Local classification of differential operators

The “forgotten” case of the local analytic theory of ordinary differential equations

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Local systems of 1st order linear differential equations

Objects: local systems of 1st order linear ODEs

- Equations: $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = A(t)X(t), \quad t \in (\mathbb{C}^1, 0), \quad A \in \mathrm{Mat}_n(\mathbb{C}(\!(t)\!)),$
- $X(t) = \big(x_1(t), \dots, x_n(t)\big)$ "unknown" variables (solutions),
- $\mathbb{C}(\!(t)\!) =$ the algebra of formal Laurent series,
- A a formal Laurent matrix series,

  $$A(t) = \sum_{k=-(r+1)}^{\infty} t^k A_{k+r+1}, \quad A_k \in \mathrm{Mat}_n(\mathbb{C}).$$
- $r =$ Poincaré rank, the "order of pole" of A ($r = 0$ if the pole is simple).
- $A_0 =$ leading term with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ ("spectrum").

Reducing to identity with formal Taylor series $\mathbb{C}[\![t]\!]$

$$t^{r+1}\partial X = A(t)X, \quad A(t) = A_0 + tA_1 + t^2A_2 + \cdots, \quad \partial = \tfrac{\mathrm{d}}{\mathrm{d}t}, \quad A_0 \neq 0.$$

Euler derivation

It is very convenient to use the Euler derivation $\epsilon = t\partial$ instead of ∂: the former is "diagonal" on monomials, the latter nilpotent.

Group action and gauge equivalence

Action: linear change of the dependent variables

1. The group: $\mathfrak{G} = \mathrm{GL}_n(\mathbb{C}[\![t]\!])$: $\quad \mathfrak{G} = \left\{ H(t) = \sum_{k=0}^{\infty} t^k H_k \right\}, \quad \det H_0 \neq 0.$
   Assuming $H_0 = E$ does not restrict generality.
2. Informally: substitute $X(t) = H(t)Y(t)$ and write a new system for Y.
3. Formally: $A(t) \sim A'(t) \iff \exists H(t) \in \mathfrak{G}$ such that
   $$A'(t) = t^r \epsilon H(t) H^{-1}(t) + H(t)\, A(t)\, H^{-1}(t).$$
4. In the autonomous case (independent of t) the classification coincides with that of linear operators by action of $\mathrm{GL}(n, \mathbb{C})$ ("Jordan normal form").

What can be achieved by the suitable gauge transformation?

Naively, one might expect that all non-leading terms can be eliminated, so that
$A'(t) = A_0$ and the system is $\quad t^r \epsilon X = A_0 X$.

True to a certain extent. Obstructions are called resonances: these are arithmetic dependencies between the eigenvalues of the leading term A_0.

Conjugacy of Fuchsian singularities

Consider first the simpler case: $r = 0$ (first order pole, Fuchsian singularities)

$$A = A_0 + tA_1 + t^2A_1 + \cdots, \quad B = B_0 + tB_1 + t^2B_2 + \cdots.$$

$$H = E + tH_1 + t^2H_2 + \ldots, \qquad A_i, B_i, H_i \in \mathrm{Mat}(n, \mathbb{C}), \quad i = 1, 2, \ldots.$$

Conjugacy equation: $AH = \epsilon H + HB$

$$(A_0 + tA_1 + t^2A_2 + \cdots)(E + tH_1 + t^2H_2 + \cdots)$$
$$= (tH_1 + 2t^2H_2 + 3t^3H_3 + \cdots) +$$
$$+ (E + tH_1 + t^2H_2 + \cdots)(B_0 + tB_1 + t^2B_2 + \cdots)$$

$$A_0 = B_0,$$
$$A_1 + A_0H_1 = H_1 + B_1 + H_1A_0,$$
$$A_2 + A_1H_1 + A_0H_2 = 2H_2 + B_2 + H_1B_1 + H_2A_0,$$
$$\cdots = \cdots$$
$$A_0H_k - H_kA_0 - kH_k = \text{previously computed terms.}$$

Homological equation and its solvability

Commutator with A: linear endomorphism on matrices

$$\mathrm{ad}_A\colon \operatorname{Mat}(n,\mathbb{C}) \to \operatorname{Mat}(n,\mathbb{C}), \qquad H \longmapsto [A,H] = AH - HA.$$

Homological equation: $\qquad [A,H] - kH = \text{r.h.s.}, \quad k = 1,2,3,\dots$

When $[A,\,\cdot\,] - kE$ is surjective? Iff $k \in \mathbb{N}$ is not an eigenvalue of ad_A.

Example: $A = \Lambda = \operatorname{diag}\{\lambda_1,\dots,\lambda_n\}$

If $E_{ij} = \begin{cases} 1, & i=j, \\ 0, & i \neq j, \end{cases}$ then $[\Lambda, E_{ij}] = (\lambda_i - \lambda_j)E_{ij}$, $\quad E_{ij}$ eigenvector of ad_Λ.

Definition: resonance between eigenvalues, resonant matrices

A matrix $A \in \operatorname{Mat}_n(\mathbb{C})$ is resonant, if there is a pair of eigenvalues with integer difference $\lambda_i - \lambda_j \in \mathbb{N}$.

If A is non-resonant, then the homological equation is solvable for all $k \in \mathbb{N}$.

Gauge classification of non-resonant Fuchsian systems

Theorem (Classification of Fuchsian non-resonant systems)

If A_0 is non-resonant, then $\epsilon X = (A_0 + tA_1 + t^2 A_2 + \cdots)X$ *is gauge equivalent to the Euler system* $\epsilon Y = B_0 Y, \quad B_0 = A_0$.

