The slow divergence integral on a Möbius band

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- We prove the finite cyclicity property of “singular” 1– and 2–homoclinic loops.
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- We give a simple sufficient condition, expressed in terms of the slow divergence integral, for the existence of a period-doubling bifurcation near the 1–canard cycle.
- We prove the finite cyclicity property of “singular” 1– and 2–homoclinic loops.
- Using an idea of Khovanskii we find optimal upper bounds for the number of limit cycles Hausdorff close to canard cycles.
Motivation

Let’s consider a simple planar slow-fast system $X_{\epsilon,b}$ (depending possibly on an extra finite dimensional parameter):

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -xy + \epsilon(b - x + O(x^2)) + O(\epsilon y^2)
\end{align*}
\]  

(1)

where $\epsilon \geq 0$ is a singular perturbation parameter and $b \sim 0$ is a breaking parameter.
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- The fast subsystem $X_{0,b}$ consists of the line of singularities $\{y = 0\}$ (the critical curve or the slow curve) and fast orbits, given by parabolas $y = -\frac{1}{2}x^2 + c$.
- All singularities of the critical curve are normally hyperbolic, except the origin where we deal with a generic nilpotent contact point.
The slow divergence integral on a Möbius band

Motivation

We distinguish between two types of limit periodic sets: the contact point \((x, y) = (0, 0)\) and canard cycles.

F. Dumortier, R. Roussarie, *Canard cycles and center manifolds*, 1996

M. Krupa, P. Szmolyan, *Relaxation oscillation and canard explosion*, 2001

F. Dumortier, *Slow divergence integral and balanced canard solutions*, 2011
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The slow divergence integral on a Möbius band
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- Our model can provide much richer dynamics if we consider it on the Möbius band.

- Besides the contact point and the canard cycles we also detect so-called 1– and 2–canard cycles consisting of a fast orbit, turning around the Möbius band, and the part of the critical curve between the $\alpha$-limit set and the $\omega$-limit set of the fast orbit.
Motivation

- Our model in the Liénard plane:

![Diagram showing the model in the Liénard plane.](image-url)
Motivation

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Definitions on the smooth Möbius band

Denote by $M$ a smooth Möbius band (“smooth” means $C^\infty$-smooth). Let $(\epsilon, \mu) \sim (0, 0) \in \mathbb{R} \times \mathbb{R}^l$, with $\epsilon \geq 0$, and let $X_{\epsilon, \mu} : M \to TM$ be a smooth $(\epsilon, \mu)$-family of vector fields on $M$. We suppose $X_{\epsilon, \mu}$ has a slow-fast structure, with a singular perturbation parameter $\epsilon$ and with a generic turning point (or equivalently, a slow-fast Hopf point) $p \in M$ for $(\epsilon, \mu) = (0, 0)$. More precisely, we suppose that there exists a local chart on $M$ around $p$ in which the vector field $X_{\epsilon, \mu}$ is locally expressed, up to smooth equivalence, as:

$$
\dot{x} = y, \\
\dot{y} = -xy + \epsilon \left( b(\mu) - x + x^2 g(x, \epsilon, \mu) \right) + \epsilon y^2 H(x, y, \epsilon, \mu).
$$

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Definitions on the smooth Möbius band

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Renato Huzak
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We further assume that $X_{0,\mu}$ has a smooth $\mu$-family of one dimensional embedded manifolds $m_\mu$ containing singularities of $X_{0,\mu}$ (in the local coordinates, $m_\mu$ is given by $\{y = 0\}$), and that $m_0 = m^- \cup \{p\} \cup m^+$, where $m^-$ (resp. $m^+$) is normally attracting (resp. normally repelling).
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We suppose that the slow dynamics is nonzero on \( m^- \cup m^+ \), pointing towards \( p \) on \( m^- \) and away from \( p \) on \( m^+ \).
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Working with such an orientable submanifold, we can choose a volume form and define the divergence of (the restriction of) the vector field $X_{\epsilon, \mu}$. 

The slow divergence integral on a Möbius band

Definitions on the smooth Möbius band

The slow divergence integral is independent of the chosen volume form and the local chart.
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- The slow divergence integral is independent of the chosen volume form and the local chart.
- The slow dynamics of $X_{\epsilon,\mu}$ along the slow curve $m_{\mu} \subset \tilde{M}$, away from the turning point, is given by $x' = f(x, \mu)$, $\mu \sim 0$, where $f$ is a smooth function and $m_{\mu}$ is parametrized by a regular parameter $x$. 

$\int_0^{\alpha(u)} \text{div} X_{0,\mu} f(x, \mu) < 0, \quad u \in \Sigma^+$. (3)
Definitions on the smooth Möbius band

- The **slow divergence integral** is independent of the chosen volume form and the local chart.

- The **slow dynamics** of \(X_{\epsilon,\mu}\) along the slow curve \(m_{\mu} \subset \tilde{M}\), away from the turning point, is given by \(x' = f(x, \mu), \mu \sim 0\), where \(f\) is a smooth function and \(m_{\mu}\) is parametrized by a regular parameter \(x\).

