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The slow divergence integral on a Möbius band

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- We prove the finite cyclicity property of "singular" 1– and 2–homoclinic loops.
- Using an idea of Khovanskii we find optimal upper bounds for the number of limit cycles Hausdorff close to canard cycles.

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- Let's consider a simple planar slow-fast system $X_{\epsilon,b}$ (depending possibly on an extra finite dimensional parameter):

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- All singularities of the critical curve are normally hyperbolic, except the origin where we deal with a generic nilpotent *contact point*.

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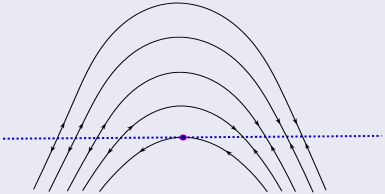

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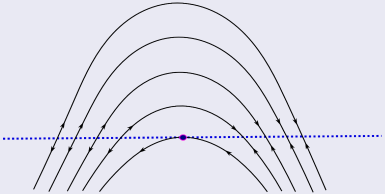

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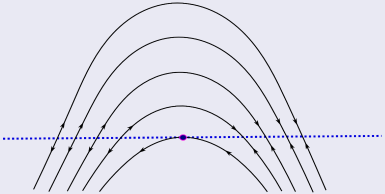

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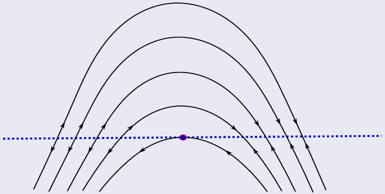

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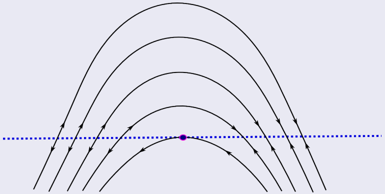

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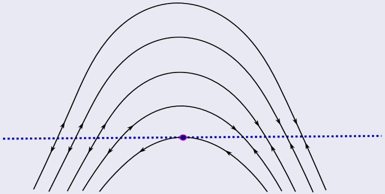

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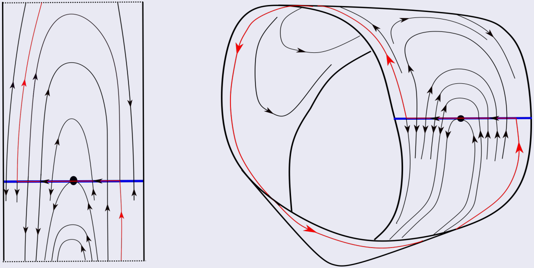
- Our model can provide much richer dynamics if we consider it on the Möbius band



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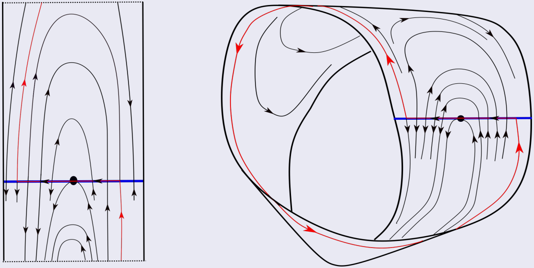
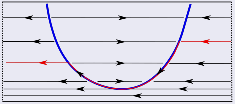
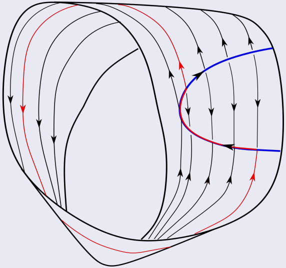


- Besides the contact point and the canard cycles we also detect so-called 1– and 2–canard cycles consisting of a fast orbit, turning around the Möbius band, and the part of the critical curve between the α-limit set and the ω-limit set of the fast orbit.

Motivation

- Our model in the Liénard plane:





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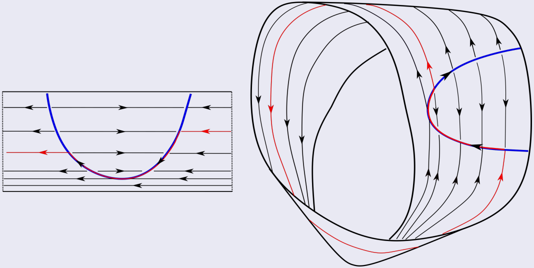

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Definitions on the smooth Möbius band

- Denote by M a smooth Möbius band ("smooth" means C^∞-smooth). Let $(\epsilon, \mu) \sim (0,0) \in \mathbb{R} \times \mathbb{R}^l$, with $\epsilon \geq 0$, and let $X_{\epsilon,\mu} : M \to TM$ be a smooth (ϵ, μ)-family of vector fields on M.

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- We suppose $X_{\epsilon,\mu}$ has a slow-fast structure, with a singular perturbation parameter ϵ and with a *generic turning point* (or equivalently, a slow-fast Hopf point) $p \in M$ for $(\epsilon, \mu) = (0,0)$.

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- We suppose $X_{\epsilon,\mu}$ has a slow-fast structure, with a singular perturbation parameter ϵ and with a *generic turning point* (or equivalently, a slow-fast Hopf point) $p\in M$ for $(\epsilon,\mu)=(0,0)$.
- More precisely, we suppose that there exists a local chart on M around p in which the vector field $X_{\epsilon,\mu}$ is locally expressed, up to smooth equivalence, as:

$$\left\{\begin{array}{rcl}\dot{x} & = & y\\ \dot{y} & = & -xy+\epsilon\big(b(\mu)-x+x^2g(x,\epsilon,\mu)\big)+\epsilon y^2H(x,y,\epsilon,\mu).\end{array}\right. \tag{2}$$

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- We assume that the family m_μ of slow curves is located in an open orientable submanifold $\widetilde{M}$ of M.
- Working with such an orientable submanifold, we can choose a volume form and define the divergence of (the restriction of) the vector field $X_{\epsilon,\mu}$.