Proof:

$$
\begin{aligned}
A_0 &= B_0, \\
A_1 + A_0 H_1 &= H_1 + B_1 + H_1 A_0, \\
A_2 + A_1 H_1 + A_0 H_2 &= 2H_2 + B_2 + H_1 B_1 + H_2 A_0, \\
\cdots &= \cdots \\
A_0 H_k - H_k A_0 - k H_k &= \text{previously computed terms}, \\
[A_0, H_k] - k H_k &= \text{anything you write here}.
\end{aligned}
$$

All equations are solvable w.r.t. H_k with $B_k = 0$ (or any other choice of B_k).

How to deal with resonances?

1. If resonances occur, then some natural values k appear as eigenvalues of ad_{A_0}, hence $[A_0, \cdot\,] - kE$ becomes non-invertible and non-surjective.
2. To ensure solvability of the homological equation, one has to study the the image of $\mathrm{ad}_{A_0} - kE$ and choose B_k from a subspace in $\mathrm{Mat}_n(\mathbb{C})$ which is complementary to the image.
3. The problem completely belongs to the realm of the linear algebra. The choice of the complements has some freedom that can be used.
4. It turns out that the direction $\mathbb{C} \cdot E_{ij}$ is complementary to the image of $\mathrm{ad}_{A_0} - kE$ when $\lambda_i - \lambda_j = k$, so choosing $B_k = b_{ijk}E_{ij}$ with suitable $b_{ijk} \in \mathbb{C}$ (resonant monomial) guarantees solvability of the homological equation in the kth row.
5. As a result, the resonant Fuchsian system turns out to a system of the form $\epsilon Y = B(t)Y, \quad B(t) = A_0 + \sum t^k B_k$ with only finitely many terms (hence polynomial) and of very special (sparse) form.
6. This resonant normal form is integrable: its solution can be explicitly written as $Y(t) = t^{A_0} \cdot P(t)$ with a matrix polynomial $P \in \mathrm{Mat}(n, \mathbb{C}[\,t\,]), \quad P(0) = E$.

Non-Fuchsian singularities

Consider the positive Poincaré rank $r > 0$. What changes in the analysis?

$$A = A_0 + tA_1 + t^2A_1 + \cdots, \quad B = B_0 + tB_1 + t^2B_2 + \cdots, \quad H = E + tH_1 + t^2H_2 + \dots.$$

Conjugacy equation revisited: $AH = t^r\epsilon H + HB$

$$\begin{aligned}(A_0 + tA_1 + t^2A_2 + \cdots)&(E + tH_1 + t^2H_2 + \cdots)\\ &= (t^{1+r}H_1 + 2t^{2+r}H_2 + 3t^{3+r}H_3 + \cdots)+\\ &\quad + (E + tH_1 + t^2H_2 + \cdots)(B_0 + tB_1 + t^2B_2 + \cdots)\end{aligned}$$

$$\begin{aligned} A_0 &= B_0,\\ A_1 + A_0H_1 &= B_1 + H_1A_0,\\ A_2 + A_1H_1 + A_0H_2 &= H_1 + B_2 + H_1B_1 + H_2A_0,\\ \cdots &= \cdots\\ A_0H_k - H_kA_0 &= \text{previously computed terms.}\end{aligned}$$

The operator $\mathrm{ad}_{A_0} - kE$ is replaced by ad_{A_0} for all $k = 1, 2, \dots$.

Solvability of the homological equation

1. The homological operator $\mathrm{ad}_{A_0} = [A_0, \cdot\,]$ is the same for all degrees k.
2. Even in the simplest diagonal case $A_0 = \Lambda = \mathrm{diag}\{\lambda_1, \dots, \lambda_n\}$ it is non-surjective for all k: the identity $\lambda_i = \lambda_i$ is always a "resonance".
3. If we assume that the eigenvalues are pairwise different, then the identities $\lambda_i - \lambda_i = 0$ are the only resonances.
4. The corresponding resonant monomials are very easy to describe: in every degree only the diagonal terms E_{ii} need to be included in the normal form.

Definition (Resonant irregular systems)

The irregular system $t^r \epsilon X = A(t)X, \quad A(t) = A_0 + tA_1 + t^2 A_2 + \cdots$ with $r > 0$ is resonant, if A_0 has eigenvalues of multiplicity $\geqslant 1$ (pairs of equal eigenvalues).

Theorem (Formal diagonalization of irregular systems)

A non-resonant irregular (non-Fuchsian) system is gauge equivalent to a diagonal system $t^r \epsilon Y = \Lambda(t)Y, \quad \Lambda(t) = \mathrm{diag}\{\lambda_1(t), \dots, \lambda_n(t)\}, \quad \Lambda(0) = A_0.$

More about irregular systems

1. A one-dimensional irregular "system" (i.e., equation) $t^r \epsilon x = \lambda(t) \cdot x$ can be transformed by a substitution $x = h(t)y$ to an equation with a polynomial (in t) right hand side size $\lambda(t) \in \mathbb{C}[t]$ of degree $\deg \lambda \leqslant r$.
2. As any scalar linear equation, this one is (Liouville) integrable, so is the diagonal normal form.
3. An irregular system with resonances (e.g., with several zero eigenvalues) is a real hassle to study. It still can be diagonalized, if we allow for ramified transformations, that is, replace the group $\mathfrak{G} = \mathrm{GL}(n, \mathbb{C}[\![t]\!])$ by the larger group $\mathfrak{G}^{1:m} = \mathrm{GL}(n, \mathbb{C}[\![t^{1/m}]\!])$ for a suitable integer ramification index $m \in \mathbb{N}$.
4. How effective is this result? Can you tell which m you will need to diagonalize a given system? Depends on whom you ask....

The real difference between Fuchsian and irregular systems

Genuine game-changer: convergence.

Assume that the series $A(t) = \sum_{k=0}^{\infty}$ converges in some disk $\{|t| < \rho\}$, $\rho > 0$.

What can be said about the system $t^r \epsilon X = A(t)X$ and its derivatives?

