Singularities of analytic dynamical systems with 1-summable normalizing transformation

Structure of the talk

- The general problem
- Examples
- The example of the saddle-node
- Common features
- Examples revisited with some results
- Going to higher codimension

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When are two germs of analytic dynamical systems equivalent in the neighborhood of a singularity under an analytic change of coordinates?

The use of normal forms

For instance, if there exists an analytic change of coordinates to a linear system.

Two steps

- look for a formal change of coordinates to normal form
- study convergence of normalizing change of coordinates.

Very often, the change of coordinates to normal form diverges.

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- look for a formal change of coordinates to normal form
- study convergence of normalizing change of coordinates.

Very often, the change of coordinates to normal form diverges.

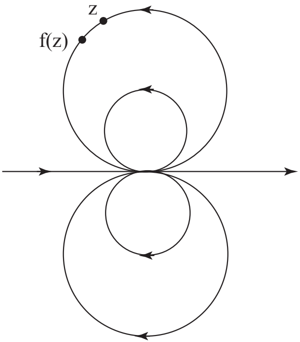
Why?

Examples where the change to normal form diverges

In all these examples the change of coordinate to normal form is 1-summable

- **Example 1.** A germ of analytic diffeomorphism $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that

$$f(z) = z + z^2 + (1-a)z^3 + o(z^3)$$



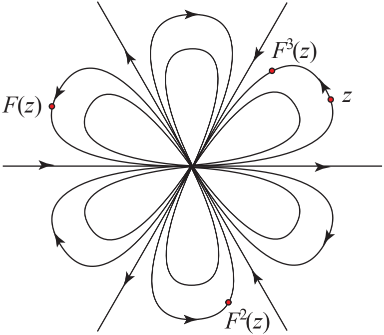
Normal form: the time-one map of the vector field

$$\dot{z} = \frac{z^2}{1+az}$$

Resonant diffeomorphism

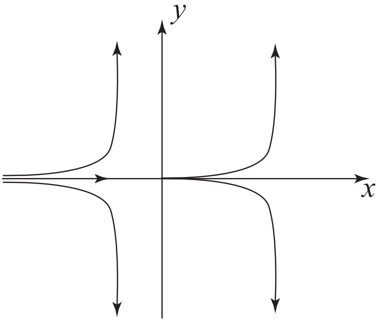
- **Example 2.** A germ of analytic diffeomorphism $f:(\mathbb{C},0)\to(\mathbb{C},0)$ such that

$$f(z)=\exp\left(\frac{2\pi ip}{q}\right)z+z^{q+1}+Az^{2q+1}+o(z^{2q+1})$$



Saddle-node

- **Example 3.** A germ of saddle-node of a planar vector field

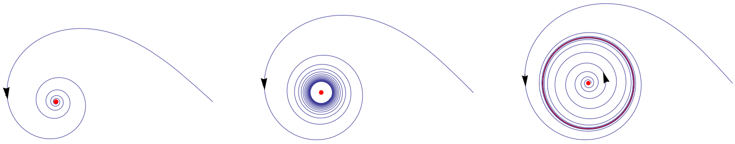


Normal form

$$\begin{cases} \dot{x} = x^2 \\ \dot{y} = y(1 + ax) \end{cases}$$

Weak focus

- **Example 4.** A Hopf bifurcation



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

Orbital normal form

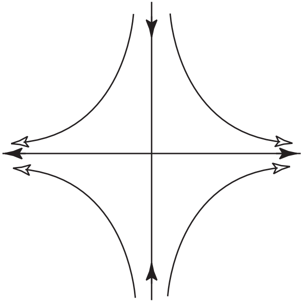
$$\dot{z} = z(i+\epsilon) - z^2\bar{z} + az^3\bar{z}^2$$

which can be rewritten

$$\begin{cases} \dot{r} = \epsilon r - r^3 + ar^5 \\ \dot{\theta} = 1 \end{cases}$$