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- We have $f < 0$. Now we can define the slow divergence integral $I_\pm(u,\mu)$ along $m^\pm$:

$$I_+(u,\mu) := \int_{\alpha(u)}^{0} \frac{\operatorname{div} X_{0,\mu} dx}{f(x,\mu)} < 0, \ \ I_-(u,\mu) := \int_{\omega(u)}^{0} \frac{\operatorname{div} X_{0,\mu} dx}{f(x,\mu)} < 0, \tag{3}$$

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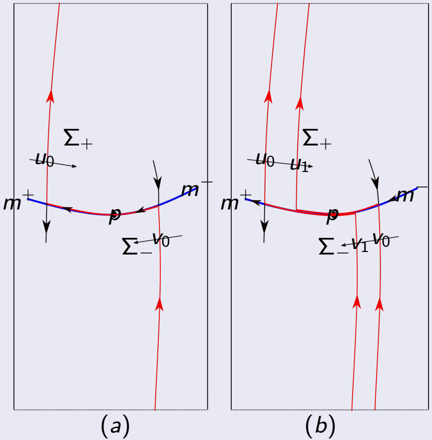

Figure: Canard cycles on the Möbius band M turning around M, at level $(\epsilon, \mu) = (0, 0)$. (a) 1–canard cycles intersect Σ_+ only once. (b) 2–canard cycles intersect Σ_+ twice.

Definitions on the smooth Möbius band

Definition (1 and 2–periodic orbits)

Let L_{u_0} and L_{u_0,u_1} be 1– and 2–canard cycles.

(a) Let $V \subset M$ be a small tubular neighborhood of L_{u_0}. Let $\mathcal{O} \subset V$ be a periodic orbit of $X_{\epsilon,\mu}$, with $\epsilon > 0$. We call $\mathcal{O}$ a 1–periodic orbit if $\mathcal{O}$ intersects the section Σ_+ only once. Isolated 1–periodic orbits are called 1–limit cycles.

(b) Let $V \subset M$ be a small tubular neighborhood of L_{u_0} or L_{u_0,u_1}. Let $\mathcal{O} \subset V$ be a periodic orbit of $X_{\epsilon,\mu}$, with $\epsilon > 0$. We call $\mathcal{O}$ a 2–periodic orbit if $\mathcal{O}$ intersects the section Σ_+ twice. Isolated 2–periodic orbits are called 2–limit cycles.

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Definitions on the smooth Möbius band

Definition (Cyclicity of L_{u_0} and L_{u_0,u_1})

Let $X_{\epsilon,\mu}$ be a smooth (ϵ,μ)-family of vector fields on M, defined above, and let L_{u_0} and L_{u_0,u_1} be the limit periodic sets. The cyclicity of L_{u_0} (resp. L_{u_0,u_1}) in the family $X_{\epsilon,\mu}$ is bounded from above by $N \in \mathbb{N}$ if there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood W of 0 in the μ-space such that $X_{\epsilon,\mu}$, with $(\epsilon,\mu) \in [0,\epsilon_0] \times W$, generates at most N limit cycles, lying each within Hausdorff distance δ_0 of L_{u_0} (resp. L_{u_0,u_1}). We call the smallest N with this property the cyclicity of L_{u_0} (resp. L_{u_0,u_1}) in the family $X_{\epsilon,\mu}$, and denote it by $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0})$ (resp. $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1})$).

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Limit cycle bifurcations Hausdorff-close to L_{u_0}

- For $(u, \mu) \sim (u_0, 0)$, the slow divergence integral along the slow curve from $\omega(u) \in m^-$ to $\alpha(u) \in m^+$ is given by:

$$I(u, \mu) = I_-(u, \mu) - I_+(u, \mu) \tag{4}$$

Theorem

Suppose that $I(u, \mu)$ is nonzero near $(u, \mu) = (u_0, 0)$. Then $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0}) = 1$ and $X_{\epsilon,\mu}$ has no 2–periodic orbits Hausdorff-close to L_{u_0}. In case $I(u_0, 0) < 0$ (resp. $I(u_0, 0) > 0$) any 1–limit cycle bifurcating from L_{u_0} is hyperbolically attracting (resp. hyperbolically repelling).

Limit cycle bifurcations Hausdorff-close to L_{u_0}

- If the function $u \to I(u,0)$ has a simple zero at $u = u_0$, then for $\lambda \sim 0$, $\epsilon \sim 0$ and $\epsilon > 0$ the b-family $X_{\epsilon,\mu} = X_{\epsilon,b,\lambda}$ undergoes, Hausdorff-close to L_{u_0}, a *period doubling bifurcation*, giving rise to a 2–limit cycle. In this case we do not need the parameter λ.

Theorem

Let us suppose that the function $u \to I(u,0)$ has a simple zero at $u = u_0$ (i.e. $I(u_0,0) = 0$ and $\frac{\partial I}{\partial u}(u_0,0) \neq 0$). Then there are continuous functions $u(\epsilon,\lambda)$ and $b(\epsilon,\lambda)$ defined for $\epsilon \geq 0$, $\epsilon \sim 0$ and $\lambda \sim 0$, smooth for $\epsilon > 0$, with $u(0,0) = u_0$ and $b(0,\lambda) = 0$, such that for each $\epsilon > 0$, $\epsilon \sim 0$ and $\lambda \sim 0$ the b-family $X_{\epsilon,b,\lambda}$ undergoes a period doubling bifurcation at $(u(\epsilon,\lambda), b(\epsilon,\lambda))$.

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Limit cycle bifurcations Hausdorff-close to L_{u_0}

- To prove that, under the same condition on I, $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0}) \leq 2$, we use a method of Khovanskii (Mamouhdi, Roussarie).

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- The case of higher multiplicity zeros in the slow divergence integral is a topic of further study.

The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to L_{u_0}

- We call the 1–canard cycle L_{u_0} a singular 1–homoclinic loop if the slow dynamics has a hyperbolic saddle at precisely one corner point: "$f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$" or "$f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$". We prove that such a limit periodic set can produce at most one limit cycle.