1. In the Fuchsian case $r = 0$ the solution has a form $X(t) = t^{A_0}U(t)$ with a convergent matrix-function U, $\quad U(0) = E$.
2. The gauge transform $H(t)$ conjugating the initial system with its integrable normal form (Euler system in the non-resonant case, more involved in the resonant case) converges as well.
3. It is safe to assume that any formal statement about Fuchsian systems remains true in the convergent (analytic) context.
4. In the irregular case the formal series $H(t) = E + \sum_{k=1}^{\infty} t^k H_k \in \mathfrak{G}$ diagonalizing the system, as a rule, diverges.
5. The obstruction to its convergence was identified as a collection of $2r$ Stokes matrices in $\mathrm{GL}(n, \mathbb{C})$ (a matrix cocycle). Convergence occurs if and only if this cocycle vanishes.

Higher order linear equations

Linear homogeneous equations involving higher order derivatives

- $a_0(t)\,y^{(n)} + a_1(t)\,y^{(n-1)} + \cdots + a_{n-1}(t)\,y^{(1)} + a_n(t)\,y^{(0)} = 0,$
- $a_0, \cdots, a_n \in \mathbb{C}[\![t]\!]$ formal coefficients, $a_0 \not\equiv 0,$
- $y^{(0)} = y(t)$ the unknown function, $y^{(k+1)} = \frac{\mathrm{d}}{\mathrm{d}t} y^{(k)}$, $k = 0, 1, \ldots, n-1.$

Apparently, should be just a particular case of the classical theory:

- Can be reduced to a system of first order linear ODEs for the variables $x_1 = y^{(0)},\ x_2 = y^{(1)},\ \ldots,\ x_n = y^{(n-1)}$.
- The system has rather specific companion form:

$$A(t) = -\frac{1}{a_0(t)} \cdot \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ a_n & a_{n-1} & \cdots & a_2 & a_1 \end{pmatrix} \in \operatorname{Mat}(n, \mathbb{C}(\!(t)\!)).$$

- Conversely, any system $\frac{\mathrm{d}}{\mathrm{d}t} X = A(t) X$ can be reduced to an nth order equation for each of the components $x_1, \ldots, x_n$.

Trouble: no natural group action.

- The linear transformations $y = h(t)z, \quad h \in \mathrm{GL}(1, \mathbb{C}[\![t]\!])$ form a group which is way too small to produce a meaningful classification.
- Alternatively, if $A(t)$ is a matrix in the companion form, then the similar matrix $B(t) = B \sim A$ in general is not in the companion form.
- One could try to enlarge the class of linear transformations

$$y = h_0(t)z^{(0)} + h_1(t)z^{(1)} + h_2(t)z^{(2)} + \cdots, \qquad h_i \in \mathbb{C}(\!(t)\!),$$

  but the following problems arise:
  - It is not clear where to stop the summation;
  - It is not clear why this transformation would be invertible;
  - Why such transformations would form a group at all?
- Note that for any finite truncation there will be a nontrivial null space (solutions of the corresponding ODE).

Change of paradigm: switch to noncommutative algebras

1. Treat $\mathbb{C}((t))$ as a differential field with the operator $\partial = \frac{\mathrm{d}}{\mathrm{d}t}$ on this field and its possible extensions.
2. The "powers" ∂^k, $k = 0, 1, 2 \dots$ are naturally defined linear operators on $\mathbb{C}((t))$.
3. The elements $a(t) = a \in \mathbb{C}((t))$ themselves can be considered as "zero order differential operators" acting on $\mathbb{C}((t))$ by multiplication, $y \longmapsto a(t)y$.
4. There are naturally defined operations of addition, and composition of differential operators (multiplication by constant is composition with the operator $y \mapsto \lambda u$).
5. This defines us the (noncommutative) algebra $\mathscr{L}$ of linear differential operators.
6. Obviously, all powers ∂^k commute between themselves, as well as all multiplication operators $a \in \mathbb{C}((t))$. All non-commutative identities in this algebra follow from Leibniz rule $\partial(ty) = t\,\partial y + \partial y$ which translates as $[\partial, t] = 1$.
7. Using this identity allows to write each element $L \in \mathscr{L}$ as a "polynomial" $L = a_0\partial^n + a_1\partial^{n-1} + \cdots + a_{n-1}\partial + a_n, \quad a_i \in \mathbb{C}((t))$.
   Note: all coefficients are to the left from all powers ∂^k. This convention ensures uniqueness of the representation.
8. The "degree" in ∂ coincides with the usual order of differential operators.

Ore theory

ØYSTEIN ORE (1933), *Theory of noncommutative polynomials* (Ann. Math.)

- Abstract algebraic construction: commutative algebra $\mathfrak{A}$ is extended by a new element δ such that $[\delta, a] \in \mathfrak{A}$ for any $a \in \mathfrak{A}$. How much $\mathfrak{A}[\delta]$ differs from $\mathfrak{A}[x]$?
- $\mathfrak{A}[\delta]$ is filtered (not graded!). The "degree" in δ is well defined.
- $\mathfrak{A}[\delta]$ is a Euclidean ring (algebra): the division with remainder works in $\mathfrak{A}[\delta]$.
- Define $D = \gcd(P, Q)$ for any two polynomials $P, Q \in \mathfrak{A}[\delta]$ as the maximal polynomial which divides P and Q from the right: $P = P'D, \quad Q = Q'D$. Can be computed by the Euclid algorithm.
- Define the dual notion of $S = \operatorname{lcm}(P, Q)$ as the smallest polynomial which is divisible from the right by both P, Q: $\quad S = UP = VQ$.
- Similarity: define $\operatorname{ad}_P Q = \operatorname{lcm}(P, Q) \cdot P^{-1} = U$ (cancellation of the factor).

If we identify polynomials with differential operators and the latter with their kernels (solutions of the homogeneous differential equations $Py = 0$):

- $\gcd(P, Q)$ is "the" operator annulling solutions of both $Py = 0$ and $Qy = 0$;
- $\operatorname{lcm}(P, Q)$ is "the" operator annulling the sum of the solution spaces.