- We have \(f < 0\). Now we can define the **slow divergence integral** \(l_{\pm}(u, \mu)\) along \(m_{\pm}\):

\[
I_+ (u, \mu) := \int_{\alpha(u)}^{0} \text{div} X_{0, \mu} dx \overline{f(x, \mu)} < 0, \quad I_- (u, \mu) := \int_{\omega(u)}^{0} \text{div} X_{0, \mu} dx \overline{f(x, \mu)} < 0,
\]

(3)
The slow divergence integral on a Möbius band

Definitions on the smooth Möbius band

Figure: Canard cycles on the Möbius band $M$ turning around $M$, at level $(\epsilon, \mu) = (0, 0)$. (a) 1–canard cycles intersect $\Sigma_+$ only once. (b) 2–canard cycles intersect $\Sigma_+$ twice.
The slow divergence integral on a Möbius band

Definitions on the smooth Möbius band

Definition (1 and 2–periodic orbits)

Let $L_{u_0}$ and $L_{u_0,u_1}$ be 1– and 2–canard cycles.

(a) Let $V \subset M$ be a small tubular neighborhood of $L_{u_0}$. Let $O \subset V$ be a periodic orbit of $X_{\epsilon,\mu}$, with $\epsilon > 0$. We call $O$ a 1–periodic orbit if $O$ intersects the section $\Sigma_+$ only once. Isolated 1–periodic orbits are called 1–limit cycles.

(b) Let $V \subset M$ be a small tubular neighborhood of $L_{u_0}$ or $L_{u_0,u_1}$. Let $O \subset V$ be a periodic orbit of $X_{\epsilon,\mu}$, with $\epsilon > 0$. We call $O$ a 2–periodic orbit if $O$ intersects the section $\Sigma_+$ twice. Isolated 2–periodic orbits are called 2–limit cycles.
Definitions on the smooth Möbius band

Definition (Cyclicity of $L_{u_0}$ and $L_{u_0, u_1}$)

Let $X_{\epsilon, \mu}$ be a smooth $(\epsilon, \mu)$-family of vector fields on $M$, defined above, and let $L_{u_0}$ and $L_{u_0, u_1}$ be the limit periodic sets. The cyclicity of $L_{u_0}$ (resp. $L_{u_0, u_1}$) in the family $X_{\epsilon, \mu}$ is bounded from above by $N \in \mathbb{N}$ if there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood $W$ of 0 in the $\mu$-space such that $X_{\epsilon, \mu}$, with $(\epsilon, \mu) \in [0, \epsilon_0] \times W$, generates at most $N$ limit cycles, lying each within Hausdorff distance $\delta_0$ of $L_{u_0}$ (resp. $L_{u_0, u_1}$). We call the smallest $N$ with this property the cyclicity of $L_{u_0}$ (resp. $L_{u_0, u_1}$) in the family $X_{\epsilon, \mu}$, and denote it by $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0})$ (resp. $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0, u_1})$).
The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to $L_{u_0}$

For $(u, \mu) \sim (u_0, 0)$, the slow divergence integral along the slow curve from $\omega(u) \in m^-$ to $\alpha(u) \in m^+$ is given by:

$$I(u, \mu) = I_-(u, \mu) - I_+(u, \mu)$$

(4)

Theorem

Suppose that $I(u, \mu)$ is nonzero near $(u, \mu) = (u_0, 0)$. Then $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0}) = 1$ and $X_{\epsilon, \mu}$ has no 2–periodic orbits Hausdorff-close to $L_{u_0}$. In case $I(u_0, 0) < 0$ (resp. $I(u_0, 0) > 0$) any 1–limit cycle bifurcating from $L_{u_0}$ is hyperbolically attracting (resp. hyperbolically repelling).
The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to $L_{u_0}$

If the function $u \rightarrow I(u, 0)$ has a simple zero at $u = u_0$, then for $\lambda \sim 0$, $\epsilon \sim 0$ and $\epsilon > 0$ the $b$-family $X_{\epsilon, \mu} = X_{\epsilon, b, \lambda}$ undergoes, Hausdorff-close to $L_{u_0}$, a period doubling bifurcation, giving rise to a 2-limit cycle. In this case we do not need the parameter $\lambda$.

Theorem

Let us suppose that the function $u \rightarrow I(u, 0)$ has a simple zero at $u = u_0$ (i.e. $I(u_0, 0) = 0$ and $\frac{\partial I}{\partial u}(u_0, 0) \neq 0$). Then there are continuous functions $u(\epsilon, \lambda)$ and $b(\epsilon, \lambda)$ defined for $\epsilon \geq 0$, $\epsilon \sim 0$ and $\lambda \sim 0$, smooth for $\epsilon > 0$, with $u(0, 0) = u_0$ and $b(0, \lambda) = 0$, such that for each $\epsilon > 0$, $\epsilon \sim 0$ and $\lambda \sim 0$ the $b$-family $X_{\epsilon, b, \lambda}$ undergoes a period doubling bifurcation at $(u(\epsilon, \lambda), b(\epsilon, \lambda))$. 

Renato Huzak
The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to $L_{u_0}$

- To prove that, under the same condition on $I$, 
  $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0}) \leq 2$, we use a method of Khovanskii (Mamouhdi, Roussarie).