Theorem

Let us suppose that $f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)[$. Then $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0}) = 1$ and $X_{\epsilon,\mu}$ has no 2–periodic orbits Hausdorff-close to L_{u_0}. When a 1–limit cycle exists, it is hyperbolic and attracting.
A similar result is true in the case $f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in]\alpha(u_0), \omega(u_0)]$. A 1–limit cycle bifurcating from L_{u_0} is hyperbolic and repelling.

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Limit cycle bifurcations Hausdorff-close to L_{u_0}

- "Regular" 1–homoclinic loops of finite codimension have been studied by Guimond, 1999.

Limit cycle bifurcations Hausdorff-close to L_{u_0,u_1}

- Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed. For $(u, \tilde{u}, \mu) \sim (u_0, u_1, 0)$, we define the so-called *total slow divergence integral* of L_{u_0,u_1}:

$$T(u, \tilde{u}, \mu) = I_-(u, \mu) - I_+(\tilde{u}, \mu) + I_-(\tilde{u}, \mu) - I_+(u, \mu). \quad (5)$$

Theorem

Suppose that T is nonzero near $(u, \tilde{u}, \mu) = (u_0, u_1, 0)$. Then $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. *In case $T(u_0, u_1, 0) < 0$ (resp. $T(u_0, u_1, 0)) > 0$) any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolically attracting (resp. hyperbolically repelling).*

Limit cycle bifurcations Hausdorff-close to L_{u_0,u_1}

1. If $I_-(u_1,0) - I_+(u_1,0) \neq 0$, then there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood W of 0 in the μ-space such that system $X_{\epsilon,\mu}$, with $(\epsilon,\mu) \in [0,\epsilon_0] \times W$, has no limit cycles lying within Hausdorff distance δ_0 of L_{u_0,u_1}.

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2. If $I_-(u_1,0) - I_+(u_1,0) = 0$ and $I_-(u_0,0) - I_+(u_0,0) \neq 0$ (this implies $T(u_0,u_1,0) \neq 0$), then we have that $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. In case $I_-(u_0,0) - I_+(u_0,0) < 0$ (resp. $I_-(u_0,0) - I_+(u_0,0) > 0$) any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and attracting (resp. repelling). Moreover, if $\frac{\partial(I_- - I_+)}{\partial u}(u_1,0) \neq 0$, then $\mathrm{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) = 1$.

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3. If $I_-(u_i,0) - I_+(u_i,0) = 0$ for $i = 0,1$ (this implies $T(u_0,u_1,0) = 0$) and $\frac{\partial(I_- - I_+)}{\partial u}(u_i,0) \neq 0$ for $i = 0,1$, then $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$.

Limit cycle bifurcations Hausdorff-close to L_{u_0,u_1}

- We allow the slow dynamics to have a hyperbolic saddle at precisely one corner point, $\omega(u_0)$ or $\alpha(u_0)$. In this case we call L_{u_0,u_1} a *singular 2–homoclinic loop*.

Theorem

Llet us suppose that $f(\omega(u_0),0)=0$, $\frac{\partial f}{\partial x}(\omega(u_0),0)\neq 0$ and that $f(x,0)<0$ for all $x\in[\alpha(u_0),\omega(u_0)[$. Then $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0,u_1})\leq 1$. Any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and attracting.
A similar result is true in the case $f(\alpha(u_0),0)=0$, $\frac{\partial f}{\partial x}(\alpha(u_0),0)\neq 0$ and $f(x,0)<0$ for all $x\in]\alpha(u_0),\omega(u_0)]$. Any 2–limit cycle bifurcating from L_{u_0,u_1} is hyperbolic and repelling.

Transition maps

- We define now the following transition maps for $(\bar{\epsilon}, B, \lambda) \sim (0,0,0)$:
  1. the forward transition map $\Delta_- : \Sigma_+ \to \Sigma_p$ along the flow of $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$;
  2. the backward transition map $\Delta_+ : \Sigma_+ \to \Sigma_p$ along the flow of $-X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$.

  The map $\Delta_\pm$ includes a passage near $m^\pm$.

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  The map $\Delta_\pm$ includes a passage near $m^\pm$.
- The system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 1–periodic orbit passing through the point $u \in \Sigma_+$ if and only if the following holds: $\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon})$.

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- The system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 1–periodic orbit passing through the point $u \in \Sigma_+$ if and only if the following holds: $\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon})$.
- Similarly, the system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 2–periodic orbit passing through the points $u, u' \in \Sigma_+$, with $u \neq u'$, if and only if the following holds: $\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u', B, \lambda, \bar{\epsilon})$ and $\Delta_-(u', B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon})$.

Transition maps

- For a regular slow dynamics, the study of the transition maps relies on [Dumortier, Roussarie,1996]. The following theorem gives the structure of $\Delta_\pm$.

Theorem

There exist $\bar{\epsilon}$-regularly smooth functions $\bar{I}_\pm$ in (u, B, λ) and $\bar{\epsilon}$-regularly smooth functions $f_\pm$ in (B, λ) such that $\bar{I}_\pm(u, B, \lambda, 0) = I_\pm(u, 0, \lambda)$, with $I_\pm$ defined in (3), and such that

$$\Delta_\pm(u, B, \lambda, \bar{\epsilon}) = f_\pm(B, \lambda, \bar{\epsilon}) \pm \exp\left(\frac{\bar{I}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right). \tag{6}$$

Furthermore, $f(0, \lambda, 0) = 0$ and $\frac{\partial f}{\partial B}(0, \lambda, 0) \neq 0$ where $f(B, \lambda, \bar{\epsilon}) := f_-(B, \lambda, \bar{\epsilon}) - f_+(B, \lambda, \bar{\epsilon})$.

Transition maps

- The following theorem gives the structure of the transition map Δ_- ([De Maesschalck,Dumortier, 2008, Huzak, De Maesschalck,Dumortier,2013]).

