Germs of analytic families of diffeomorphisms unfolding a parabolic point and similar problems

Christiane Rousseau

Work done with C. Christopher, P. Mardešić, R. Roussarie and L. Teyssier following pioneering work by A. Douady, A. Glutsyuk, P. Lavaurs, R. Oudkerk

Structure of the lecture

- Statement of the problem
- Finite normal form
- Preparation of the family
- Construction of a modulus of analytic classification in codimension 1
- What we learn from the modulus
- Going to higher codimension

Statement of the problem

We consider germs of generic analytic k-parameter families f_ϵ of diffeomorphisms unfolding a parabolic point of codimension k

$$f_0(z) = z + z^{k+1} + o(z^{k+1})$$

When are two such germs conjugate?

Statement of the problem

More generally, whe are two germs of generic analytic k-parameter families of *dynamical systems* with a “finite”normal form analytically “equivalent”?

Dynamical systems could be fixed points of diffeomorphisms, singular points of ordinary differential equations, singular points of systems of linear differential equations, etc.

Analytic “equivalence” could be orbital equivalence, conjugacy, etc.

Conjugacy of two germs of families

Two germs of analytic families of diffeomorphisms f_ϵ and $\tilde{f}_{\tilde{\epsilon}}$ are conjugate it there exists $r, \rho > 0$ and analytic functions

$$h : \mathbb{D}_\rho \to \mathbb{C}, \qquad H : \mathbb{D}_r \times \mathbb{D}_\rho \to \mathbb{C}$$

such that

- h is a diffeomorphism and, for each fixed ϵ, $H_\epsilon = H(\cdot, \epsilon)$ is a diffeomorphism;
- for all $\epsilon \in \mathbb{D}_\rho$ and for all $z \in \mathbb{D}_r$, then

$$\tilde{f}_{h(\epsilon)} = H_\epsilon \circ f_\epsilon \circ (H_\epsilon)^{-1}$$

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The difficulty is the change of parameters... Hence, we *prepare* the families to a *canonical parameter* so that a conjugacy between them preserves the parameter (i.e. h is the identity.)

Preparation of the family

We are considering families with a "finite" normal form.

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The idea is to change coordinates and parameters so that the families have the same equilibrium positions and/or “special objects” (limit cycles, leaves, etc.) as the normal form.

Families unfolding a germ of parabolic diffeomorphism

Let $\tilde{f}_{\tilde{\epsilon}}$ be a "generic" k-parameter unfolding of a germ of diffeormorphism

$$f_0(\tilde{z}) = \tilde{z} + \tilde{z}^{k+1} + O(\tilde{z}^{k+2})$$

The (finite) formal normal form is the time-one map of the vector field

$$\frac{P_\epsilon(z)}{1 + a(\epsilon)z^k} \frac{\partial}{\partial z},$$

where

$$P_\epsilon(z) = z^{k+1} + \epsilon_{k-1}z^{k-1} + \cdots + \epsilon_1 z + \epsilon_0$$

is the universal unfolding of z^{k+1}

The parameter is canonical

When $k = 1$, the formal normal form is the time-one map of the vector field

$$\frac{z^2 - \epsilon}{1 + a(\epsilon)z} \frac{\partial}{\partial z}.$$

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Hence the multipliers at the fixed points are $\lambda_\pm = \exp(\mu_\pm)$, where $\mu_\pm = \pm\frac{2\sqrt{\epsilon}}{1+a(\epsilon)\sqrt{\epsilon}}$ are the eigenvalues at the singular points $\pm\sqrt{\epsilon}$ of the vector field.

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$$\frac{1}{\mu_+} - \frac{1}{\mu_-} = \frac{1}{2\sqrt{\epsilon}}$$

The parameter is an analytic invariant!