Theorem

Let us suppose that $u \rightarrow I(u, 0)$ has a simple zero at $u = u_0$. Then 
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The case of higher multiplicity zeros in the slow divergence integral is a topic of further study.
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- The case of higher multiplicity zeros in the slow divergence integral is a topic of further study.
We call the 1–canard cycle $L_{u_0}$ a singular 1–homoclinic loop if the slow dynamics has a hyperbolic saddle at precisely one corner point: “$f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$” or “$f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$”. We prove that such a limit periodic set can produce at most one limit cycle.

**Theorem**

Let us suppose that $f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)]$. Then $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0}) = 1$ and $X_{\epsilon, \mu}$ has no 2–periodic orbits Hausdorff-close to $L_{u_0}$. When a 1–limit cycle exists, it is hyperbolic and attracting. A similar result is true in the case $f(\alpha(u_0), 0) = 0, \frac{\partial f}{\partial x}(\alpha(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)]$. A 1–limit cycle bifurcating from $L_{u_0}$ is hyperbolic and repelling.
“Regular” 1–homoclinic loops of finite codimension have been studied by Guimond, 1999.
The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to \( L_{u_0,u_1} \)

Let \( u_0, u_1 \in \Sigma_+, \) with \( u_0 < u_1, \) be arbitrary but fixed. For \((u, \tilde{u}, \mu) \sim (u_0, u_1, 0),\) we define the so-called total slow divergence integral of \( L_{u_0,u_1}:\)

\[
T(u, \tilde{u}, \mu) = I_-(u, \mu) - I_+(\tilde{u}, \mu) + I_-(\tilde{u}, \mu) - I_+(u, \mu). \tag{5}
\]

Theorem

Suppose that \( T \) is nonzero near \((u, \tilde{u}, \mu) = (u_0, u_1, 0).\) Then \( \text{Cycl}(X_{\epsilon, \mu}, L_{u_0,u_1}) \leq 1.\) In case \( T(u_0, u_1, 0) < 0 \) (resp. \( T(u_0, u_1, 0) > 0)\) any 2-limit cycle bifurcating from \( L_{u_0,u_1} \) is hyperbolically attracting (resp. hyperbolically repelling).
The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to $L_{u_0,u_1}$

1. If $I_-(u_1,0) - I_+(u_1,0) \neq 0$, then there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood $W$ of 0 in the $\mu$-space such that system $X_{\epsilon,\mu}$, with $(\epsilon, \mu) \in [0, \epsilon_0] \times W$, has no limit cycles lying within Hausdorff distance $\delta_0$ of $L_{u_0,u_1}$. 

2. If $I_-(u_0,0) - I_+(u_0,0) = 0$ and $I_-(u_0,0) - I_+(u_0,0) \neq 0$, then we have that $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. In case $I_-(u_0,0) - I_+(u_0,0) < 0$ (resp. $I_-(u_0,0) - I_+(u_0,0) > 0$) any 2-limit cycle bifurcating from $L_{u_0,u_1}$ is hyperbolic and attracting (resp. repelling). Moreover, if $\partial(I_-(u_0,0) - I_+(u_0,0)) \neq 0$, then $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) = 1$.

3. If $I_-(u_i,0) - I_+(u_i,0) = 0$ for $i = 0, 1$ (this implies $T(u_0,u_1,0) = 0$) and $\partial(I_-(u_i,0) - I_+(u_i,0)) \neq 0$ for $i = 0, 1$, then $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$. 

4. Renato Huzak
1 If $l_-(u_1,0) - l_+(u_1,0) \neq 0$, then there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood $W$ of 0 in the $\mu$-space such that system $X_{\epsilon,\mu}$, with $(\epsilon, \mu) \in [0, \epsilon_0] \times W$, has no limit cycles lying within Hausdorff distance $\delta_0$ of $L_{u_0,u_1}$.

2 If $l_-(u_1,0) - l_+(u_1,0) = 0$ and $l_-(u_0,0) - l_+(u_0,0) \neq 0$ (this implies $T(u_0, u_1, 0) \neq 0$), then we have that $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 1$. In case $l_-(u_0,0) - l_+(u_0,0) < 0$ (resp. $l_-(u_0,0) - l_+(u_0,0) > 0$) any 2–limit cycle bifurcating from $L_{u_0,u_1}$ is hyperbolic and attracting (resp. repelling). Moreover, if $\frac{\partial(l_--l_+)}{\partial u}(u_1,0) \neq 0$, then $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) = 1$. 

3 If $I^-(u_1,0) = I^+(u_1,0) = 0$ for $i = 0, 1$ (this implies $T(u_0, u_1, 0) = 0$) and $\frac{\partial(I^--I^+)}{\partial u}(u_i,0) \neq 0$ for $i = 0, 1$, then $\text{Cycl}(X_{\epsilon,\mu}, L_{u_0,u_1}) \leq 2$. 

Renato Huzak
The slow divergence integral on a Möbius band

Limit cycle bifurcations Hausdorff-close to $L_{u_0,u_1}$

1. If $I_-(u_1, 0) - I_+(u_1, 0) \neq 0$, then there exists $\epsilon_0 > 0$, $\delta_0 > 0$ and a neighborhood $W$ of 0 in the $\mu$-space such that system $X_{\epsilon, \mu}$, with $(\epsilon, \mu) \in [0, \epsilon_0] \times W$, has no limit cycles lying within Hausdorff distance $\delta_0$ of $L_{u_0,u_1}$.