Theorem

For all $k > 0$ there exists $\bar{\epsilon}_k > 0$ so that Δ_- is C^∞ on $U_- \cap \{\bar{\epsilon} \leq \bar{\epsilon}_k\}$ and has a C^k-extension to the closure of $U_- \cap \{\bar{\epsilon} \leq \bar{\epsilon}_k\}$. Furthermore,

$$\frac{\partial \Delta_-}{\partial u}(u, B, \lambda, \bar{\epsilon}) = -\exp\left(\frac{\mathcal{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right), \tag{7}$$

where $(u, B, \lambda, \bar{\epsilon}) \in U_- \cap \{\bar{\epsilon} \leq \bar{\epsilon}_k\}$, $\mathcal{I}_-$ is $\bar{\epsilon}$-regularly C^k in (u, B, λ), $\mathcal{I}_-(u, B, \lambda, \bar{\epsilon}) \to -\infty$ as $(u, B, \lambda, \bar{\epsilon}) \to (u_0, 0, 0, 0)$ and $\frac{\partial \mathcal{I}_-}{\partial u}(u, B, \lambda, \bar{\epsilon}) > 0$.

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Transition maps

- Using Theorem 9, the equation for 1-limit cycles can be written as:

$$\exp\left(\frac{\bar{I}_-(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)+\exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)=f(B,\lambda,\bar{\epsilon}), \quad (8)$$

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- and the system for 2-limit cycles can be written as:

$$\begin{cases} \exp\left(\frac{\bar{I}_-(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)+\exp\left(\frac{\bar{I}_+(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)=f(B,\lambda,\bar{\epsilon}) \\ \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)+\exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)=f(B,\lambda,\bar{\epsilon}). \end{cases} \quad (9)$$

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Transition maps

- Instead of working with (9), it is sometimes more convenient to use the equation for the fixed points $\{P_{B,\lambda,\bar{\epsilon}} \circ P_{B,\lambda,\bar{\epsilon}}(u) = u\}$, where $P_{B,\lambda,\bar{\epsilon}}(u) = \Delta_+^{-1} \circ \Delta_-(u)$ is the 1–return map, or to use the difference equation $\{\Delta_{B,\lambda,\bar{\epsilon}}(u) = 0\}$ where $\Delta_{B,\lambda,\bar{\epsilon}}(u) = P_{B,\lambda,\bar{\epsilon}}(u) - P^{-1}_{B,\lambda,\bar{\epsilon}}(u)$.

Proof of Theorem 3

- Let I be nonzero near $(u, \mu) = (u_0, 0, 0)$ (i.e. $I_-(u_0, 0, 0) \neq I_+(u_0, 0, 0)$).

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Proof of Theorem 3

- Let I be nonzero near $(u, \mu) = (u_0, 0, 0)$ (i.e. $I_-(u_0, 0, 0) \neq I_+(u_0, 0, 0)$).Let us suppose that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 2–periodic orbit intersecting Σ_+ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_0$, with $\bar{u} < \tilde{u}$.

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- Then $\Delta_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \Delta_{B,\lambda,\bar{\epsilon}}(\tilde{u}) = 0$, $P_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B,\lambda,\bar{\epsilon}}(\tilde{u}) = \bar{u}$ and $P_{B,\lambda,\bar{\epsilon}}([\bar{u}, \tilde{u}]) = [\bar{u}, \tilde{u}]$.

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- Then $\Delta_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \Delta_{B,\lambda,\bar{\epsilon}}(\tilde{u}) = 0$, $P_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B,\lambda,\bar{\epsilon}}(\tilde{u}) = \bar{u}$ and $P_{B,\lambda,\bar{\epsilon}}([\bar{u}, \tilde{u}]) = [\bar{u}, \tilde{u}]$.
- The derivative of $\Delta_{B,\lambda,\bar{\epsilon}}$ can be written as:

$$\Delta'_{B,\lambda,\bar{\epsilon}}(u) = -\exp\left(\frac{I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right) + \exp\left(\frac{I_+(u) - I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right), \quad u \in [\bar{u}, \tilde{u}].$$

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Proof of Theorem 3

- This implies that the equation $\{\Delta'_{B,\lambda,\bar{\epsilon}} = 0\}$ is equivalent, for $\bar{\epsilon} > 0$ and $u \in [\bar{u}, \tilde{u}]$, to the following equation:

$$I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) - I_+(u) + o(1) = 0, \quad (10)$$

for a new $o(1)$-term.

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for a new $o(1)$-term.

- Since $I_\pm$ are smooth and $u, P_{B,\lambda,\bar{\epsilon}}(u), P^{-1}_{B,\lambda,\bar{\epsilon}}(u) \approx u_0$ for all $u \in [\bar{u}, \tilde{u}]$, we have:

$$\begin{aligned} I_-(u) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + I_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u)) - I_+(u) \\ \approx I_-(u_0) - I_+(u_0) + I_-(u_0) - I_+(u_0) \\ = 2(I_-(u_0) - I_+(u_0)) \neq 0, \end{aligned}$$

for $u \in [\bar{u}, \tilde{u}]$.

Proof of Theorem 4

- Let $I(u_0,0,0)=0$ and $\frac{\partial I}{\partial u}(u_0,0,0)\neq 0$. The 1–return map $P_{B,\lambda,\bar{\epsilon}}$ fulfils the conditions of the following theorem:

Theorem (period doubling bifurcation)

Let $p_B:\mathbb{R}\to\mathbb{R}$ be a smooth one-parameter family of mappings such that p_{B_0} has a fixed point x_0 with eigenvalue -1. Assume

(PD1) $\frac{\partial p}{\partial B}\frac{\partial^2 p}{\partial x^2}+2\frac{\partial^2 p}{\partial x\partial B}\neq 0$ *at* $(x,B)=(x_0,B_0)$;

(PD2) $a:=\frac{1}{2}\left(\frac{\partial^2 p}{\partial x^2}\right)^2+\frac{1}{3}\frac{\partial^3 p}{\partial x^3}\neq 0$ *at* $(x,B)=(x_0,B_0)$.