- "the": There is a delicate point concerning the leading terms, since P and $aP, \quad a \in \mathfrak{A}$ have the same kernel.

Equivalence of differential operators

Template definition

Two operators $L, M \in \mathscr{L} = \mathbb{C}((t)) \otimes_{\mathbb{C}} \mathbb{C}[\partial]$ are said to be Ore equivalent, if there exists an operator $H \in \mathscr{L}$, which sends solutions of the equation $Mz = 0$ to solutions of $Ly = 0$, $\quad y = h_0(t)z^{(0)} + h_1(t)z^{(1)} + h_2(t)z^{(2)} + \cdots, \quad h_i \in \mathbb{C}((t))$, bijectively.

- If we denote $y = Hz$, the above definition means that $LHz = 0$ on $\{Mz = 0\}$, that is, LH is divisible by M from the right: $\quad LH = KM \quad$ for some $K \in \mathscr{L}$.
- The bijectivity means that H never vanishes on $\{Mz = 0\}$, that is, $\gcd(H, M) = 1$.
- Caveat: solutions of differential equations only rarely belong to the field $\mathbb{C}((t))$, one has to extend the action of $\mathscr{L}$ on extensions of $\mathbb{C}((t))$. Yet the identities between operators persist over passing to the extensions.

Theorem (Ore)

Despite its obvious asymmetry, this definition produces a genuine equivalence relation, in particular, it is symmetric and transitive.

Meet the small devils which are in details

A few things need to be specified before launching the classification machinery:

1. There are two groups acting on systems of 1st order equatons, $\mathfrak{G} = \mathrm{GL}(n, \mathbb{C}[\![t]\!])$ and a larger group $\mathfrak{M} = \mathrm{GL}(n, \mathbb{C}(\!(t)\!)) \supsetneq \mathfrak{G}$. The latter "does not feel" resonances for Fuchsian systems.
2. For systems there is the dichotomy Fuchsian/non-Fuchsian which is somewhat artificial: $\mathfrak{M}$-equivalence can turn a Fuchsian system to a non-Fuchsian one.
3. What is a *morally correct* analog of Fuchsian systems in $\mathscr{L}$?
4. What are *morally correct* extra conditions on the pair of operators H, K conjugating L and M via the identity $LH = KM$, to imitate the group actions of $\mathfrak{G}$ and $\mathfrak{M}$?
5. The common curse/blessing in disguise: the operators L and aL, $0 \not\equiv a \in \mathbb{C}(\!(t)\!)$, define the same solutions. How to build a "projective invariant" theory?

Fuchsian equations of higher order

1. First, change the base differentiation of the field $\mathbb{C}((t))$ and introduce $\epsilon = t\partial$, so that $\partial = t^{-1}\epsilon$. It is called the Euler derivation.
2. The algebras $\mathbb{C}((t))[\partial]$ and $\mathbb{C}((t))[\epsilon]$ coincide, although $\mathbb{C}[\![t]\!][\epsilon] \neq \mathbb{C}[\![t]\!][\partial]$.
3. A Fuchsian system can be written in the operator form as $(\epsilon - A)X = 0$, where $A = A(t) \in \operatorname{Mat}(n, \mathbb{C}[\![t]\!])$ is non-singular matrix function.

Definition: Fuchsian operators, Fuchsian equations

An operator $L \in \mathscr{L}$ is Fuchsian, if it has the "monic" form

$$L = \epsilon^n + a_1(t)\epsilon^{n-1} + \cdots + a_{n-1}(t)\epsilon + a_n(t)$$

with nonsingular coefficients $a_1, \ldots, a_n \in \mathbb{C}[\![t]\!]$. An equation $Ly = 0$ is Fuchsian, if $L = a_0 F$, where $0 \not\equiv a_0(t) \in \mathbb{C}((t))$ and $F \in \mathbb{C}[\![t]\!][\epsilon]$ is a Fuchsian operator.

1. All solutions of Fuchsian equations grow moderately as $t \to 0$.
2. The converse holds: if all solutions of a linear equation grow moderately, then the equation is Fuchsian.

Example. The Euler equation: $P(\epsilon)y = 0$ with "constant" coefficients, $P \in \mathbb{C}[\epsilon]$.

Expansion in formal series

1. Any operator from $\mathbb{C}[\![t]\!][\epsilon]$ of order n can be expanded in the formal Taylor series
$$L = \sum_{k=0}^{\infty} t^k p_k(\epsilon), \qquad p_k \in \mathbb{C}[\epsilon], \quad \deg p_k \leqslant n, \quad k = 0, 1, 2, 3, \dots .$$
2. The operator is Fuchsian, if $p_0(0) = 1$, i.e., $\deg p_0 = n = \max_k \deg p_k$.
3. The *canonical form* collects powers of t to the left from powers of ϵ.
4. The commutation law $[\partial, t] = 1$ implies that $[\epsilon, t] = t$ and allows especially simple form for the iterates $\epsilon^j t^k = t^k(\epsilon + k)^j$.
5. Consequently, $p(\epsilon)t^k = t^k p^{[k]}(\epsilon)$, where $p^{[j]}$ is p with the argument shifted by k.
6. This allows to reduce all "algebraic" equations in $\mathbb{C}[\![t]\!][\epsilon]$ to infinite systems of equations in the (commutative) ring of polynomials $\mathbb{C}[\epsilon]$.

Fuchsian equivalence between Fuchsian operators

After experimentation with various definitions, the most interesting was identified.

Definition. Fuchsian equivalence

A pair of operators $L, M \in \mathbb{C}[\![t]\!][\epsilon]$ of the same order is Fuchsian equivalent, if there exist Fuchsian operators H, K such that

$$LH = KM, \qquad \gcd(H, M) = 1.$$

Remark. What were the other choices and why they were discarded

Requirement of H, K being nonsingular was way too restrictive.

On the other hand, dropping all constraints on H, K resulted in the classification similar to $\mathfrak{M}$-classification of systems: the only invariant was the monodromy operator, no resonances felt.