The parameter is almost canonical

Theorem [RT] Let $(z, \epsilon) \mapsto (\check{z}, \check{\epsilon})$ map a vector field $\dot{z} = v_\epsilon(z) = \frac{P_\epsilon(z)}{1+a(\epsilon)z^k}$ to $\dot{\check{z}} = \check{v}_{\check{\epsilon}}(\check{z}) = \frac{\check{P}_{\check{\epsilon}}(\check{z})}{1+\check{a}(\check{\epsilon})\check{z}^k}$. Then there exists $\tau = \exp(2\pi i m/k)$ and $t(\epsilon)$ such that the change has the form

$$\begin{cases} \check{z} = \tau \Phi_{v_\epsilon}^{t(\epsilon)}(z), \\ \check{\epsilon}_j = \tau^{1-j} \epsilon_j, \end{cases}$$

where $\Phi_{v_\epsilon}^{t(\epsilon)}$ is the flow of v_ϵ for the time $t(\epsilon)$.

Preparation of the family

Let $\tilde{f}_{\tilde{\epsilon}}$ be an unfolding of a germ of diffeormorphism

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By the Weierstrass preparation theorem

$$f_{\tilde{\epsilon}}(\tilde{z}) - \tilde{z} = \widetilde{P}_{\tilde{\epsilon}}(\tilde{z})\tilde{h}(\tilde{z}, \tilde{\epsilon})$$

with $P_{\tilde{\epsilon}}(\tilde{z}) = \tilde{z}^{k+1} + \eta_k(\tilde{\epsilon})\tilde{z}^k + \cdots + \eta_1(\tilde{\epsilon})\tilde{z} + \eta_0(\tilde{\epsilon})$ and $\tilde{h}(\tilde{z}, \tilde{\epsilon}) = 1 + O(|\tilde{z}, \tilde{\epsilon}|)$

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A translation $\tilde{z} \mapsto \check{z} = \tilde{z} + \frac{1}{k+1}\eta_k(\tilde{\epsilon})$ allows bringing

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The family is generic if the change of parameters $\tilde{\epsilon} \mapsto (\check{\epsilon}_0, \ldots, \check{\epsilon}_{k-1})$ is invertible.

Preparation of the family

In these new $(\check{z}, \check{\epsilon})$, the family becomes

$$f_{\check{\epsilon}}(\check{z}) = \check{z} + P_{\check{\epsilon}}(\check{z}) h(\check{z}, \check{\epsilon})$$

We write

$$h(\check{z}, \check{\epsilon}) = c_0(\check{\epsilon}) + c_1(\check{\epsilon})\check{z} + \cdots + c_k(\check{\epsilon})\check{z}^k + P_{\check{\epsilon}}(\check{z}) g(\check{z}, \check{\epsilon})$$

with $c_0(\check{\epsilon}) = 1 + O(\check{\epsilon})$. Then the multipliers at the fixed points are independent of g:

$$\lambda_j = 1 + P'_{\check{\epsilon}}(\check{z}_j)\left(c_0(\check{\epsilon}) + c_1(\check{\epsilon})\check{z}_j + \cdots + c_k(\check{\epsilon})\check{z}_j^k\right)$$

Preparation of the family

There exists a polynomial $S_{\check{\epsilon}}(\check{z})$ of degree k with $S_{\check{\epsilon}}(0) = 1 + O(\check{\epsilon})$ such that

$$\log \lambda_j = \mu_j = P'_{\check{\epsilon}}(\check{z}_j) S_{\check{\epsilon}}(\check{z}_j)$$

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Kostov's Theorem gives a change $(\check{z}, \check{\epsilon}) \mapsto (z, \epsilon)$ transforming $P_{\check{\epsilon}}(\check{z}) S_{\check{\epsilon}}(\check{z}) \frac{\partial}{\partial \check{z}}$ into $\frac{P_\epsilon(z)}{1+a(\epsilon)z^k} \frac{\partial}{\partial z}$

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We apply the change $(\check{z}, \check{\epsilon}) \mapsto (z, \epsilon)$ to the diffeomorphism. This finishes the preparation. $\square$

The equivalence problem when $k = 1$

A family under the form

$$f_\epsilon(z) = z + (z^2 - \epsilon)h(z, \epsilon)$$

with *canonical parameter* is called *prepared*.