Then there is a smooth curve of fixed points of p_B passing through (x_0,B_0), the stability of which changes at (x_0,B_0). There is also a smooth curve γ passing through (x_0,B_0) so that $\gamma\setminus\{(x_0,B_0)\}$ is a union of hyperbolic period 2 orbits. The curve γ has a quadratic tangency with the line $B=B_0$ at (x_0,B_0). If a is positive (resp. negative), the period 2 orbits are attracting (resp. repelling).

Proof of Theorem 4

- The derivative of $P_{B,\lambda,\bar{\epsilon}}$ w.r.t. u is given by

$$\frac{\partial P_{B,\lambda,\bar{\epsilon}}}{\partial u}(u) = \frac{\frac{\partial \Delta_-}{\partial u}(u, B, \lambda, \bar{\epsilon})}{\frac{\partial \Delta_+}{\partial u}(P_{B,\lambda,\bar{\epsilon}}(u), B, \lambda, \bar{\epsilon})}, \qquad (11)$$

with

$$\frac{\partial \Delta_\pm}{\partial u}(u, B, \lambda, \bar{\epsilon}) = \pm \exp\left(\frac{\hat{I}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right)$$

where functions $\hat{I}_\pm$ are $\bar{\epsilon}$-regularly smooth in (u, B, λ) and $\hat{I}_\pm(u, B, \lambda, 0) = I_\pm(u, 0, \lambda)$.

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$$\frac{\partial\Delta_\pm}{\partial u}(u,B,\lambda,\bar\epsilon)=\pm\exp\left(\frac{\hat I_\pm(u,B,\lambda,\bar\epsilon)}{\bar\epsilon^2}\right)$$

  where functions $\hat I_\pm$ are $\bar\epsilon$-regularly smooth in (u,B,λ) and $\hat I_\pm(u,B,\lambda,0)=I_\pm(u,0,\lambda)$.
- Since the function $u\to I_-(u,0,0)-I_+(u,0,0)$ has a simple zero at $u=u_0$, $f(0,0,0)=0$ and $\frac{\partial f}{\partial B}(0,0,0)\neq 0$, we can apply the Implicit Function Theorem to the following $\bar\epsilon$-regularly smooth in (u,B,λ) system:

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Proof of Theorem 4

- $$\begin{cases} \Delta_-(u, B, \lambda, \bar{\epsilon}) - \Delta_+(u, B, \lambda, \bar{\epsilon}) = 0 \\ \hat{I}_-(u, B, \lambda, \bar{\epsilon}) - \hat{I}_+(u, B, \lambda, \bar{\epsilon}) = 0, \end{cases}$$

  and find a solution $(\lambda, \bar{\epsilon}) \to (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))$, $\bar{\epsilon}$-regularly smooth in λ, with $u(0, 0) = u_0$ and $B(0, 0) = 0$.

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Proof of Theorem 4

- $$\begin{cases} \Delta_-(u,B,\lambda,\bar\epsilon) - \Delta_+(u,B,\lambda,\bar\epsilon) = 0 \\ \hat I_-(u,B,\lambda,\bar\epsilon) - \hat I_+(u,B,\lambda,\bar\epsilon) = 0, \end{cases}$$

  and find a solution $(\lambda,\bar\epsilon) \to (u(\lambda,\bar\epsilon), B(\lambda,\bar\epsilon))$, $\bar\epsilon$-regularly smooth in λ, with $u(0,0) = u_0$ and $B(0,0) = 0$.
- From this and (11) follows

$$P_{B(\lambda,\bar\epsilon),\lambda,\bar\epsilon}(u(\lambda,\bar\epsilon)) = u(\lambda,\bar\epsilon) \text{ and } \frac{\partial P_{B(\lambda,\bar\epsilon),\lambda,\bar\epsilon}}{\partial u}(u(\lambda,\bar\epsilon)) = -1,$$

for all $(\lambda,\bar\epsilon) \sim (0,0)$ and $\bar\epsilon > 0$.

Proof of Theorem 4

- $$\begin{cases} \Delta_-(u, B, \lambda, \bar{\epsilon}) - \Delta_+(u, B, \lambda, \bar{\epsilon}) = 0 \\ \hat{I}_-(u, B, \lambda, \bar{\epsilon}) - \hat{I}_+(u, B, \lambda, \bar{\epsilon}) = 0, \end{cases}$$

  and find a solution $(\lambda, \bar{\epsilon}) \to (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))$, $\bar{\epsilon}$-regularly smooth in λ, with $u(0,0) = u_0$ and $B(0,0) = 0$.
- From this and (11) follows

  $$P_{B(\lambda,\bar{\epsilon}),\lambda,\bar{\epsilon}}(u(\lambda, \bar{\epsilon})) = u(\lambda, \bar{\epsilon}) \text{ and } \frac{\partial P_{B(\lambda,\bar{\epsilon}),\lambda,\bar{\epsilon}}}{\partial u}(u(\lambda, \bar{\epsilon})) = -1,$$

  for all $(\lambda, \bar{\epsilon}) \sim (0,0)$ and $\bar{\epsilon} > 0$.
- Thus, for each $(\lambda, \bar{\epsilon}) \sim (0,0)$ and $\bar{\epsilon} > 0$, $P_{B(\lambda,\bar{\epsilon}),\lambda,\bar{\epsilon}}$ has a fixed point $u(\lambda, \bar{\epsilon})$ with eigenvalue -1.

Proof of Theorem 4

- The quantity (PD1) becomes:

$$\frac{\frac{\partial(\Delta_- - \Delta_+)}{\partial B}(u)\left(\frac{\partial I_-}{\partial u}(u) - \frac{\partial I_+}{\partial u}(u)\right) + o(1)}{\bar{\epsilon}^2 \frac{\partial \Delta_-}{\partial u}(u)}, \tag{12}$$

where $(u, B) = (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))$.

Proof of Theorem 4

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where $(u, B) = (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))$.

