,
- $\mid f''(\varphi)\mid\leq \overline{M}_3\,\varphi^{-\alpha}$, for $\varphi \geq \varphi_1$.

Remark

These conditions were weakned in the later articles in order to cover bigger class of spirals e.g. f not decreasing, but radially decreasing.... e.g. Bessel spiral, type of so called wavy spiral

Limit cycle

Theorem

Let the system (1) have limit cycle $r = a$ of multiplicity m, $1 \leq m \leq l$; Γ_1 and Γ_2 the parts of two trajectories of (1) near the limit cycle from outside and inside respectively.
(a) Then Γ_1 and Γ_2 are comparable with exponential spirals $r = a \pm e^{-\beta\varphi}$ when $m = 1$, $\beta \neq 0$ (depending only on coefficients a_i, $0 \leq i \leq l-1$);
(b) Γ_1 and Γ_2 are comparable with power spirals $r = a \pm \varphi^{-1/(m-1)}$ when $m > 1$.
In both cases we have

$$d := \dim_B \Gamma_i = 2 - \frac{1}{m}, \quad i = 1, 2.$$

For $m = 1$ we have $\mathcal{M}^d(h, \Gamma_i) = 2/\beta$, $i = 1, 2$, where $h(\varepsilon) := \varepsilon(\log(1/\varepsilon))^{-1}$.
For $m > 1$ the spirals are Minkowski measurable.

About proof

- The proof is based on formula for limit cycle spiral $\Gamma ... r(\varphi) = 1 \pm \varphi^{-\alpha}$

$$d := \dim_B \Gamma = 1 + \frac{1}{1+\alpha} = \frac{2+\alpha}{1+\alpha}.$$

- This formula could be computed directly estimating $\varepsilon-$neighborhood of a spiral.
- Also, the formula comes from the fact that box dimension of the limit cycle spiral is equal to the box dimension of the set A consisting of concentric circles, tending to some fixed radius, with asymptotics $S \ldots k^{-\alpha}$, $\alpha > 0$.
- There is sectorial bilipshitz map between limit cycle spiral and the set of concentric circles (flow-sector theorem).
- Box dimension is $\dim_B A = 1 + \dim_B S$
- We obtain a criteria for limit cycle spiral $r(\varphi) = 1 \pm f(\varphi)$ to have the same box dimension as $r(\varphi) = 1 \pm \varphi^{-\alpha}$.

Example $l = 2$, $a_1 = -2$

$$\begin{cases} \dot{r} = r(r^4 + a_1 r^2 + a_0), \\ \dot{\varphi} = 1. \end{cases} \quad (2)$$

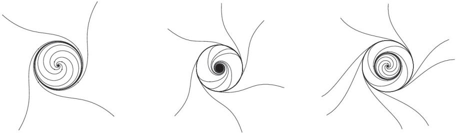
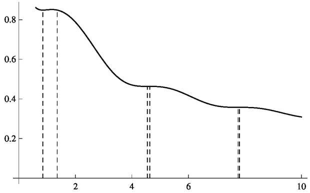
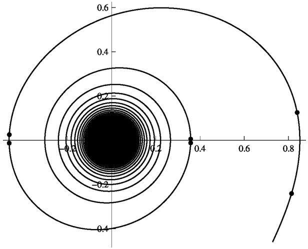


Figure: $a_0 < 0$, $a_0 = 0$, $a_0 \in (0,1)$

Figure: $a_0 = 1$, $a_0 > 1$

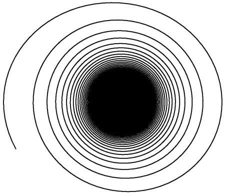
Weakned conditions for spirals





- Weakned conditions for the first derivative.
- Derivative could be zero and bigger than zero on a small intervals.
- No condition on the second derivative.

Application-Bessel spiral



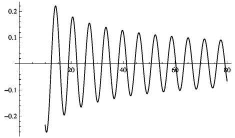


Figure: Curve $\Gamma = \{(x(t), \dot{x}(t)) : t \in [t_0, \infty)\}$, $\dim_B \Gamma = \frac{4}{3}$ and a graph of function $x(t) = J_\alpha(t)$, $\alpha \in \mathbb{R}$, (D. V., L. K., V.Ž.).