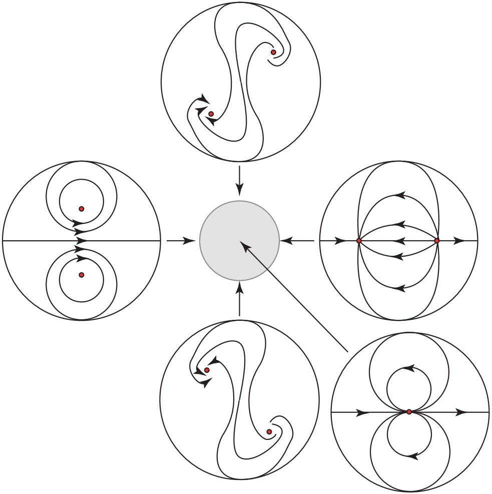
When considering whether two families are conjugate, we can always limit ourselves to prepared families.

We have also identified a normal form, namely the time one map of the vector field $\dot{z} = \frac{z^2-\epsilon}{1+a(\epsilon)z}$

The conjugacy problem for prepared families

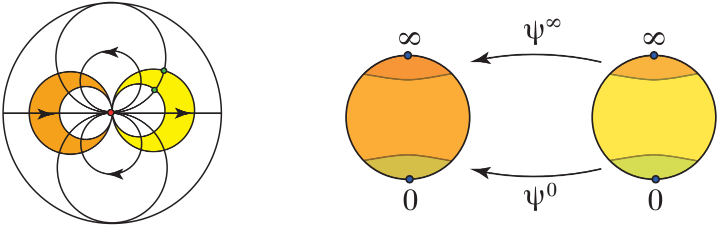
The normal form is unique. Hence if two prepared families have the same normal form, then they would be conjugate if the change of coordinates to the normal form were convergent.

But it is not. However, topologically the family f_ϵ behaves as the time one map of the vector field $\dot{z} = \frac{z^2-\epsilon}{1+a(\epsilon)z}$



The classifying object will be the "space of orbits"

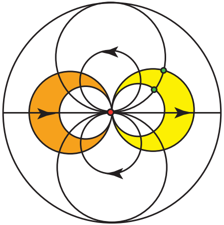
Let us consider the case $\epsilon = 0$

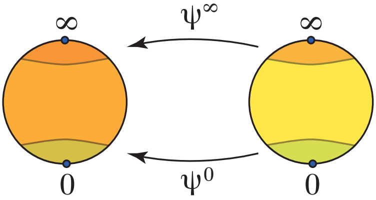


Two fundamental domains are necessary to cover all orbits.

If we identify the two sides of the crescent, the corresponding Riemann surface is conformally equivalent to a sphere minus two points.

The classifying object is called the Ecalle-Voronin modulus

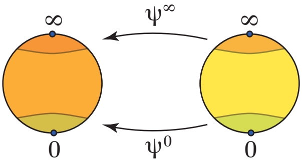




Each sphere ($\mathbb{CP}^1$) has an almost unique coordinate (up to linear map). The *Ecalle-Voronin* modulus is given by the identifying maps (ψ^0, ψ^∞) in the neighborhoods of 0 and ∞.

The maps ψ^0 and ψ^∞ are germs of analytic diffeomorphisms.

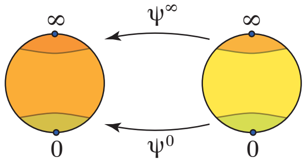
The Ecalle-Voronin modulus



- Two germs of parabolic diffeomorphisms f and $\tilde{f}$ with same formal normal form are conjugate if and only if they have the same Ecalle-Voronin modulus up to linear maps

$$\begin{cases} \psi^0 = L_C \circ \tilde{\psi}^0 \circ L_{C'} \\ \psi^\infty = L_C \circ \tilde{\psi}^\infty \circ L_{C'} \end{cases}$$

The Ecalle-Voronin modulus



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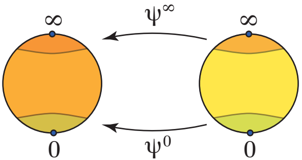
- A germ of parabolic diffeomorphism f is conjugate to its normal form iff ψ^0 and ψ^∞ are both linear.