- The quantity (PD2) becomes

$$a = \frac{\left(\frac{\partial I_-}{\partial u}(u)\right)^2 - \left(\frac{\partial I_+}{\partial u}(u)\right)^2 + o(1)}{6\bar{\epsilon}^4}, \ (u, B) = (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon})).$$

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Proof of Theorem 4

Lemma

Let $m \in \mathbb{N}$, $m \geq 1$. Then we have:

$$\bar{\epsilon}^{2m} \frac{\partial^{m+1} \Delta_\pm}{\partial u^{m+1}}(u) = \pm \left(\left(\frac{\partial I_\pm}{\partial u}(u) \right)^m + o(1) \right) \exp \left(\frac{\hat{I}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right),$$

where $\hat{I}_\pm(u, B, \lambda, \bar{\epsilon})$ are defined after (11), $I_\pm(u) = I_\pm(u, 0, \lambda)$ and the $o(1)$-term is $\bar{\epsilon}$-regularly smooth in (u, B, λ).

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where $\hat{I}_{\pm}(u, B, \lambda, \bar{\epsilon})$ are defined after (11), $I_{\pm}(u) = I_{\pm}(u, 0, \lambda)$ and the $o(1)$-term is $\bar{\epsilon}$-regularly smooth in (u, B, λ).

- Thus, putting all the informations together, we have proved that for each fixed $(\lambda, \bar{\epsilon}) \sim (0, 0)$, $\bar{\epsilon} > 0$, the B-family $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ undergoes a period doubling bifurcation at $(u, B) = (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))$.

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Proof of Theorem 5

- We consider

$$\begin{cases} \exp\left(\frac{\bar{I}_-(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = f(B,\lambda,\bar{\epsilon}) \\ \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = f(B,\lambda,\bar{\epsilon}). \end{cases}$$

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- The main difficulty lies in the fact that the limit $\bar{\epsilon} = 0$ of this system is degenerate. Our goal is, therefore, to replace the system with a new system, non-singular for $\bar{\epsilon} = 0$, using [Mamouhdi, Roussarie,2012].

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Proof of Theorem 5

- Let us suppose that $\Psi(u,u')$ and $\Phi(u,u')$ are two smooth functions defined on a rectangle $R = [\bar{U}_1, \tilde{U}_1] \times [\bar{U}_2, \tilde{U}_2]$ and let us suppose that $\frac{\partial \Psi}{\partial u}$, $\frac{\partial \Psi}{\partial u'}$, $\frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial u'}$ are nonzero for all $(u,u') \in R$.

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- Let us suppose that $\Psi(u,u')$ and $\Phi(u,u')$ are two smooth functions defined on a rectangle $R=[\bar{U}_1,\tilde{U}_1]\times[\bar{U}_2,\tilde{U}_2]$ and let us suppose that $\frac{\partial\Psi}{\partial u}$, $\frac{\partial\Psi}{\partial u'}$, $\frac{\partial\Phi}{\partial u}$ and $\frac{\partial\Phi}{\partial u'}$ are nonzero for all $(u,u')\in R$.
- We further assume that the equation $\{\det(\Psi,\Phi)(u,u')=0\}$ for contact points is equivalent on R to an equation $\{E(u,u')=0\}$, where E is a smooth function on R, and where $\frac{\partial E}{\partial u}$ and $\frac{\partial E}{\partial u'}$ are nonzero for all $(u,u')\in R$. (Equivalent means $\det(\Psi,\Phi)=F.E$, where the factor F is a smooth nowhere zero function on R.)

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- Now we can define a *regular pair of foliations* $(\widetilde{\Psi},\widetilde{\Phi})$ on R as follows: the curves $\{\Psi(u,u')=\alpha\}$ (resp. $\{\Phi(u,u')=\beta\}$) are the leaves of foliation $\tilde{\Psi}$ (resp. $\tilde{\Phi}$).

Proof of Theorem 5

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- We further assume that the equation $\{\det(\Psi, \Phi)(u, u') = 0\}$ for contact points is equivalent on R to an equation $\{E(u, u') = 0\}$, where E is a smooth function on R, and where $\frac{\partial E}{\partial u}$ and $\frac{\partial E}{\partial u'}$ are nonzero for all $(u, u') \in R$. (Equivalent means $\det(\Psi, \Phi) = F.E$, where the factor F is a smooth nowhere zero function on R.)
- Now we can define a *regular pair of foliations* $(\widetilde{\Psi}, \widetilde{\Phi})$ on R as follows: the curves $\{\Psi(u, u') = \alpha\}$ (resp. $\{\Phi(u, u') = \beta\}$) are the leaves of foliation $\tilde{\Psi}$ (resp. $\tilde{\Phi}$).
- Each leaf and the curve $\{E(u, u') = 0\}$ are simple connected

Proof of Theorem 5

- We relate the number of intersection points of two leaves $\{\Psi(u, u') = \alpha\}$ and $\{\Phi(u, u') = \beta\}$ in R with the number of intersection points of the curve $\{E(u, u') = 0\}$ and one of these two leaves in R.

Lemma

Let $(\widetilde{\Psi}, \widetilde{\Phi})$ be a regular pair of foliations on R as defined above and let $\alpha, \beta \in \mathbb{R}$ be arbitrary but fixed. Let $\mathcal{N}(\alpha, \beta)$ be the number of intersection points of $\{\Psi(u, u') = \alpha\}$ with $\{\Phi(u, u') = \beta\}$ in R, counting multiplicity, and let $\mathcal{N}(\beta)$ be the number of intersection points of $\{E(u, u') = 0\}$ with $\{\Phi(u, u') = \beta\}$ in R, counting multiplicity. If $\mathcal{N}(\beta)$ is finite, then

$$\mathcal{N}(\alpha, \beta) \leq \mathcal{N}(\beta) + 1. \tag{13}$$

Proof of Theorem 5

- To find at most 3 solutions of the system in $[\bar{u},\tilde{u}]\times[\bar{u},\tilde{u}]$, for each $(B,\lambda,\bar{\epsilon})\sim(0,0,0)$, with $\bar{\epsilon}>0$, we use the lemma twice. The system (9) is a special case of the more general system

$$\left\{\begin{array}{l} \exp\left(\frac{\bar{I}_-(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)+\exp\left(\frac{\bar{I}_+(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)=\alpha \\ \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)+\exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right)=\beta \end{array}\right. \tag{14}$$

where $\alpha,\beta\in\mathbb{R}$, and it suffices to prove that (14) has at most 3 solutions in $[\bar{u},\tilde{u}]\times[\bar{u},\tilde{u}]$, for each fixed $(B,\lambda,\bar{\epsilon})\sim(0,0,0)$, with $\bar{\epsilon}>0$, and $\alpha,\beta\in\mathbb{R}$.