Resonant saddle

- **Example 5.** A germ of resonant saddle of a planar vector field of order 1 with quotient of eigenvalues $-\frac{p}{q}$

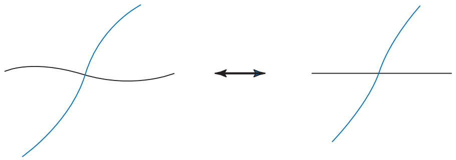


Orbital normal form

$$\begin{cases} \dot{x} = x \\ \dot{y} = y\left(-\frac{p}{q} + x^p y^q + a x^{2p} y^{2q}\right) \end{cases}$$

Curvilinear angle

- **Example 6.** A germ of curvilinear angle



When are two germs of curvilinear angles conformally equivalent?

We will consider the case where the angle is of the form $2\pi\frac{p}{q}$, which we call a *rational angle*.

Linear differential system

- **Example 7.** A nonresonant irregular singular point of Poincaré rank 1 of a linear differential system

$$x^2\frac{dy}{dx} = A(x)y, \qquad y \in \mathbb{C}^n$$

Normal form

$$x^2\frac{dy}{dx} = (D_0 + D_1 x)y$$

with D_0, D_1 diagonal

Formal normal form at a saddle-node

$$\dot{x} = x^2$$
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Formal normal form at a saddle-node

$$\begin{aligned}\dot{x} &= x^2\\ \dot{y} &= y(1+Ax)\end{aligned}$$

- We cannot do better: what is the meaning of A?
- Generically the change to normal form diverges:

Why?

Answers to the divergence

1. We need to extend x, y to be in $\mathbb{C}$.
2. The saddle-node is a multiple singular point. Hence it is natural to unfold. The formal normal form of a generic unfolding

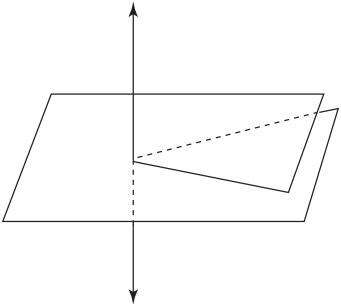
$$\begin{aligned} \dot{x} &= x^2 - \epsilon \\ \dot{y} &= y(1 + A(\epsilon)x) \end{aligned}$$

In the unfolding there are rigid *models* near each of the two singular points.

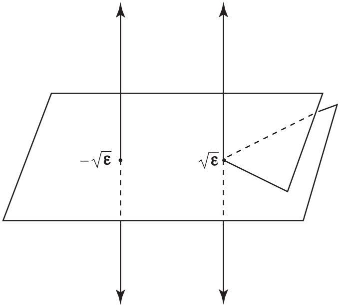
Generically these models mismatch till the merging of the singular points, yielding divergence at the limit.

One example of mismatch

Consider the unfolding of a saddle-node with normal form
$\dot{x} = x^2 - \epsilon, \dot{y} = y(1 + Ax)$



(d) $\epsilon = 0$

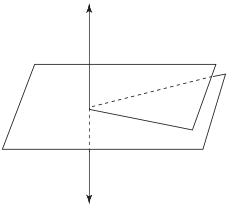


(e) $\epsilon \neq 0$

Generically a saddle-node has no analytic center manifold

The parametric resurgence phenomenon

Conclusion 1: When we unfold a system with no analytic center manifold, then the analytic separatrices of the two singular points do not match.



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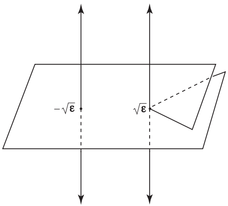
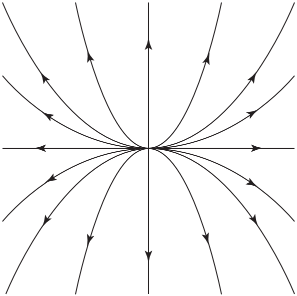


(g) $\epsilon \neq 0$

Conclusion 2: When we unfold a system with no analytic center manifold then the node is non linearizable as soon as resonant.