2. If $I_-(u_1, 0) - I_+(u_1, 0) = 0$ and $I_-(u_0, 0) - I_+(u_0, 0) \neq 0$ (this implies $T(u_0, u_1, 0) \neq 0$), then we have that $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0,u_1}) \leq 1$. In case $I_-(u_0, 0) - I_+(u_0, 0) < 0$ (resp. $I_-(u_0, 0) - I_+(u_0, 0) > 0$) any 2–limit cycle bifurcating from $L_{u_0,u_1}$ is hyperbolic and attracting (resp. repelling). Moreover, if $\frac{\partial(I_- - I_+)}{\partial u}(u_1, 0) \neq 0$, then $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0,u_1}) = 1$.

3. If $I_-(u_i, 0) - I_+(u_i, 0) = 0$ for $i = 0, 1$ (this implies $T(u_0, u_1, 0) = 0$) and $\frac{\partial(I_- - I_+)}{\partial u}(u_i, 0) \neq 0$ for $i = 0, 1$, then $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0,u_1}) \leq 2$. 
Limit cycle bifurcations Hausdorff-close to $L_{u_0,u_1}$

- We allow the slow dynamics to have a hyperbolic saddle at precisely one corner point, $\omega(u_0)$ or $\alpha(u_0)$. In this case we call $L_{u_0,u_1}$ a singular 2–homoclinic loop.

**Theorem**

Let us suppose that $f(\omega(u_0),0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0),0) \neq 0$ and that $f(x,0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)]$. Then $\text{Cycl}(X_\epsilon, \mu, L_{u_0,u_1}) \leq 1$. Any 2–limit cycle bifurcating from $L_{u_0,u_1}$ is hyperbolic and attracting.

A similar result is true in the case $f(\alpha(u_0),0) = 0$, $\frac{\partial f}{\partial x}(\alpha(u_0),0) \neq 0$ and $f(x,0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)]$. Any 2–limit cycle bifurcating from $L_{u_0,u_1}$ is hyperbolic and repelling.
We define now the following transition maps for $(\bar{\epsilon}, B, \lambda) \sim (0, 0, 0)$:

1. the forward transition map $\Delta_- : \Sigma_+ \to \Sigma_p$ along the flow of $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$;
2. the backward transition map $\Delta_+ : \Sigma_+ \to \Sigma_p$ along the flow of $-X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$.

The map $\Delta_{\pm}$ includes a passage near $m_{\pm}$. 
The slow divergence integral on a Möbius band

Transition maps

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- The system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 1–periodic orbit passing through the point $u \in \Sigma_+$ if and only if the following holds: $\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon})$. 
We define now the following transition maps for $(\bar{\epsilon}, B, \lambda) \sim (0, 0, 0)$:

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The map $\Delta_{\pm}$ includes a passage near $m_{\pm}$.

The system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 1–periodic orbit passing through the point $u \in \Sigma_+$ if and only if the following holds:

$\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon})$.

Similarly, the system $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 2–periodic orbit passing through the points $u, u' \in \Sigma_+$, with $u \neq u'$, if and only if the following holds: $\Delta_-(u, B, \lambda, \bar{\epsilon}) = \Delta_+(u', B, \lambda, \bar{\epsilon})$ and $\Delta_-(u', B, \lambda, \bar{\epsilon}) = \Delta_+(u, B, \lambda, \bar{\epsilon})$. 
Transition maps

For a regular slow dynamics, the study of the transition maps relies on [Dumortier, Roussarie, 1996]. The following theorem gives the structure of $\Delta_{\pm}$.

Theorem

There exist $\bar{\epsilon}$-regularly smooth functions $\bar{l}_{\pm}$ in $(u, B, \lambda)$ and $\bar{\epsilon}$-regularly smooth functions $f_{\pm}$ in $(B, \lambda)$ such that $\bar{l}_{\pm}(u, B, \lambda, 0) = l_{\pm}(u, 0, \lambda)$, with $l_{\pm}$ defined in (3), and such that

$$
\Delta_{\pm}(u, B, \lambda, \bar{\epsilon}) = f_{\pm}(B, \lambda, \bar{\epsilon}) \pm \exp \left( \frac{\bar{l}_{\pm}(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right). \tag{6}
$$

Furthermore, $f(0, \lambda, 0) = 0$ and $\frac{\partial f}{\partial B}(0, \lambda, 0) \neq 0$ where $f(B, \lambda, \bar{\epsilon}) := f_{-}(B, \lambda, \bar{\epsilon}) - f_{+}(B, \lambda, \bar{\epsilon})$. 
Transition maps

The following theorem gives the structure of the transition map $\Delta_-$ ([De Maesschalck, Dumortier, 2008, Huzak, De Maesschalck, Dumortier, 2013]).