Conjugacy between Fuchsian operators

Expand all operators in the conjugacy equation:

$$L = \sum_{k=0}^{\infty} t^k p_k(\epsilon), \quad M = \sum_{k=0}^{\infty} t^k q_k(\epsilon), \quad H = \sum_{k=0}^{\infty} t^k h_k(\epsilon), \quad K = \sum_{k=0}^{\infty} t^k u_k(\epsilon)$$

Substitute to $LK = HM$ and reduce noncommutative products to the canonical form. Uniqueness of the canonical form implies identities between right coefficients in $\mathbb{C}[\epsilon]$.

$$\begin{aligned} p_0 u_0 &= h_0 q_0, \\ p_0 u_1^{[1]} + p_1 u_0 &= h_1 q_0^{[1]} + h_0 q_1, \\ p_0 u_2^{[2]} + p_1 u_1^{[1]} + p_2 u_0^{[2]} &= h_2 q_0^{[2]} + h_1 q_1^{[1]} + h_0 q_2, \\ \cdots &= \cdots \\ p_0 u_k^{[k]} + \cdots &= h_k q_0^{[k]} + \cdots . \end{aligned}$$

Here p_k are given, q_k should be as simple as possible, h_k, u_k unknown.
Let $q_0 = p_0$ and $u_0 = h_0$ with $\gcd(u_0, p_0) = 1$. What next?

Homological equation: a closer look

$$
\begin{aligned}
p_0 h_0 &= h_0 p_0,\\
p_0 u_1^{[1]} + p_1 h_0 &= h_1 p_0^{[1]} + h_0 q_1,\\
p_0 u_2^{[2]} + p_1 u_1^{[1]} + p_2 h_0^{[2]} &= h_2 p_0^{[2]} + h_1 q_1^{[1]} + h_0 q_2,\\
\cdots &= \cdots\\
p_0 u_k^{[k]} + \cdots &= h_k p_0^{[k]} + \cdots .
\end{aligned}
$$

The general equation is $p_0 u_k^{[k]} - p_0^{[k]} h_k =$ previously computed terms.

The Sylvester map $\mathbb{C}[\epsilon] \to \mathbb{C}[\epsilon]$, , cf. with $\operatorname{ad}_A - k \colon H \longmapsto [A_0, H] - kH$

Let $a, b \in \mathbb{C}^{n+1} \simeq \{\mathbb{C}[\epsilon] : \deg \leqslant n\}$ be two polynomials of degree n.
The Sylvester map is the linear map $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^{2n}$,

$$(u, v) \longmapsto S_{a,b}(u, v) = au + bv, \qquad u, v \in \mathbb{C}[\epsilon], \quad \deg u, v \leqslant n - 1.$$

Theorem (Sylvester? in any case 1st year Linear algebra course)

The Sylvester map $S_{a,b}$ is bijective if and only if $\gcd(a, b) = 1$, that is, a and b have no common roots.

Eulerization

When $p(\cdot)$ and its shift $p^{[k]}(\cdot) = p(\cdot\ + k)$, $k = 1, 2, 3, \ldots$ have no common roots?

Definition.

A Fuchsian operator $L = \sum_{k=0}^{\infty} t^k p_k(\epsilon)$ is nonresonant, if the leading polynomial $p_0 \in \mathbb{C}[\epsilon]$ has nonresonant roots, i.e., *no two roots differ by an integer number*.

Guarantees solvability of all conjugacy equations for $M = p_0(\epsilon)$.

NB: The same as the nonresonance condition on eigenvalues for Fuchsian systems.

Theorem. Eulerization of non-resonant Fuchsian equations

A non-resonant Fuchsian equation is equivalent to its principal (Euler) part $p_0(\epsilon) \in \mathbb{C}[\epsilon] \subseteq \mathbb{C}[\![t]\!][\epsilon]$.

Remark. An Euler operator is a composition of commuting 1-st order operators,

$$p_0(\epsilon) = (\epsilon - \lambda_1) \cdots (\epsilon - \lambda_n), \qquad \lambda_i \in \mathbb{C}.$$

Resonant normal forms

- If the principal Euler part of an operator $L = p_0 + tp_1 + t^2 p_2 + \cdots$ is resonant, then for some k the homological operator is not surjective.
- The complement to the image of the Sylvester operator is not canonically defined. In particular, one can always choose M as a polynomial in t, but this is a quite useless claim.
- Much more useful is the following result, an *integrable* polynomial normal form.

Theorem (Shira Tanny, S.Y., 2015)

A resonant Fuchsian operator is equivalent to a composition of 1st order operators

$$M = \big(\epsilon - r_1(t)\big) \cdots \big(\epsilon - r_n(t)\big), \qquad r_i \in \mathbb{C}[\,t\,], \quad r_i(0) = \lambda_i,$$

where the support of the polynomials $r_i(t) \in \mathbb{C}[\,t\,]$ depending on resonances.

In particular, $\deg r_i \leqslant \max_k \{k \in \mathbb{N} \colon \lambda_i + k = \lambda_j\}$ (cf. with non-resonant case).

Remark. The equation $My = 0$ with M as above is Liouville integrable: all solutions can be obtained by quadratures and their exponentials.

Factorization as an analog of diagonalization for systems

- The mere possibility of representing a Fuchsian operator L as a composition $L = L_1 \cdots L_n$ of 1st order operators $L_i = \epsilon - r_i(t), \quad r_i \in \mathbb{C}[\![t]\!]$ is a simple fact: it *does not require conjugacy* in $\mathscr{L}$.
- Such "noncommutative factorization" allows to reduce solution of nth order equation $Ly = 0$ to 1st order (non-homogeneous) equations, an easy job.
- In the same way diagonal systems of 1st order equations can be explicitly solved in quadratures.
- The Ore theory for $\mathfrak{A}[\delta]$ tells a lot about the general problem of factorization in the ring of non-commutative polynomials.