The node is a very simple point!



We have convergence to the orbital normal form

$$\begin{cases} \dot{x} = \lambda x \\ \dot{y} = y \end{cases}$$

as soon as $\lambda \notin 1/\mathbb{N}$.

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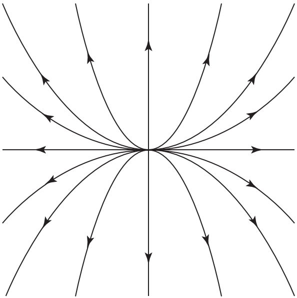

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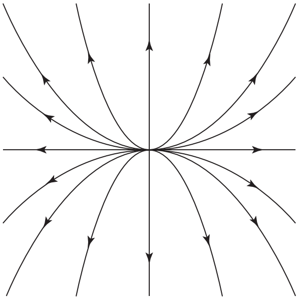

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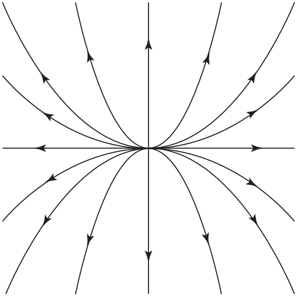
The generic situation



The leaves are given by $y = Cx^{1/\lambda}$. Only $y = 0$ is non ramified.

Hence the generic situation is that the stable manifold of the saddle does not coincide with the non ramified leaf.

Explanation of Conclusion 2

If the node is resonant then the local model at the node is the convergent normal form

$$\begin{aligned} \dot{x} &= \frac{x}{n} \\ \dot{y} &= y + Ax^n. \end{aligned}$$

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This case is obviously impossible when unfolding a system with ramification for $\epsilon = 0$ and we are forced to have $A \neq 0$, yielding that all solutions (except $x = 0$) are of the form

$$y = nAx^n \ln x + Cx^n$$

The center-manifold of a saddle-node

We have understood why divergence is the norm and convergence is the exception.

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This is just one example of the many mismatches that occur within analytic dynamical systems.

And what about the formal invariant?

$$\dot{x} = x^2 - \epsilon, \dot{y} = y(1 + Ax)$$

The ratios of eigenvalues at the singular points are

$$\mu_\pm = \pm \frac{2\sqrt{\epsilon}}{1 \pm A\sqrt{\epsilon}}$$

Then

$$\frac{1}{\mu_+} + \frac{1}{\mu_-} = A$$

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This will be a general rule.

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- To understand why we have divergence, we unfold Two objects: codimension 1 will lead to 1-summability in the limit.
- In the unfolding, generically the divergence can be seen as the limit of the gluing of the two local models which are rigid. Hence, divergence is the rule and convergence is exceptional.

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- In all cases we have a finite parameter family representing a formal normal form: “the model family”.
- The extra formal parameter(s) are present to match the need of independent multipliers or eigenvalues in the unfolding.
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- In all cases we observe a "parametric resurgence phenomenon", i.e. the unfolded singular points have pathologies on discrete sequences of parameter values $\{\epsilon_n\}$ converging to $\epsilon = 0$.

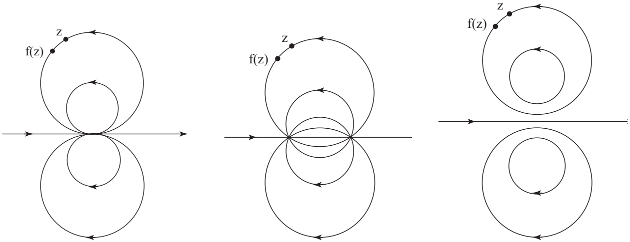
The parabolic point: coallescence of two fixed points