**Theorem**

For all $k > 0$ there exists $\bar{\epsilon}_k > 0$ so that $\Delta_-$ is $C^\infty$ on $U_- \cap \{\bar{\epsilon} \leq \bar{\epsilon}_k\}$ and has a $C^k$-extension to the closure of $U_- \cap \{\bar{\epsilon} \leq \bar{\epsilon}_k\}$. Furthermore,

$$
\frac{\partial \Delta_-}{\partial u}(u, B, \lambda, \bar{\epsilon}) = -\exp \left( \frac{I_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right),
$$

(7)

where $(u, B, \lambda, \bar{\epsilon}) \in U_- \cap \{\bar{\epsilon} \leq \bar{\epsilon}_k\}$, $I_-$ is $\bar{\epsilon}$-regularly $C^k$ in $(u, B, \lambda)$, $I_-(u, B, \lambda, \bar{\epsilon}) \to -\infty$ as $(u, B, \lambda, \bar{\epsilon}) \to (u_0, 0, 0, 0)$ and $\frac{\partial I_-}{\partial u}(u, B, \lambda, \bar{\epsilon}) > 0$. 

Renato Huzak
Using Theorem 9, the equation for 1-limit cycles can be written as:

\[
\exp\left(\frac{\bar{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right) + \exp\left(\frac{\bar{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2}\right) = f(B, \lambda, \bar{\epsilon}), \quad (8)
\]
The slow divergence integral on a Möbius band

Transition maps

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\exp \left( \frac{\bar{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) = f(B, \lambda, \bar{\epsilon}), \quad (8)
\]

- and the system for 2-limit cycles can be written as:

\[
\begin{align*}
\exp \left( \frac{\bar{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) &= f(B, \lambda, \bar{\epsilon}) \\
\exp \left( \frac{\bar{I}_-(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) &= f(B, \lambda, \bar{\epsilon}).
\end{align*}
\]

(9)
Transition maps

Instead of working with (9), it is sometimes more convenient to use the equation for the fixed points
\[ \{ P_{B,\lambda,\bar{\epsilon}} \circ P_{B,\lambda,\bar{\epsilon}}(u) = u \}, \]
where \( P_{B,\lambda,\bar{\epsilon}}(u) = \Delta_{+}^{-1} \circ \Delta_{-}(u) \) is the 1–return map, or to use the difference equation
\[ \{ \Delta_{B,\lambda,\bar{\epsilon}}(u) = 0 \} \]
where \( \Delta_{B,\lambda,\bar{\epsilon}}(u) = P_{B,\lambda,\bar{\epsilon}}(u) - P_{B,\lambda,\bar{\epsilon}}^{-1}(u) \).
Proof of Theorem 3

Let \( I \) be nonzero near \((u, \mu) = (u_0, 0, 0)\) (i.e. \( I(u_0, 0, 0) \neq I(0, 0, 0) \)).
Proof of Theorem 3

Let $I$ be nonzero near $(u, \mu) = (u_0, 0, 0)$ (i.e. $I_-(u_0, 0, 0) \neq I_+(u_0, 0, 0)$). Let us suppose that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 2–periodic orbit intersecting $\Sigma_+$ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_0$, with $\bar{u} < \tilde{u}$. 

Renato Huzak
Proof of Theorem 3

Let $l$ be nonzero near $(u, \mu) = (u_0, 0, 0)$ (i.e. $l_-(u_0, 0, 0) \neq l_+(u_0, 0, 0)$). Let us suppose that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has a 2–periodic orbit intersecting $\Sigma_+$ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_0$, with $\bar{u} < \tilde{u}$.

Then $\Delta_{B, \lambda, \bar{\epsilon}}(\bar{u}) = \Delta_{B, \lambda, \bar{\epsilon}}(\tilde{u}) = 0$, $P_{B, \lambda, \bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B, \lambda, \bar{\epsilon}}(\tilde{u}) = \bar{u}$ and $P_{B, \lambda, \bar{\epsilon}}([\bar{u}, \tilde{u}]) = [\bar{u}, \tilde{u}]$. 
Proof of Theorem 3

Let $I$ be nonzero near $(u, \mu) = (u_0, 0, 0)$ (i.e. $I_-(u_0, 0, 0) \neq I_+(u_0, 0, 0)$). Let us suppose that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon} B, \lambda}$ has a 2–periodic orbit intersecting $\Sigma_+$ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_0$, with $\bar{u} < \tilde{u}$.

Then $\Delta_{B, \lambda, \bar{\epsilon}}(\bar{u}) = \Delta_{B, \lambda, \bar{\epsilon}}(\tilde{u}) = 0$, $P_{B, \lambda, \bar{\epsilon}}(\bar{u}) = \tilde{u}$, $P_{B, \lambda, \bar{\epsilon}}(\tilde{u}) = \bar{u}$ and $P_{B, \lambda, \bar{\epsilon}}([\bar{u}, \tilde{u}]) = [\bar{u}, \tilde{u}]$.