Poincaré map near weak focus

Function $d(s) = P(s) - s$, where $P(\cdot)$ is the Poincaré map and s small enough, is the displacement function.
System with weak focus

$$\begin{aligned} \dot{x} &= -y + p(x,y) \\ \dot{y} &= x + q(x,y), \end{aligned} \tag{3}$$

$p(x,y)$ and $q(x,y)$ are C^1 functions and $|p(x,y)| \leq C(x^2+y^2)$, $|q(x,y)| \leq C(x^2+y^2)$ for some $C > 0$ and (x,y) near the origin.

Poincaré map near weak focus, (V. Ž., D. Ž.)

Theorem

Assume: Γ *a spiral trajectory of a system of class* C^1*;* $P_\sigma(s)$ *is the Poincaré map with respect to an axis* σ*,* $P_\sigma(s) = s + d_\sigma(s)$ *for each* σ*; the displacement function* $d_\sigma(\cdot) : (0, r_\sigma) \to (-\infty, 0)$ *monotonically nonincreasing;* $-d_\sigma(s) \simeq s^\alpha$ *as* $s \to 0$*, for* $\alpha > 1$*.*

(a) If Γ *is a limit cycle spiral, then*

$$\dim_B \Gamma = 2 - \frac{1}{\alpha}.$$

(b) If Γ *is a focus spiral of (3), then*

$$\dim_B \Gamma = \begin{cases} 2 - \frac{2}{\alpha} & \text{for } \alpha > 2, \\ 1 & \text{for } 1 < \alpha \leq 2. \end{cases}$$

About proof of focus case-lipeomorphism

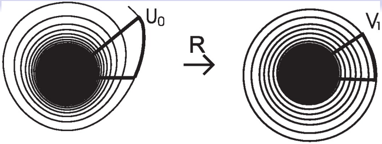


Figure: Flow-sector theorem: weak focus flow near the singular point is lipeomorphically equivalent to the annulus flow.

Theorem

(Flow-sector *Let* $U_0 \subset \mathbb{R}^2$ *be an open sector with the vertex at the origin, opening angle is in* $(0, 2\pi)$*, the boundary of* U_0 *consists of a part of a trajectory and of intervals on two rays emanating from the origin. If the diameter of* U_0 *is sufficiently small, then the system (3) restricted to* U_0 *is lipeomorphically equivalent to*

$$\begin{cases} \dot{r} &= 0 \\ \dot{\varphi} &= 1, \end{cases}$$

defined on the sector $V_0 = \{(r, \varphi) : 0 < r < 1,\ 0 < \varphi < \pi/2\}$ *in polar coordinates* (r, φ).

About proof-discrete system

$f(0) = 0$ and $x > f(x) > 0$, for $x \in (0, d)$.

$$g = id - f. \tag{4}$$

Orbit $S^g(x_0)$ of $0 < x_0 < d$ by g:

$$S^g(x_0) = \{x_n \mid n \in \mathbb{N}\}, \quad x_{n+1} = g(x_n). \tag{5}$$

Corollary

(N. E., V. Ž., D. Ž.)
f enough differentiable on $[0, d)$, positive, strictly increasing on $(0, d)$, $f(x) \simeq x^k$, $g = id - f$ then $|A_\varepsilon(S^g(x_0))| \simeq \varepsilon^{1/k}$ and $\dim_B(S^g(x_0)) = 1 - \frac{1}{k}$

Discrete system-more details

Theorem

Let $g = id - f$, 0 a fixed point of g, $1 < \mu_0^{fix}(g) < \infty$. Let $x_0 \in (0, d)$, $S^g(x_0)$ be an orbit, let $|A_\varepsilon(S^g(x_0))|$ be the length of the ε-neighborhood of the orbit $S^g(x_0)$, $\varepsilon > 0$.
Then

$$|A_\varepsilon(S^g(x_0))| \simeq \varepsilon^{1/\ \mu_0^{fix}(g)},\ \text{as } \varepsilon \to 0. \tag{6}$$

If $\mu_0^{fix}(g) = 1$, $f(x) < x$ on $(0, d)$, then

$$|A_\varepsilon(S^g(x_0))| \simeq \begin{cases} \varepsilon(-\log \varepsilon), & \text{if } f'(0) < 1 \\ \varepsilon \log(-\log \varepsilon), & \text{if } f'(0) = 1 \end{cases},\ \text{as } \varepsilon \to 0. \tag{7}$$