  Problem. Are there any specific features of the theory in the case where $\mathfrak{A} = \mathbb{C}[\![t]\!]$ is a graded ring and $\delta = \epsilon$ a derivation which preserves this grading?
- Looks like an easier task: instead of the equation $LH = KM$ we need to study a simpler equation $L = PQ$ (but still nonlinear). What are obstructions to its solution in the algebra $\mathbb{C}[\![t]\!][\epsilon]$, in which $[\epsilon, t] = t \in \mathbb{C}[\![t]\!]$ is "morally small"?
- Wait... what do we know about the commutative analog of the problem? What about factorization in the commutative ring $\mathbb{C}[\![t]\!][\xi]$ where $[\xi, t] = 0$?

Pseudopolynomials and their support

Pseudopolynomials: commutative $\mathbb{C}$-algebra $\mathscr{P} = \mathbb{C}[\![t]\!][\xi]$ of semi-infinite sums

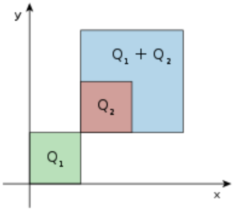
$$P(t,\xi) = \sum_{k=0}^{\infty}\sum_{j=0}^{n} p_{kj}\, t^k \xi^j, \quad p_{kj} \in \mathbb{C} \qquad \text{(think of } t \text{ as "small", } \xi \text{ "large").}$$

Support is the lattice subset $\operatorname{supp} P = \{(j,k) \in \mathbb{Z}^2_+ : p_{jk} \neq 0\}$.
Monomial: an element $M \in \mathscr{P} = ct^k\xi^j$ whose support is a single point, $c \neq 0$.

"Logarithmic behavior" of supports

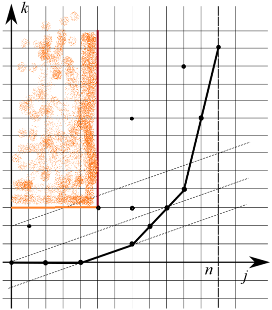
If $P = QR \in \mathscr{P}$, then $\operatorname{supp} P \subseteq \operatorname{supp} Q + \operatorname{supp} R$, where the plus stands for the Minkowski sum of two subsets of $\mathbb{Z}^2_+$. **Proof.** Obvious for the monomials.
Remainder: the Minkowsky sum $U + V$ is defined as $\{u + v \colon u \in U,\ v \in V\} \subseteq \mathbb{R}^2$.

- Why only $\subseteq$, and not equality, as for the monomials? Because some terms may accidentally cancel each other, if the same point $w \in \mathbb{Z}^2_+$ can be represented by two different ways $w = u + v = u' + v'$.
- Only the "leading" terms, corresponding to the "extremities" of $\operatorname{supp} P$, are guaranteed to survive.



The Newton polygon

Let $P(t,\xi) \in \mathscr{P}$ be a pseudopolynomial.



- With any $(j,k) \in \operatorname{supp} P$ associate the infinite semistrip $S_{jk} = \{(j',k') \in \mathbb{Z}^2_+ : j' < j,\ k' > k\}$ of "smaller" (inferior, subordinate) exponents.
- Consider the union (possibly infinite)

$$S_P = S_{00} \cup \bigcup_{(j,k)\in \operatorname{supp} P} S_{jk}.$$

- Finally, denote $\varDelta_P = \operatorname{conv} S_P$ its convex hull in $\mathbb{R}^2_+$. It is a set closed by shifts North and East in $\mathbb{R}^2_+$. This (unbounded) set is called the Newton polygon of P.

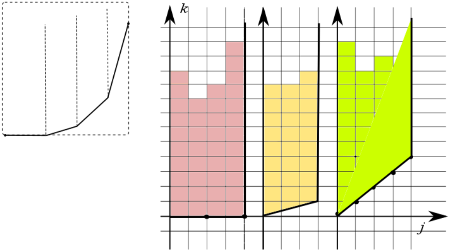
Newton diagram of $P = \sum_{k=0}^{\infty} t^k p_k(\xi)$

$\varDelta_P$ is an *epi*-graph of a piecewise affine monotone function $\chi\colon [0,n] \to \mathbb{R}_+$, $n = \deg_\xi P$, $\chi(0) = 0$, with vertices only at the lattice points.

Each affine segment has a rational slope $r_i \in \mathbb{Q}_+$. Call such polygons admissible.

The inverse function χ^{-1} is a concave majorant for the function $k \longmapsto \deg p_k$.

Cutting the Newton polygon into admissible pieces



- $\Delta_{PQ} = \Delta_P + \Delta_Q$.

  Equality, not inclusion!
- Any admissible polygon can be sliced into single slope admissible polygons.
- For single-slope polygons χ is linear.

(In)decomposability of single-slope polygons Δ of width n

1. If $\gcd\big(n, \chi(n)\big) = 1$, then there are no lattice points on the graph of χ (except for the endpoints), hence the Newton polynomial is indecomposable.
2. Otherwise, Δ can be represented as Minkowski sum of several copies of a polygon with smaller width and the same slope.
3. Their number is equal to the number of nonzero lattice points on the graph of χ.

First factorization theorem

Conjecture

A pseudopolynomial $P \in \mathscr{P}$ with a Newton polygon $\varDelta_P$ factors as a product $P = P_1 \cdots P_m$ with single-slope pseudopolynomials $P_i \in \mathscr{P}$ with pairwise different slopes $r_1, \cdots, r_m \in \mathbb{Q}_+$. The slopes are those of $\varDelta_P$.

Theorem (First theorem on factorization of pseudopolynomials)

The above conjecture is true.

Proof.

- The analogous result for formal or converging series $f(t, s)$ in two variables $t, s \in (\mathbb{C}, 0)$ is well known in the Singularity theory.
- The modern proof uses the desingularization (blow-up, resolution).
- The case of pseudopolynomials can be reduced to the above problem by the change of variables $f(t, s) = s^n P(t, \frac{1}{s})$, $n = \deg P$.