The derivative of $\Delta_{B, \lambda, \bar{\epsilon}}$ can be written as:

$$
\Delta'_{B, \lambda, \bar{\epsilon}}(u) = - \exp \left( \frac{l_-(u) - l_+(P_{B, \lambda, \bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2} \right) + \exp \left( \frac{l_+(u) - l_-(P_{B, \lambda, \bar{\epsilon}}^{-1}(u)) + o(1)}{\bar{\epsilon}^2} \right), \quad u \in [\bar{u}, \tilde{u}].
$$
Proof of Theorem 3

This implies that the equation \( \{ \Delta'_B, \lambda, \bar{\epsilon} = 0 \} \) is equivalent, for \( \bar{\epsilon} > 0 \) and \( u \in [\bar{u}, \tilde{u}] \), to the following equation:

\[
I_-(u) - I_+(P_B, \lambda, \bar{\epsilon}(u)) + I_-(P_B^{-1}, \lambda, \bar{\epsilon}(u)) - I_+(u) + o(1) = 0, \quad (10)
\]

for a new \( o(1) \)-term.
Proof of Theorem 3

This implies that the equation \( \{ \Delta'_{B,\lambda,\bar{\epsilon}} = 0 \} \) is equivalent, for \( \bar{\epsilon} > 0 \) and \( u \in [\bar{u}, \tilde{u}] \), to the following equation:

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\]

for a new \( o(1) \)-term.

Since \( I_\pm \) are smooth and \( u, P_B,\lambda,\bar{\epsilon}(u), P_B^{-1},\lambda,\bar{\epsilon}(u) \approx u_0 \) for all \( u \in [\bar{u}, \tilde{u}] \), we have:

\[
I_-(u) - I_+(P_B,\lambda,\bar{\epsilon}(u)) + I_-(P_B^{-1},\lambda,\bar{\epsilon}(u)) - I_+(u) \\
\approx I_-(u_0) - I_+(u_0) + I_-(u_0) - I_+(u_0) \\
= 2(I_-(u_0) - I_+(u_0)) \neq 0,
\]

for \( u \in [\bar{u}, \tilde{u}] \).
Proof of Theorem 4

Let \( I(u_0, 0, 0) = 0 \) and \( \frac{\partial I}{\partial u}(u_0, 0, 0) \neq 0 \). The 1–return map \( P_{B, \lambda, \bar{\epsilon}} \) fulfills the conditions of the following theorem:

Theorem (period doubling bifurcation)

Let \( p_B : \mathbb{R} \to \mathbb{R} \) be a smooth one-parameter family of mappings such that \( p_{B_0} \) has a fixed point \( x_0 \) with eigenvalue \(-1\). Assume

(PD1) \[ \frac{\partial p}{\partial B} \frac{\partial^2 p}{\partial x^2} + 2 \frac{\partial^2 p}{\partial x \partial B} \neq 0 \text{ at } (x, B) = (x_0, B_0); \]

(PD2) \[ a := \frac{1}{2} \left( \frac{\partial^2 p}{\partial x^2} \right)^2 + \frac{1}{3} \frac{\partial^3 p}{\partial x^3} \neq 0 \text{ at } (x, B) = (x_0, B_0). \]

Then there is a smooth curve of fixed points of \( p_B \) passing through \((x_0, B_0)\), the stability of which changes at \((x_0, B_0)\). There is also a smooth curve \( \gamma \) passing through \((x_0, B_0)\) so that \( \gamma \setminus \{(x_0, B_0)\} \) is a union of hyperbolic period 2 orbits. The curve \( \gamma \) has a quadratic tangency with the line \( B = B_0 \) at \((x_0, B_0)\). If \( a \) is positive (resp. negative), the period 2 orbits are attracting (resp. repelling).
Proof of Theorem 4

The derivative of $P_{B,\lambda,\bar{\epsilon}}$ w.r.t. $u$ is given by

$$\frac{\partial P_{B,\lambda,\bar{\epsilon}}(u)}{\partial u} = \frac{\partial \Delta_-}{\partial u}(u, B, \lambda, \bar{\epsilon}),$$

with

$$\frac{\partial \Delta_{\pm}}{\partial u}(u, B, \lambda, \bar{\epsilon}) = \pm \exp \left( \frac{\hat{I}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right)$$

where functions $\hat{I}_\pm$ are $\bar{\epsilon}$-regularly smooth in $(u, B, \lambda)$ and $\hat{I}_\pm(u, B, \lambda, 0) = I_\pm(u, 0, \lambda)$. 

\[ \frac{\partial \Delta_{\pm}}{\partial u} \]
Proof of Theorem 4

The derivative of $P_{B,\lambda,\bar{\epsilon}}$ w.r.t. $u$ is given by

$$\frac{\partial P_{B,\lambda,\bar{\epsilon}}(u)}{\partial u}(u) = \frac{\partial \Delta_-(u, B, \lambda, \bar{\epsilon})}{\partial u}(u, B, \lambda, \bar{\epsilon}) = \frac{\partial \Delta_+(P_{B,\lambda,\bar{\epsilon}}(u), B, \lambda, \bar{\epsilon})}{\partial u},$$

(11)

with

$$\frac{\partial \Delta_\pm(u, B, \lambda, \bar{\epsilon})}{\partial u} = \pm \exp \left( \frac{\hat{l}_\pm(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right)$$

where functions $\hat{l}_\pm$ are $\bar{\epsilon}$-regularly smooth in $(u, B, \lambda)$ and $\hat{l}_\pm(u, B, \lambda, 0) = l_\pm(u, 0, \lambda)$.