The Ecalle-Voronin modulus



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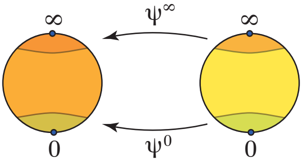
- A germ of parabolic diffeomorphism f is conjugate to its normal form iff ψ^0 and ψ^∞ are both linear.
- Any pair of germs (ψ^0, ψ^∞) in the neighborhoods of 0 and ∞ is realizable as the modulus of a germ of parabolic diffeomorphism.

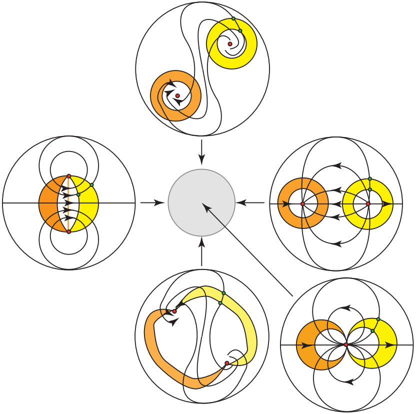
The Ecalle-Voronin modulus



- Any pair of germs (ψ^0, ψ^∞) in the neighborhoods of 0 and ∞ is realizable as the modulus of a germ of parabolic diffeomorphism.

This means that the modulus space is enormous: it is infinite-dimensional. What does this modulus mean?

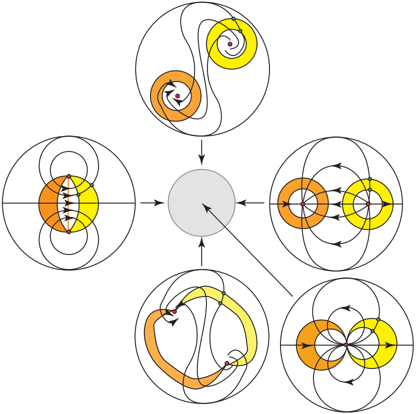
To understand we unfold the modulus



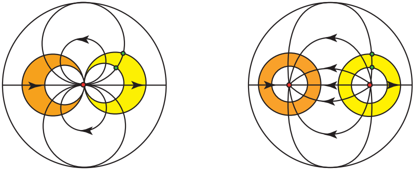
For $\epsilon \neq 0$, there are too natural ways to unfold the crescents:

- as crescents which, once the sides are identified, will have the conformal structure of a sphere;
- as annuli which, once the sides are identified, will have the conformal structure of a torus.

This could suggest two charts in parameter space.



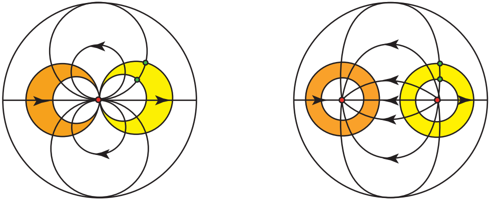
The first chart



For $\epsilon \neq 0$ the diffeomorphism can be conjugated to the model in the neighborhood of each singular point. But generically the two conjucacies are not analytic continuation one of another. If this obstruction persists till the limit $\epsilon = 0$, then the transformation to normal form may be divergent at the limit.

Conversely, if the transformation to normal form is divergent at the limit, then necessarily the two conjucacies are not analytic continuation one of another for small $\epsilon \neq 0$.