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Proof of Theorem 5

- We denote by $\Psi_{B,\lambda,\bar{\epsilon}}(u,u'), \Phi_{B,\lambda,\bar{\epsilon}}(u,u')$ the functions on the left-hand side of (14). We have

$$\det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) = \exp\left(\frac{I_-(u)+I_-(u')+o(1)}{\bar{\epsilon}^2}\right) - \exp\left(\frac{I_+(u)+I_+(u')+o(1)}{\bar{\epsilon}^2}\right).$$

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- This implies that the the set $\{\det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}})(u,u')=0\}$ of the contact points between the two foliations $\widetilde{\Psi}_{B,\lambda,\bar{\epsilon}}$ and $\widetilde{\Phi}_{B,\lambda,\bar{\epsilon}}$ is equivalent for $\bar{\epsilon}>0$ to $\{E_{B,\lambda,\bar{\epsilon}}(u,u')=0\}$ with

$$E_{B,\lambda,\bar{\epsilon}}(u,u') = I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1).$$

Proof of Theorem 5

- We define the following system:

$$\begin{cases} E_{B,\lambda,\bar{\epsilon}}(u,u') = I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0 \\ \Phi_{B,\lambda,\bar{\epsilon}}(u,u') = \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \beta. \end{cases} \tag{15}$$

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- We define the following system:

$$\begin{cases} E_{B,\lambda,\bar{\epsilon}}(u,u') = I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0 \\ \Phi_{B,\lambda,\bar{\epsilon}}(u,u') = \exp\left(\frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2}\right) = \beta. \end{cases} \tag{15}$$

- Following Lemma 13, if we denote by $\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha,\beta)$ (resp. $\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta)$) the number of solutions of (14) (resp. (15)), counting multiplicity, in $[\bar{u},\tilde{u}] \times [\bar{u},\tilde{u}]$, then

$$\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha,\beta) \leq 1 + \mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta).$$

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Proof of Theorem 5

- We have

$$
\begin{aligned}
&\det(E_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) = \\
&\quad \left(\frac{\partial I_-}{\partial u}(u) - \frac{\partial I_+}{\partial u}(u) + o(1)\right) \exp\left(\frac{I_-(u') + o(1)}{\bar{\epsilon}^2}\right) \\
&\quad - \left(\frac{\partial I_-}{\partial u}(u') - \frac{\partial I_+}{\partial u}(u') + o(1)\right) \exp\left(\frac{I_+(u) + o(1)}{\bar{\epsilon}^2}\right).
\end{aligned}
$$

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Proof of Theorem 5

- We have

$$\det(E_{B,\lambda,\bar\epsilon}, \Phi_{B,\lambda,\bar\epsilon}) =$$
$$\left(\frac{\partial I_-}{\partial u}(u) - \frac{\partial I_+}{\partial u}(u) + o(1)\right) \exp\left(\frac{I_-(u') + o(1)}{\bar\epsilon^2}\right)$$
$$- \left(\frac{\partial I_-}{\partial u}(u') - \frac{\partial I_+}{\partial u}(u') + o(1)\right) \exp\left(\frac{I_+(u) + o(1)}{\bar\epsilon^2}\right).$$

Clearly, the equation $\{\det(E_{B,\lambda,\bar\epsilon}, \Phi_{B,\lambda,\bar\epsilon})(u,u') = 0\}$ is equivalent for $\bar\epsilon > 0$ to $\{\bar E_{B,\lambda,\bar\epsilon}(u,u') = 0\}$ where

$$\bar E_{B,\lambda,\bar\epsilon}(u,u') = I_-(u') - I_+(u) + o(1).$$

Proof of Theorem 5

- Lemma 13 implies that

$$\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta) \leq 1 + \mathcal{N}_{B,\lambda,\bar{\epsilon}},$$

where $\mathcal{N}_{B,\lambda,\bar{\epsilon}}$ is the number of solutions (counting multiplicity) of the system $\{I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0, I_-(u') - I_+(u) + o(1) = 0\}$, or equivalently the system

$$\begin{cases} I_-(u) - I_+(u') + o(1) = 0 \\ I_-(u') - I_+(u) + o(1) = 0. \end{cases} \tag{16}$$

Thus, we have proved that

$$\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha, \beta) \leq 2 + \mathcal{N}_{B,\lambda,\bar{\epsilon}},$$

for each $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$, and $\alpha, \beta \in \mathbb{R}$.

Proof of Theorem 6

- Assume that $f(\omega(u_0),0)=0$, $\frac{\partial f}{\partial x}(\omega(u_0),0)\neq 0$ and $f(x,0)<0$ for all $x\in[\alpha(u_0),\omega(u_0)[$.

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Proof of Theorem 6

- Assume that $f(\omega(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)[$.
- First, let us prove that $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0}) \leq 1$, i.e. there are no 2-periodic orbits Hausdorff close to L_{u_0}.

Proof of Theorem 6

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- First, let us prove that $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})\leq 1$, i.e. there are no 2-periodic orbits Hausdorff close to L_{u_0}.
- Suppose, on the contrary, that for $(B,\lambda,\bar{\epsilon})\sim(0,0,0)$, $\bar{\epsilon}>0$, $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has a 2–periodic orbit intersecting Σ_+ in two points $\bar{u}\sim u_0$ and $\tilde{u}\sim u_0$, with $u(B,\lambda,\bar{\epsilon})<\bar{u}<\tilde{u}$.