Since the function $u \rightarrow l_-(u, 0, 0) - l_+(u, 0, 0)$ has a simple zero at $u = u_0$, $f(0, 0, 0) = 0$ and $\frac{\partial f}{\partial B}(0, 0, 0) \neq 0$, we can apply the Implicit Function Theorem to the following $\bar{\epsilon}$-regularly smooth in $(u, B, \lambda)$ system:
Proof of Theorem 4

\[ \begin{align*}
\Delta_-(u, B, \lambda, \bar{\epsilon}) - \Delta_+(u, B, \lambda, \bar{\epsilon}) &= 0 \\
\hat{I}_-(u, B, \lambda, \bar{\epsilon}) - \hat{I}_+(u, B, \lambda, \bar{\epsilon}) &= 0,
\end{align*} \]

and find a solution \((\lambda, \bar{\epsilon}) \to (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))\), \(\bar{\epsilon}\)-regularly smooth in \(\lambda\), with \(u(0, 0) = u_0\) and \(B(0, 0) = 0\).
Proof of Theorem 4

\[
\begin{align*}
\Delta_- (u, B, \lambda, \bar{\epsilon}) - \Delta_+ (u, B, \lambda, \bar{\epsilon}) &= 0 \\
\hat{I}_- (u, B, \lambda, \bar{\epsilon}) - \hat{I}_+ (u, B, \lambda, \bar{\epsilon}) &= 0,
\end{align*}
\]

and find a solution \((\lambda, \bar{\epsilon}) \rightarrow (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))\), \(\bar{\epsilon}\)-regularly smooth in \(\lambda\), with \(u(0, 0) = u_0\) and \(B(0, 0) = 0\).

From this and (11) follows

\[
P_B(\lambda, \bar{\epsilon}, \lambda, \bar{\epsilon}) u(\lambda, \bar{\epsilon})) = u(\lambda, \bar{\epsilon})\text{ and } \frac{\partial P_B(\lambda, \bar{\epsilon}, \lambda, \bar{\epsilon})}{\partial u}(u(\lambda, \bar{\epsilon})) = -1,
\]

for all \((\lambda, \bar{\epsilon}) \sim (0, 0)\) and \(\bar{\epsilon} > 0\).
Proof of Theorem 4

\[
\begin{aligned}
\Delta_-(u, B, \lambda, \bar{\epsilon}) - \Delta_+(u, B, \lambda, \bar{\epsilon}) &= 0 \\
\hat{I}_-(u, B, \lambda, \bar{\epsilon}) - \hat{I}_+(u, B, \lambda, \bar{\epsilon}) &= 0,
\end{aligned}
\]

and find a solution \((\lambda, \bar{\epsilon}) \rightarrow (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))\), \(\bar{\epsilon}\)-regularly smooth in \(\lambda\), with \(u(0, 0) = u_0\) and \(B(0, 0) = 0\).

From this and (11) follows

\[
P_{B(\lambda, \bar{\epsilon}), \lambda, \bar{\epsilon}}(u(\lambda, \bar{\epsilon})) = u(\lambda, \bar{\epsilon}) \quad \text{and} \quad \frac{\partial P_{B(\lambda, \bar{\epsilon}), \lambda, \bar{\epsilon}}}{\partial u}(u(\lambda, \bar{\epsilon})) = -1,
\]

for all \((\lambda, \bar{\epsilon}) \sim (0, 0)\) and \(\bar{\epsilon} > 0\).

Thus, for each \((\lambda, \bar{\epsilon}) \sim (0, 0)\) and \(\bar{\epsilon} > 0\), \(P_{B(\lambda, \bar{\epsilon}), \lambda, \bar{\epsilon}}\) has a fixed point \(u(\lambda, \bar{\epsilon})\) with eigenvalue \(-1\).
The quantity (PD1) becomes:

\[
\frac{\partial (\Delta_+-\Delta_+)}{\partial B}(u) \left( \frac{\partial I_+}{\partial u} (u) - \frac{\partial I_-}{\partial u} (u) \right) + o(1) \nonumber
\]

where \((u, B) = (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon}))\).
Proof of Theorem 4

- The quantity (PD1) becomes:

$$\frac{\partial(\Delta_- - \Delta_+)}{\partial B}(u) \left( \frac{\partial I_-}{\partial u}(u) - \frac{\partial I_+}{\partial u}(u) \right) + o(1)$$

$$\bar{\varepsilon}^2 \frac{\partial \Delta_-}{\partial u}(u),$$

where \((u, B) = (u(\lambda, \bar{\varepsilon}), B(\lambda, \bar{\varepsilon}))\).