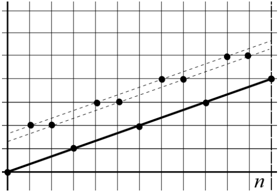
□

How to construct such factorization explicitly, at least formally?

Quasihomogeneity: the way of organizing monomials

Method of indeterminate coefficients to solve the equation $P = QR$

Given $P \in \mathscr{P}$, find Q, R subject to certain conditions involving the “principal” terms of P, Q, R. Infinite system of linear equations on the coefficients of Q, R separately, but nonlinear jointly in Q, R. There is a natural way to “linearize” this system.



Organizing 2D array into 1D strings

Let $w = -\frac{p}{q}$, $\gcd(p,q) = 1$, be a nonpositive rational number, assigned as the weight of ξ. A p/polynomial $P \in \mathscr{P}$ is quasihomogeneous of weight $\alpha \in \mathbb{Q}$, if $\operatorname{supp} P \subseteq \{(i,j) : i + wj = \alpha\}$.

- Any $P \in \mathscr{P}$ can be expanded as an ordered infinite sum $P = \sum_{\alpha \in \mathbb{Q}} P_\alpha$ of q/homogeneous p/polynomials of increasing weights.
- Any monomial of weight zero has the form cu^k, $c \neq 0, \quad u = t^q \xi^p$.
- Any q/homogeneous $P_\alpha \in \mathscr{P}$ can be represented as $P_\alpha = t^k \xi^s \cdot f(u)$ with $k - ws = \alpha$ and a suitable univariate polynomial $f \in \mathbb{C}[u]$.
- The representation is unique if $f(0) \neq 0$.

Factorization in $\mathscr{P}$: homological equation

1. Given $P \in \mathscr{P}$ with $\Delta = \Delta_P$ and a decomposition $\Delta = \Delta' + \Delta''$, find Q, R so that $\Delta_Q = \Delta'$, $\Delta_R = \Delta''$.
2. To do this, choose a weight w (in a smart way) and expand $P = \sum P_\gamma$ as a sum of q/homogeneous terms starting from the leading term P_{γ^*}.
3. Write $Q = \sum_\alpha Q_\alpha$, $R = \sum_\beta R_\beta$ with unknown q/homogeneous terms.
4. Knowing w, Δ' and Δ'', determine the support of the leading terms Q_{α^*} and R_{β^*}.
5. Plugging the expansions into *nonlinear* equation $P = QR$, we obtain an infinite system of *linear* equations between q/homogeneous ($\approx$ univariate) polynomials.

$$P_{\gamma^*} = Q_{\alpha^*} R_{\beta^*}, \qquad \alpha^* + \beta^* = \gamma^*.$$

$$\cdots = \cdots$$

$$P_\gamma = Q_{\alpha^*} R_{\gamma-\alpha} + \sum_{\alpha>\alpha^*,\ \beta>\beta^*} Q_\alpha R_\beta + Q_{\gamma-\beta^*} R_{\beta^*}, \qquad \alpha + \beta = \gamma > \gamma^*.$$

Homological operator (first approximation)

Given two q/homogeneous polynomials Q^*, R^* of different weights, describe the image of the “Sylvester map” $(U, V) \longmapsto Q^* U + R^* V$ between the spaces of pairs of appropriately chosen q/homogeneous pairs.

Fine print and the bottom line

1. The “Sylvester map” should be constrained by the requirement that the support of the r.h.s. Δ and q/homogeneous solutions U, V must belong to the corresponding Newton polygons Δ', Δ''.
2. The choice of the weight is crucial. For a random choice of w the leading terms Q^*, R^* will be monomial, and the “Sylvester map” will be non-surjective.
3. If $-w$ is one of the slopes of the Newton diagram of P, Q is a single-slope p/polynomial associated with this slope, then the surjectivity is possible.
4. Still the structure of the equations is delicate (as it depends on the order irregularly) and the proof of their solvability is highly nontrivial.
5. Isaac Newton, who invented this method, went all the way to the complete solution. Solution exists, all “Sypvester maps” are surjective.
6. The First Factorization theorem gives an indirect proof of solvability: since factorization is possible, the system is solvable, “Sylvester maps” are surjective.

Theorem

The “Sylvester map”, properly defined, is always surjective if Δ' and Δ'' have no edges with the same slope.

Back to the noncommutative algebra $\mathscr{W} = \mathbb{C}[\![t]\!][\epsilon]$

- Elements: semi-infinite series $L = \sum c_{kj} t^k \epsilon^j, \quad c_{kj} \in \mathbb{C}$.
- The coefficients are uniquely determined if we insist that powers of t always to the left from powers of ϵ (canonical form).
- Define $\operatorname{supp} L$, and the Newton polygon Δ_L literally as before.
- The commutation law in $\mathscr{W}$ is $\epsilon^j t^k = t^k(\epsilon + k)^j$, which allows to reduce the composition back to the canonical form. Note that $t^k\epsilon^j - \epsilon^j t^k$ contains only "inferior terms"!
- **Theorem.** *For any* $L, M \in \mathscr{W}$, $\qquad \Delta_{LM} = \Delta_{ML} = \Delta_L + \Delta_M$.
- This allows to translate the formal construction of decomposition $P = QR$ to the non-commutative case $L = MN$:
  - Choose a decomposition $\Delta_L = \Delta' + \Delta''$ on the level of the Newton polygons,
  - Choose a suitable weight $w = -\frac{p}{q}$ and expand everything into sums of q/homogeneous components $L = \sum L_\gamma, M = \sum M_\alpha, N = \sum N_\beta$,
  - Substitute these expansions into the equation and identify the corresponding "Sylvester maps" in each degree of the q/homogeneity.
- What can go wrong? $\mathscr{W}_\gamma \neq \sum_{\alpha+\beta=\gamma} \mathscr{W}_\alpha \mathscr{W}_\beta$, only the inclusion $\subseteq$ holds.
- However, the difference consists of "inferior" terms ($\mathscr{W}$ is filtered by $\mathscr{W}_\gamma$ and not graded). All "Sylvester maps" remain the same as in the commutative case $\mathscr{P}$.
- **Conclusion.** Surjectivity of "Sylvester maps" implies solvability of all equations.