Proof of Theorem 6

- Assume that $f(\omega(u_0),0)=0$, $\frac{\partial f}{\partial x}(\omega(u_0),0)\neq 0$ and $f(x,0)<0$ for all $x\in[\alpha(u_0),\omega(u_0)[$.
- First, let us prove that $\mathrm{Cycl}(X_{\epsilon,\mu},L_{u_0})\leq 1$, i.e. there are no 2-periodic orbits Hausdorff close to L_{u_0}.
- Suppose, on the contrary, that for $(B,\lambda,\bar{\epsilon})\sim(0,0,0)$, $\bar{\epsilon}>0$, $X_{\bar{\epsilon}^2,\bar{\epsilon}B,\lambda}$ has a 2–periodic orbit intersecting Σ_+ in two points $\bar{u}\sim u_0$ and $\tilde{u}\sim u_0$, with $u(B,\lambda,\bar{\epsilon})<\bar{u}<\tilde{u}$.
- We have for $u\in[\bar{u},\tilde{u}]$:

$$\Delta'_{B,\lambda,\bar{\epsilon}}(u) = -\exp\left(\frac{\mathcal{I}_-(u,B,\lambda,\bar{\epsilon}) - I_+(P_{B,\lambda,\bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2}\right) + \exp\left(\frac{I_+(u) - \mathcal{I}_-(P^{-1}_{B,\lambda,\bar{\epsilon}}(u),B,\lambda,\bar{\epsilon}) + o(1)}{\bar{\epsilon}^2}\right).$$

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Proof of Theorem 7

- Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed, and let us suppose that $T(u_0, u_1, 0) \neq 0$, where T is the total slow divergence integral.

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Proof of Theorem 7

- Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed, and let us suppose that $T(u_0, u_1, 0) \neq 0$, where T is the total slow divergence integral.
- Suppose, on the contrary, that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has two 2–periodic orbits, one intersecting Σ_+ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_1$, and the other in $\bar{\bar{u}} \sim u_0$ and $\tilde{\tilde{u}} \sim u_1$.

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- Then we have $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$ or $\bar{\bar{u}} < \bar{u} < \tilde{u} < \tilde{\tilde{u}}$.

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- Then we have $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$ or $\bar{\bar{u}} < \bar{u} < \tilde{u} < \tilde{\tilde{u}}$.
- Suppose without loss of generality that $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$.

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- Suppose, on the contrary, that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has two 2–periodic orbits, one intersecting Σ_+ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_1$, and the other in $\bar{\bar{u}} \sim u_0$ and $\tilde{\tilde{u}} \sim u_1$.
- Then we have $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$ or $\bar{\bar{u}} < \bar{u} < \tilde{u} < \tilde{\tilde{u}}$.
- Suppose without loss of generality that $\bar{u} < \bar{\bar{u}} < \tilde{\tilde{u}} < \tilde{u}$. Then $\Delta_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \Delta_{B,\lambda,\bar{\epsilon}}(\bar{\bar{u}}) = 0$, $P_{B,\lambda,\bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B,\lambda,\bar{\epsilon}}(\bar{\bar{u}}) = \tilde{\tilde{u}}$, $P_{B,\lambda,\bar{\epsilon}}([\bar{u}, \bar{\bar{u}}]) = [\tilde{\tilde{u}}, \tilde{u}]$ and $P^{-1}_{B,\lambda,\bar{\epsilon}}([\bar{u}, \bar{\bar{u}}]) = [\tilde{\tilde{u}}, \tilde{u}]$.

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Proof of Theorem 7

- We get

$$\Delta'_{B,\lambda,\bar\epsilon}(u) = -\exp\left(\frac{I_-(u) - I_+(P_{B,\lambda,\bar\epsilon}(u)) + o(1)}{\bar\epsilon^2}\right) + \exp\left(\frac{I_+(u) - I_-(P^{-1}_{B,\lambda,\bar\epsilon}(u)) + o(1)}{\bar\epsilon^2}\right),$$

where $u \in [\bar u, \bar{\bar u}]$.

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  where $u \in [\bar u, \bar{\bar u}]$.

- The equation $\{\Delta'_{B,\lambda,\bar\epsilon}(u) = 0\}$ is equivalent for $\bar\epsilon > 0$ and $u \in [\bar u, \bar{\bar u}]$ to an equation given in (10). Since $T(u_0, u_1, 0) \neq 0$, $u \sim u_0$, $P_{B,\lambda,\bar\epsilon}(u), P^{-1}_{B,\lambda,\bar\epsilon}(u) \sim u_1$ for all $u \in [\bar u, \bar{\bar u}]$, (10) has no solutions w.r.t. $u \in [\bar u, \bar{\bar u}]$.

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  where $u \in [\bar{u}, \bar{\bar{u}}]$.

- The equation $\{\Delta'_{B,\lambda,\bar{\epsilon}}(u) = 0\}$ is equivalent for $\bar{\epsilon} > 0$ and $u \in [\bar{u}, \bar{\bar{u}}]$ to an equation given in (10). Since $T(u_0, u_1, 0) \neq 0$, $u \sim u_0$, $P_{B,\lambda,\bar{\epsilon}}(u), P^{-1}_{B,\lambda,\bar{\epsilon}}(u) \sim u_1$ for all $u \in [\bar{u}, \bar{\bar{u}}]$, (10) has no solutions w.r.t. $u \in [\bar{u}, \bar{\bar{u}}]$.
- This is a contradiction with $\Delta'_{B,\lambda,\bar{\epsilon}}(u') = 0$. Thus, $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$.

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Future research

- The cyclicity of 1– and 2-canard cycles in the case of higher multiplicity zero in the slow divergence integral

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- The cyclicity of 1– and 2-canard cycles if the slow dynamics has singularities including at the contact point (non-generic contact point)