The first chart (studied by Glutsyuk)



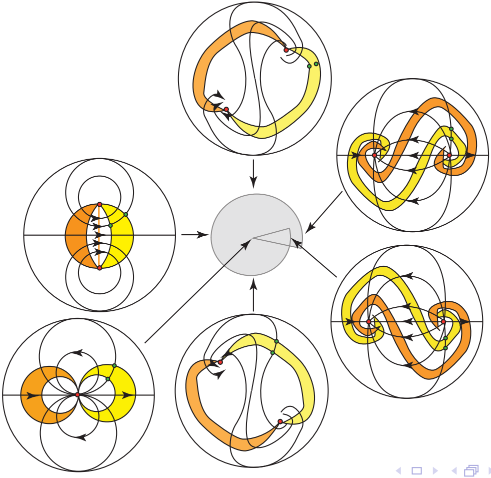
$\varepsilon = 0$ $\varepsilon \neq 0$

Unfolding has allowed us to understand why we have divergence at the limit.

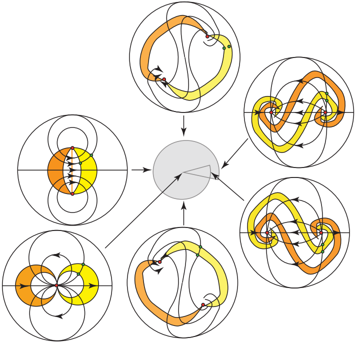
But this point of view does not apply to all values of $\epsilon \neq 0$. Indeed, when $|f_\epsilon'(\pm\sqrt{\epsilon})| = 1$, then the fixed points may not be linearizable... Also, the domains where we can bring to the model may not intersect. Hence the need for a second point of view.

The second chart can be pushed to cover all values of ϵ, but in a ramified way

The idea of unfolding the crescents as crescents goes back to Douady and Lavaurs.



The classifying object for the family of diffeomorphisms



All crescents with sides identified have the conformal structure of spheres. The identifying maps in the neighborhoods of 0 and ∞ form a continuous family $(\psi^0_{\hat{\epsilon}}, \psi^\infty_{\hat{\epsilon}})_{\hat{\epsilon}\in V}$.

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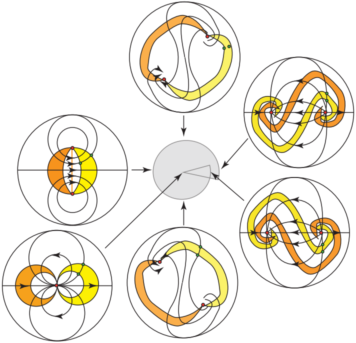


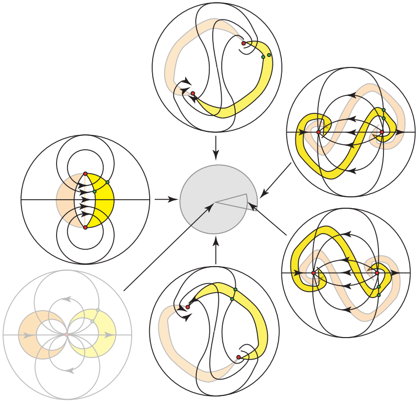
All crescents with sides identified have the conformal structure of spheres. The identifying maps in the neighborhoods of 0 and ∞ form a continuous family $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V}$.

Two families with same formal normal forms are conjugate if and only they have equivalent *modulus* $(\psi^0_{\hat\epsilon}, \psi^\infty_{\hat\epsilon})_{\hat\epsilon\in V}$ (equivalence under linear changes of coordinates).

The lessons

- The dynamics is closely related to that of the vector field $\frac{z^2-\epsilon}{1+a(\epsilon)z}\frac{\partial}{\partial z}$.
- For each $\epsilon \neq 0$, one crescent is enough to describe the dynamics.
- The parametric resurgence phenomenon also occurs.