- **Example 1.** Unfolding

$$f_\epsilon(z) = z + (z^2 - \epsilon)(1 + O(z, \epsilon))$$

The model family is the time-one map of

$$\dot{z} = \frac{z^2 - \epsilon}{1 + a(\epsilon)z}$$



Coallescence of a fixed point and a periodic orbit of period q

- **Example 2.** An unfolding of

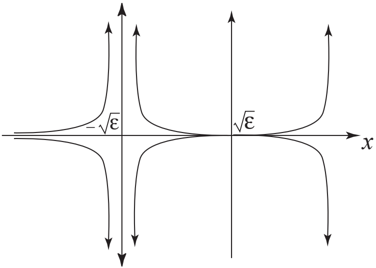
$$f(z) = \exp\left(\frac{2\pi i p}{q}\right) z + \frac{1}{q} z^{q+1} + a z^{2q+1} + o(z^{2q+1})$$

can be taken so that

$$f_\epsilon^{\circ q}(z) = z + z(z^q - \epsilon)(1 + O(z, \epsilon))$$

Coallescence of a saddle and a node

- **Example 3.**



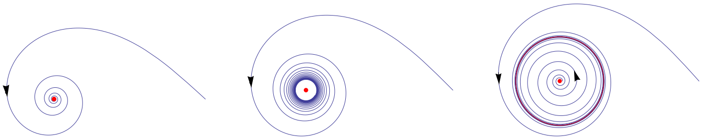
Orbital normal form for the unfolding

$$\begin{cases} \dot{x} = x^2 - \epsilon \\ \dot{y} = y(1 + ax) \end{cases}$$

Coallescence of a focus and a limit cycle

- **Example 4.** A Hopf bifurcation with orbital normal form

$$\dot{z} = z(i + (z\bar{z} - \epsilon)(1 + az\bar{z}))$$



(a) $\epsilon < 0$ (b) $\epsilon = 0$ (c) $\epsilon > 0$

A complex weak focus is orbitally the same as a saddle with ratio of eigenvalues equal to -1

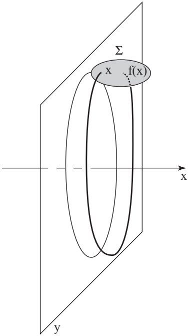
- **Example 4.** Taking $w = \overline{z}$, the system can be rewritten

$$\dot{z} = z(i + (zw - \epsilon)(1 + azw))$$
$$\dot{w} = w(-i + (zw - \epsilon)(1 + azw))$$

  which is orbitally the same as a complex saddle. The complex curve $zw = \epsilon$ is a special *leaf*, which has non trivial homology.

Coallescence of the invariants manifolds of a saddle point with a distinguished invariant manifold

- **Example 5.**



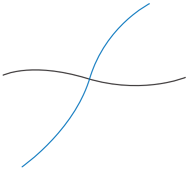
Orbital normal form

$$\begin{cases} \dot{x} = x \\ \dot{y} = y\left(-\frac{p}{q}(1+\epsilon) + x^p y^q + a x^{2p} y^{2q}\right) \end{cases}$$

Two families are orbitally equivalent if and only if the holonomies of their y-separatrices are conjugate.

Curvilinear angles

- **Example 6.** For a curvilinear angle we have a Schwarz symmetry $z \mapsto \Sigma_j(z)$ associated to each curve.



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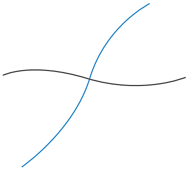

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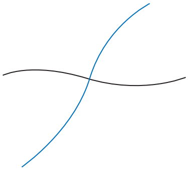


Let

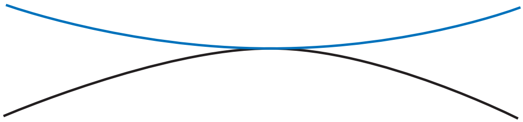
$$f = \Sigma_2 \circ \Sigma_1$$

f is a germ of analytic diffeomorphism:

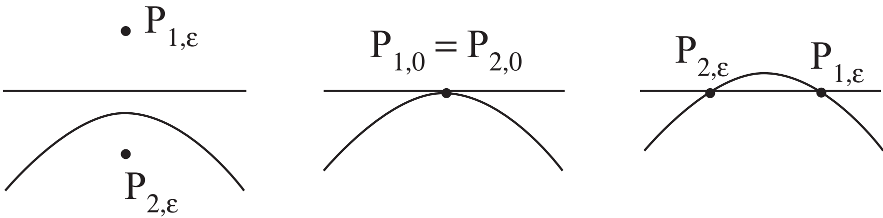
$$f(z) = e^{4\pi i \frac{p}{q}} z + o(z), \qquad \Sigma_1 \circ f = f^{-1} \circ \Sigma_1$$

Case of the horn

It is a special case of Example 1 and the coallescence of two intersection points of the analytic arcs



Unfolding the horn



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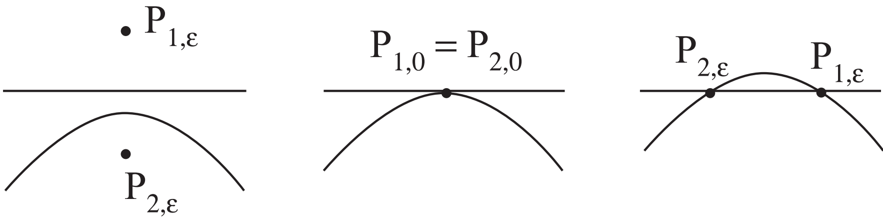

Formal invariant a: a limit of a measure of a shift between the two angles

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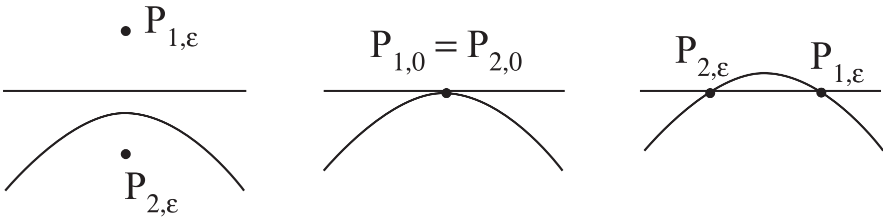

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The confluence of two regular singular points

- **Example 7.**

Normal form

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In the unfolding we have, generically, at each regular singular point a basis of solutions that are eigenvectors of the monodromy around the regular singular point.

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For the special resonant values of the parameter at one singular point, it forces the existence of solutions with logarithmic terms.

Going to higher codimension

When we have the confluence of $k+1$ *special objects* we often observe k-summability of the normalizing changes of coordinates.

Example 7 bis

A nonresonant irregular singular point of Poincaré rank k of a linear differential system

$$x^{k+1}\frac{dy}{dx} = A(x)y, \qquad y \in \mathbb{C}^n$$

with formal normal form of the unfolding

$$P_\epsilon(x)\frac{dy}{dx} = (D_0 + D_1 x + \ldots D_k x^k)y$$

where

$$P_\epsilon(x) = x^{k+1} + \epsilon_{k-1}x^{k-1} + \cdots + \epsilon_1 x + \epsilon_0$$

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$(k+1)n$ parameters to control the eigenvalues at $k+1$ singular points

The mismatch in this example

$$P_\epsilon(x)\frac{dy}{dx} = (D_0 + D_1 x + \dots D_k x^k)y$$

At each regular singular point there is a basis of eigensolutions for the monodromy operator around the singular point associated with the eigenvalues of D_0

The mismatch here: the eigensolutions at the different singular points are not analytic solutions one of the other.

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At each regular singular point there is a basis of eigensolutions for the monodromy operator around the singular point associated with the eigenvalues of D_0

The mismatch here: the eigensolutions at the different singular points are not analytic solutions one of the other.

There is also a *parametric resurgence phenomenon* when the monodromy operator ceases to be diagonalizable on convergent sequences of parameter values.

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