Moreover, for $1 \leq \mu_0^{fix}(g) < \infty$,

$$\mu_0^{fix}(g) = \frac{1}{1 - \dim_B(g)}. \tag{8}$$

Non-analytic Poincaré map

EXAMPLE

- $g_1(x) = x - x^2$, (diff. generators)
- $g_2(x) = x - x^2(-\log x)$, $g_3(x) = x - x^2 \log(-\log x)$ (nondiff. generators)
- $|A_\varepsilon(S^{g_1}(x_0))| \simeq \varepsilon^{1/2}$ – power-type behaviour!
- $\lim_{\varepsilon \to 0} \frac{|A_\varepsilon(S^{g_{2,3}})|}{\varepsilon^{1/2}} = +\infty$, $\lim_{\varepsilon \to 0} \frac{|A_\varepsilon(S^{g_{2,3}})|}{\varepsilon^{1/(2+\delta)}} = 0$, $\forall \delta > 0$ – noncomparable to any power!
- $\dim_B S^{g_1}(x_0) = \frac{1}{2}$, but also $\dim_B S^{g_{2,3}}(x_0) = \frac{1}{2}$
- Minkowski content

$$\mathcal{M}^{1/2}(S^{g_1}(x_0)) > 0, \text{ but both } \mathcal{M}^{1/2}(S^{g_{2,3}}(x_0)) = 0$$

- In nondiff. case find appropriate gauge functions (instead of powers) to compare $|A_\varepsilon|$ with $\to$ generalized Minkowski content (Lapidus)

ε-neighbourhood

Theorem (Mardešić, Resman, Županović, 2012)

$f \in C^r(0,d)$, continuous on $[0,d)$, positive on $(0,d)$, $f(0) = f'(0) = 0$, f sublinear:

$$m \le x \cdot (\log f)'(x),\ \ x \in (0,d),\ \ m > 1.$$

Then

$$|A_\varepsilon(S^g(x_0))| \simeq f^{-1}(\varepsilon) \text{ as } \varepsilon \to 0.$$

$\star$ e.g. $f(x) = \frac{x}{-\log x}$ not sublinear, $\frac{|A_\varepsilon(S^g(x_0))|}{f^{-1}(\varepsilon)} \to \infty$ as $\varepsilon \to 0$.

Parabolic germs

Pavao Mardešić will talk about

- Mardešić, Resman, Rolin, Ž,
  The Fatou coordinate for parabolic Dulac germs, Journal of Differential Equations 2018
- Mardešić, Resman, Rolin, Ž,
  Length of epsilon-neighborhoods of orbits of Dulac maps, preprint ArXiv
- We consider the class of parabolic Dulac germs of hyperbolic polycycles.
- In view of formal or analytic characterization of such a germ f by fractal properties of several of its orbits, we study the length $A_f(x_0, \varepsilon)$ of ε-neighborhoods of orbits of f with initial points x_0.

Slow-fast systems

- Renato Huzak, Box dimension and cyclicity of canard cycles. Qual. Theory 15 Dyn. Syst. (2017)
- Renato Huzak, Domagoj Vlah, Fractal analysis of canard cycles with two breaking parameters and applications, to appear in Communications on Pure and Applied Analysis
- Authors apply *fractal analysis of bifurcations*, to singular perturbation theory
- In the articles a bound on the cyclicity of canard cycles of a class of systems is given in terms of the derivative of the slow divergence integral and related to the box dimensions (of the derivative of the slow divergence integral).
- Furthermore, there is numerical computation of box dimension in the second article.

Fractal zeta functions

Goran Radunović will talk about his articles with Michel L. Lapidus and D. Žubrinić

- Fractal zeta functions represent a bridge connecting the geometry of fractal sets with complex analysis
- Their poles, called complex dimensions of fractal sets, are important for understanding the oscillatory nature of the inner geometry of fractal sets
- The theory could be applied to sets which are orbits of dynamical systems.
- We expect to read asymptotic expansion of the length of $\varepsilon-$neighborhood of orbit by zeta function.