One crescent is enough to describe the dynamics



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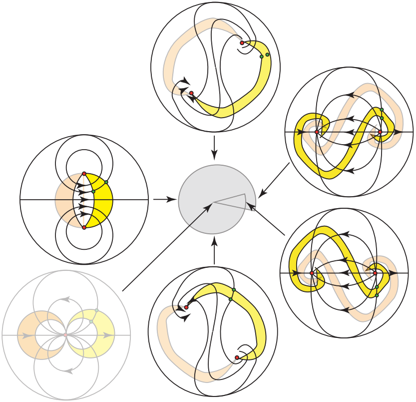
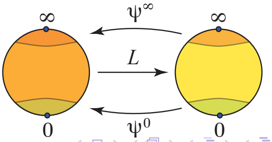
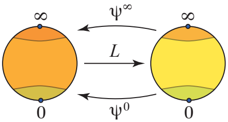


This is because a global diffeomorphism exists between the two crescents, the *Lavaurs map*.



The renormalized return maps



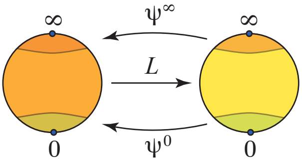
In the spherical coordinates the Lavaurs map is linear. It is possible to study the dynamics of the fixed points by the renormalized return maps

$$\begin{cases} L_{\hat\epsilon}\circ\psi^\infty_{\hat\epsilon}, & \text{near } \sqrt{\hat\epsilon}, \\ L_{\hat\epsilon}\circ\psi^0_{\hat\epsilon}, & \text{near } -\sqrt{\hat\epsilon}. \end{cases}$$

Another lesson

- $\psi^\infty_{\hat\epsilon}$ controls the dynamics near $+\sqrt{\hat\epsilon}$ and $\psi^0_{\hat\epsilon}$ controls the dynamics near $-\sqrt{\hat\epsilon}$.

The decomposition of the dynamics



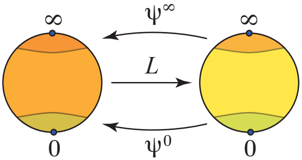
We have decomposed the dynamics into

- a wild linear part, which depends only on $\hat{\epsilon}$ and $a(\epsilon)$ (i.e. the formal part!) and has no limit when $\epsilon \to 0$;
- and a nonlinear part which has a limit when $\epsilon \to 0$.

The Lavaurs map has the form

$$L_{\hat{\epsilon}}(w) = \exp\left(-\frac{2\pi i}{\sqrt{\hat{\epsilon}}}\right) c(\hat{\epsilon}) w$$

The parametric resurgence phenomenon

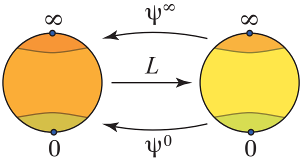


It can occur at any fixed point. Let us consider $-\sqrt{\hat{\epsilon}}$ with renormalized return map

$$\kappa_{\hat{\epsilon}} = L_{\hat{\epsilon}} \circ \psi^0_{\hat{\epsilon}}$$

We can choose coordinates on the spheres to that $\psi'_{\hat{\epsilon}}(0) = 1$. We consider sequences ϵ_n such that $\kappa'_{\hat{\epsilon}}(0) = \exp(2\pi i p/q)$.

The parametric resurgence phenomenon



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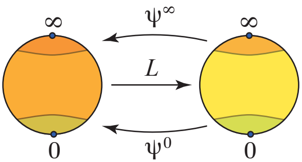

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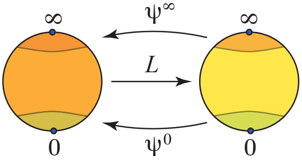
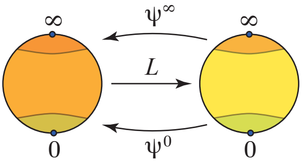
We can choose coordinates on the spheres to that $\psi'_{\hat{\epsilon}}(0) = 1$. We consider sequences ϵ_n such that $\kappa'_{\hat{\epsilon}}(0) = \exp(2\pi ip/q)$. If $\exp(2\pi ip/q)\psi^0_0$ is nonlinearizable (has a nonzero resonant term), then so does κ_{ϵ_n} for n sufficiently large. For ϵ close to ϵ_n orbit(s) of period q appear(s) for $\kappa_{\hat{\epsilon}}$, corresponding to orbits of large period for f_ϵ.