First factorization theorem for differential operators

- Recall: a system of 1st order ODE's has the form $\big(t^r\epsilon - A(t)\big)x = 0, A(0) \neq 0$, $r \in \mathbb{Z}_+$ the Poincaré rank.
- A single slope operator $L \in \mathscr{W}$ takes the form $L = p(t^r\epsilon) + \cdots$ for some univariate polynomial $p \in \mathbb{C}[u]$, $r \in \mathbb{Q}_+$, of course, involving only the terms $(t^r\epsilon)^k$ with integer exponents for t, ϵ, and dots standing for inferior terms (support strictly inside Δ_L).

First Factorization Theorem for Differential Operators

An operator $L \in \mathscr{W}$ whose Newton polygon consists of pairwise different slopes $r_1, \dots, r_k$, factors as a noncommutative composition $L = L_1 \cdots L_k$ of general single slope operators $L_i = p_i(t^{r_i}\epsilon) + \cdots, \quad p_i(0) \neq 0$.

Informal conclusions:

- For general single slope operator $L \in \mathscr{W}$ the slope r plays role of Poincaré rank.
- Unlike classical Poincaré rank, its analog r is only rational.
- A classical analog of the FFT should be a result on block diagonalization of a system into blocks with different Poincaré ranks. Such results are unknown.

Factorization of single slope pseudopolynomials

Consider $P \in \mathscr{P}$ with a single slope $r > 0$ and the principal part $p(t^r \xi)$, $\quad p \in \mathbb{C}[u]$.

- **Definition.** Let $k \in \mathbb{N}$ be the minimal exponent such that $(t^r \xi)^k$ is a genuine monomial, i.e., $kr \in \mathbb{N}$. The monomial $U = t^{kr} \xi^k \in \mathbb{C}[t, \xi]$ will be called generating monomial.
- By construction, $p(u) = \sigma(U)$. The degree $\deg \sigma \geqslant 1$ is the number of lattice points (minus one) on the principal edge of $\varDelta_P$, and $\sigma(0) \neq 0$.
- If $\deg \sigma = 1$, then $\varDelta_P$ is clearly indecomposable, hence P must be irreducible. If $\deg \sigma = d > 2$, then $\varDelta_P$ can be represented as sum of d identical copies of a single slope diagram $\varDelta_P = \varDelta' + \cdots + \varDelta'$.
- When decomposition of $\varDelta_P$ can be transformed into decomposition of P in $\mathscr{P}$?
- Let $\lambda_1, \dots, \lambda_d \in \mathbb{C}^*$ be the roots of the polynomial σ. L is called non-resonant, if λ_i are pairwise different.

Factorization theorem for single-slope pseudopolynomials

A non-resonant single-slope pseudopolynomial $P \in \mathscr{P}$ can be decomposed as $P = cP_1 \cdots P_d$, where $c \in \mathbb{C}^*$ and $P_i = (t^{kr}\xi^k - \lambda_i + \cdots) \in \mathscr{P}$ are *irreducible* pseudopolynomials. **Proof.** By resolution of singularities on $\{t = 0\}$. □

Second factorization theorem for differential operators

Second Factorization Theorem for Differential Operators.

A non-resonant single-slope operator $L \in \mathscr{W}$ can be decomposed as $L = cL_1 \cdots L_d$, where $c \in \mathbb{C}^*$ and $L_i = (t^{kr}\epsilon^k - \lambda_i + \cdots) \in \mathscr{W}$ are *irreducible* operators.

Proof.

The same as for First Factorization Theorem:

1. Consider an analogous problem for $\mathscr{P}$, write an infinite system of linear algebraic equations, introduce homological operator, "Sylvester maps"....
2. The "commutative" theorem guarantees solvability of all these equations for any collection of right hand sides.
3. In turn, this implies that all relevant "Sylvester maps" are surjective.
4. The formal equations for the noncommutative factorization problem has the same triangular structure.
5. Its solvability follows from the surjectivity of all "Sylvester maps".

This result is a perfect analog in $\mathscr{W}$ of the diagonalization of nonresonant non-Fuchsian singularities of systems of 1st order equations.

Instead of the conclusion

- One can go one step further and consider factorization of single-slope operators from $\mathscr{W}$ into first order terms. This would be analogous to expanding pseudopolynomials into Puiseaux-type series in fractional powers $t^{1/\mu}$ (work in progress with Midory Komatsudani Quispe).
- A **great challenge** is to identify obstructions to analytic factorization for the algebra $\mathscr{O}(t)[\epsilon]$ of non-Fuchsian operators with holomorphic coefficients. For systems of 1st order equations this obstruction is the Stokes cocycle, a collection of automorphisms of the formal normal form.
- Another (completely different but not less aching) **challenge** is to "explain" a surprising similarity between two apparently completely different algebras,
  - $\mathrm{Mat}(n, \mathbb{C}[\![t]\!]) = \mathbb{C}[\![t]\!] \otimes \mathrm{GL}(n, \mathbb{C})$ of formal series in t with matrix coefficients, where t commutes with the coefficients from $\mathrm{GL}(n, \mathbb{C})$ which in turn do not commute between themselves, and
  - $\mathscr{W} = \mathbb{C}[\![t]\!] \otimes \mathbb{C}[\epsilon]$ of formal series in t with coefficients in $\mathbb{C}[\epsilon]$ which commute between themselves, but do not commute with the variable t.

References

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2. L. Mezuman and S. Yakovenko, *Formal factorization of higher order irregular linear differential operators*, Preprint `arXiv:1805.02210 [math.CA]`, May 2018.

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