Oscillatory integrals

We also study oscillatory integrals

$$\int_{\mathbb{R}^n} e^{i\tau f(\mathrm{x})}\Phi(\mathbf{x})\, d\mathbf{x}$$

by curve defined parametrically by the oscillatory integral

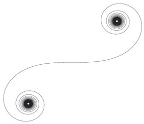


Figure: clothoid, $f(x) = x^2$

Oscillatory integrals and fractal dimensions

Rolin, Vlah, Ž, 2017, preprint ArXiv

- We analyze critical points ($\nabla f = 0$) of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$.
- Standard approach is analysis of asymptotic behavior of oscillatory integrals

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(x)} \Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty,$$

  with respect to parameter $\tau \in \mathbb{R}$.
- We examine geometrical properties of curve in the complex plane generated by $I(\tau)$ for $\tau \geq \tau_0 > 0$, and also of graphs of real and imaginary parts of $I(\tau)$.

$$X(\tau) := \mathrm{Re}\, I(\tau) = \int_{\mathbb{R}^n} \cos(\tau f(\mathbf{x}))\Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty,$$

$$Y(\tau) := \mathrm{Im}\, I(\tau) = \int_{\mathbb{R}^n} \sin(\tau f(\mathbf{x}))\Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty.$$

Standard assumptions on functions f and Φ

$$I(\tau) = \int_{\mathbb{R}^n} e^{i\tau f(\mathrm{x})} \Phi(\mathbf{x})\, d\mathbf{x}, \quad \tau \to \infty, \quad \tau \in \mathbb{R}.$$

- Function $\Phi : \mathbb{R}^n \to \mathbb{R}$
  - is called the *amplitude function*,
  - is of class C^∞,
  - is a function with compact support,
  - point $\mathbf{0}$ is inside the compact support of function Φ.
- Function $f : \mathbb{R}^n \to \mathbb{R}$
  - is called the *phase function*,
  - point $\mathbf{0} \in \mathbb{R}^n$ is the critical point of function f,
  - is a *real analytic* function in the neighborhood of its critical point $\mathbf{0}$,
  - point $\mathbf{0}$ is *the only* critical point of function f inside the compact support of function Φ.

Oscillatory and curve dimensions

Under the standard assumptions on functions f and Φ we determine:

- *Oscillatory dimensions* of functions $X(\tau)$ and $Y(\tau)$, which are defined as the box dimension of graphs of functions

$$x(t) := X(1/t), \quad t \to 0, \qquad y(t) := Y(1/t), \quad t \to 0,$$

  and associated Minkowski contents.
- *Curve dimension* of function $I(\tau)$, which is defined as the box dimension of the curve defined in the complex plane by $I(\tau)$, for $\tau \geq \tau_0 > 0$, and associated Minkowski contents.
- We show result for $n = 1$.
- For $n \geq 2$ dimensional case we use resolution of singularities and Newton diagram.

Phase function of a single variable

Theorem ($n = 1$)

Let the standard assumptions on f and Φ hold, and let $f(0) \neq 0$. Let $f'(0) = f''(0) = \cdots = f^{(p-1)}(0) = 0$ and $f^{(p)}(0) \neq 0$, $p \geq 2$ (p is the order of degeneracy). Using asymptotic $I(\tau) \sim C_1 \cdot e^{i\tau f(0)} \cdot \tau^{-1/p}$, as $\tau \to \infty$, it follows:

- *Oscillatory dimension of both $X(\tau)$ and $Y(\tau)$ is $d' = \frac{3p-1}{2p}$ and associated graphs are Minkowski nondegenerate. Bounds on d'-dimensional Minkowski content depend only on $f(0)$, p and C_1.*
- *Curve dimension of $I(\tau)$ is $d = \frac{2p}{p+1}$, and associated curve has d-dimensional Minkowski content*

$$\mathcal{M}^d(\Gamma) = C_1^{\frac{2p}{p+1}} \cdot \pi \cdot \left(\frac{\pi}{p \cdot f(0)} \right)^{-\frac{2}{p+1}} \cdot \frac{p+1}{p-1}.$$

Two variables-Newton diagram, logarithms in the asymptotic expansion of the integral,....

Some plans for future

Some plans for the future:

- Bifurcations by 1-parametric ε-neighborhood
- Bifurcations by fractal zeta functions
- Non-analytic phase maps-construction of toric manifold based on Varchenko results for analytic maps.