The parametric resurgence phenomenon



On sequences of parameter values, the mismatch is carried by the fixed points themselves, which are forced to be nonlinearizable.

What about a fixed point when the multiplier is an irrational rotation?



Suppose that $\kappa'_\epsilon(0) = \exp(2\pi i\alpha)$ with α irrational. Then ϵ is very close to values ϵ' for which $\kappa'_{\epsilon'}(0) = \exp(2\pi ip/q)$. When perturbing close to ϵ', generically an orbit of large period appears.

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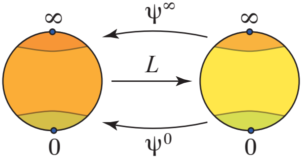
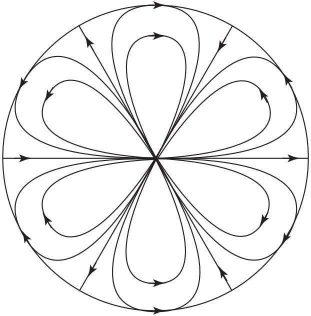


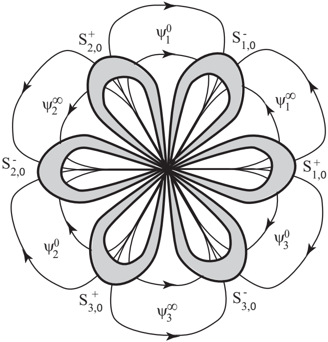
Suppose that $\kappa'_\epsilon(0) = \exp(2\pi i\alpha)$ with α irrational. Then ϵ is very close to values ϵ' for which $\kappa'_{\epsilon'}(0) = \exp(2\pi ip/q)$. When perturbing close to ϵ', generically an orbit of large period appears. If α is Liouvillian (well approximated by the rationals) this accumulation of periodic points may lead to the nonlinearizability of the fixed point at ϵ.

(This obstruction to linearizability for multipliers of the form $\exp(2\pi i\alpha)$ was studied by Ilyashenko-Pjartli and Yoccoz.)

Moving to codimension $k > 1$

The modulus space for $\epsilon = 0$

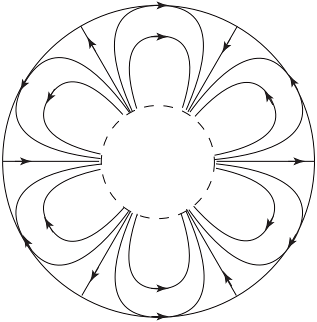




The diffeomorphism now has $2k$ petals near the parabolic point. Hence we need $2k$ fundamental domains and the modulus space has $2k$ components $(\psi^0_1, \psi^\infty_1, \dots, \psi^0_k, \psi^\infty_k)$

We need to unfold that

The behavior stays the same near the boundary.

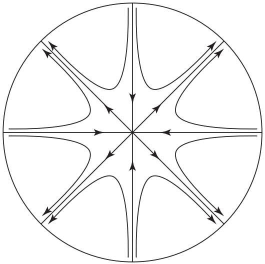


How do we take the fundamental domains inside?

Some lessons for codimension 1

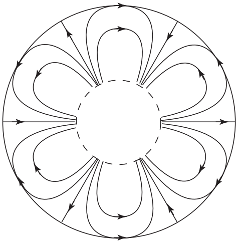
- The geometry of the vector field $\frac{z^2-\epsilon}{1+a(\epsilon)z}\frac{\partial}{\partial z}$, itself very close to the geometry of the vector field of $(z^2-\epsilon)\frac{\partial}{\partial z}$
- We cannot unfold the Ecalle-Voronin modulus in a uniform way on the parameter space
- We identified crescents on which the space of orbits had the conformal structure of $\mathbb{CP}^1 \setminus \{0,\infty\}$

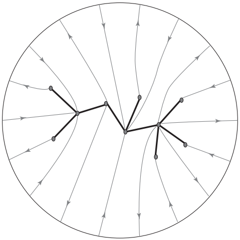
The geometry of the vector field $\frac{P_\epsilon(z)}{1+a(\epsilon)z^k}\frac{\partial}{\partial z}$



- It is very close to that of the polynomial vector field $P_\epsilon(z)\frac{\partial}{\partial z}$, which has been studied by Douady and Sentenac
- The organizing center is a pole at infinity with $2k$ separatrices.

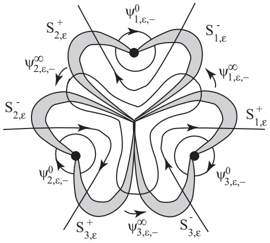
The geometry of the vector field $P_\epsilon(z)\,\frac{\partial}{\partial z}$





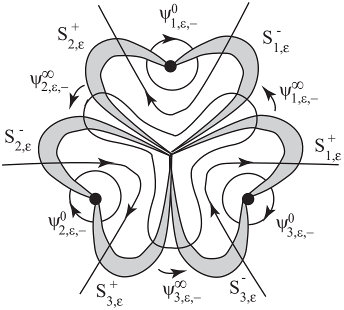
- Generically, on $C(k) = \frac{\binom{2k}{k}}{k+1}$ open sets (DS-domain) in parameter space, the separatrices land at the singular points which are simple.
- In that case the singular points are all linked by trajectories.

Building the orbit space



- The crescents are bounded by trajectories of the rotated vector field $e^{i\theta}P_\epsilon(z)\frac{\partial}{\partial z}$ for some θ such that $|\theta| < \frac{\pi}{2} - \delta$ for some positive δ
- The crescents could be spiraling when approaching the singular points.
- We can enlarge the DS domains so as to cover all parameter values outside the discriminantal set $\Delta = 0$

The modulus of analytic classification on a DS-domain in parameter space



It is given by an unfolding

$$(\psi^0_{1,\epsilon}, \psi^\infty_{1,\epsilon}, \dots, \psi^0_{k,\epsilon}, \psi^\infty_{k,\epsilon})$$

The classification theorem

Two prepared germs of families of diffeomorphisms unfolding a parabolic point of codimension k are conjugate if and only if

- They have the same formal normal form
- On each DS domain they have equivalent

$$(\psi^0_{1,\epsilon}, \psi^\infty_{1,\epsilon}, \dots, \psi^0_{k,\epsilon}, \psi^\infty_{k,\epsilon})_{\epsilon \in S_s}$$

  up to bounded and bounded away from zero linear changes of coordinates on the spheres.

The realization

Any

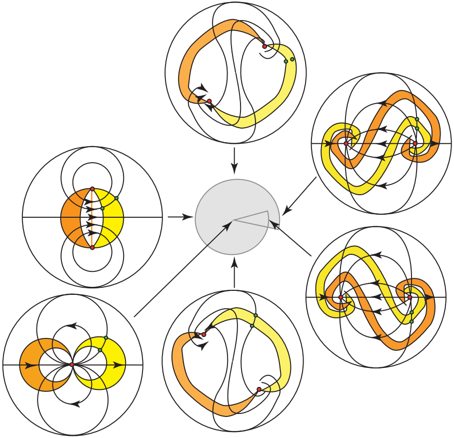
$$(\psi^0_{1,\epsilon,s}, \psi^\infty_{1,\epsilon,s}, \dots, \psi^0_{k,\epsilon,s}, \psi^\infty_{k,\epsilon,s})_{\epsilon \in S_s}$$

on a DS domain S_s, depending analytically on ϵ with continuous limit at the boundary can be realized as a germ of family of diffeomorphisms unfolding a parabolic point of codimension k on S_s.

An additional condition is necessary to glue together the realizations over the different S_s in a uniform family.

This has been done with Christopher when $k = 1$. It is still open when $k > 1$.

The additional necessary condition



The condition expresses that the two realizations over the self-intersection in parameter space are conjugate. .

Thank you!