- The quantity (PD2) becomes

$$a = \frac{\left( \frac{\partial I_-}{\partial u}(u) \right)^2 - \left( \frac{\partial I_+}{\partial u}(u) \right)^2 + o(1)}{6\bar{\varepsilon}^4}, \quad (u, B) = (u(\lambda, \bar{\varepsilon}), B(\lambda, \bar{\varepsilon}))$$.
The slow divergence integral on a Möbius band

Proof of Theorem 4

Lemma

Let \( m \in \mathbb{N}, m \geq 1 \). Then we have:

\[
\bar{\epsilon}^{2m} \frac{\partial^{m+1} \Delta_{\pm}}{\partial u^{m+1}}(u) = \pm \left( \left( \frac{\partial I_{\pm}}{\partial u}(u) \right)^{m} + o(1) \right) \exp \left( \frac{\hat{I}_{\pm}(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^{2}} \right),
\]

where \( \hat{I}_{\pm}(u, B, \lambda, \bar{\epsilon}) \) are defined after (11), \( I_{\pm}(u) = I_{\pm}(u, 0, \lambda) \) and the \( o(1) \)-term is \( \bar{\epsilon} \)-regularly smooth in \((u, B, \lambda)\).
Proof of Theorem 4

Lemma

Let \( m \in \mathbb{N}, \ m \geq 1 \). Then we have:

\[
\bar{\epsilon}^2 m \frac{\partial^{m+1} \Delta_{\pm}}{\partial u^{m+1}}(u) = \pm \left( \left( \frac{\partial l_{\pm}}{\partial u}(u) \right)^m + o(1) \right) \exp \left( \frac{\hat{I}_{\pm}(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right),
\]

where \( \hat{I}_{\pm}(u, B, \lambda, \bar{\epsilon}) \) are defined after (11), \( l_{\pm}(u) = l_{\pm}(u, 0, \lambda) \) and the \( o(1) \)-term is \( \bar{\epsilon} \)-regularly smooth in \( (u, B, \lambda) \).

Thus, putting all the informations together, we have proved that for each fixed \( (\lambda, \bar{\epsilon}) \sim (0, 0), \ \bar{\epsilon} > 0 \), the \( B \)-family \( X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda} \) undergoes a period doubling bifurcation at 
\( (u, B) = (u(\lambda, \bar{\epsilon}), B(\lambda, \bar{\epsilon})) \).
Proof of Theorem 5

We consider

\[
\begin{align*}
\exp \left( \frac{\bar{I}_-(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2} \right) &= f(B, \lambda, \bar{\epsilon}) \\
\exp \left( \frac{\bar{I}_-(u',B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u,B,\lambda,\bar{\epsilon})}{\bar{\epsilon}^2} \right) &= f(B, \lambda, \bar{\epsilon}).
\end{align*}
\]
Proof of Theorem 5

We consider

\[
\exp \left( \frac{\tilde{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\tilde{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) = f(B, \lambda, \bar{\epsilon})
\]

\[
\exp \left( \frac{\tilde{I}_-(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\tilde{I}_+(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) = f(B, \lambda, \bar{\epsilon}).
\]

The main difficulty lies in the fact that the limit \( \bar{\epsilon} = 0 \) of this system is degenerate. Our goal is, therefore, to replace the system with a new system, non-singular for \( \bar{\epsilon} = 0 \), using [Mamouhdi, Roussarie, 2012].
Proof of Theorem 5

Let us suppose that $\Psi(u, u')$ and $\Phi(u, u')$ are two smooth functions defined on a rectangle $R = [\bar{U}_1, \tilde{U}_1] \times [\bar{U}_2, \tilde{U}_2]$ and let us suppose that $\frac{\partial \Psi}{\partial u}, \frac{\partial \Psi}{\partial u'}, \frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial u'}$ are nonzero for all $(u, u') \in R$. 

We further assume that the equation
\[
\det(\Psi, \Phi)(u, u') = 0
\]
for contact points is equivalent on $R$ to an equation
\[
E(u, u') = 0
\]
where $E$ is a smooth function on $R$, and where $\frac{\partial E}{\partial u}$ and $\frac{\partial E}{\partial u'}$ are nonzero for all $(u, u') \in R$. (Equivalent means $\det(\Psi, \Phi) = F \cdot E$, where the factor $F$ is a smooth nowhere zero function on $R$.)

Now we can define a regular pair of foliations $(\tilde{\Psi}, \tilde{\Phi})$ on $R$ as follows: the curves \{
$\Psi(u, u') = \alpha$
\}(resp. \{
$\Phi(u, u') = \beta$
\}) are the leaves of foliation $\tilde{\Psi}$ (resp. $\tilde{\Phi}$). Each leaf and the curve \{ 
$E(u, u') = 0$
\} are simple connected.
Proof of Theorem 5

Let us suppose that $\Psi(u, u')$ and $\Phi(u, u')$ are two smooth functions defined on a rectangle $R = [\bar{U}_1, \tilde{U}_1] \times [\bar{U}_2, \tilde{U}_2]$ and let us suppose that $\frac{\partial \psi}{\partial u}, \frac{\partial \psi}{\partial u'}, \frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial u'}$ are nonzero for all $(u, u') \in R$.

We further assume that the equation \{\det(\psi, \phi)(u, u') = 0\} for contact points is equivalent on $R$ to an equation \{\text{\textit{E}}(u, u') = 0\}, where $\text{\textit{E}}$ is a smooth function on $R$, and where $\frac{\partial \text{\textit{E}}}{\partial u}$ and $\frac{\partial \text{\textit{E}}}{\partial u'}$ are nonzero for all $(u, u') \in R$. (Equivalent means \det(\psi, \phi) = F.E, where the factor $F$ is a smooth nowhere zero function on $R$.)}
Proof of Theorem 5

- Let us suppose that $\Psi(u, u')$ and $\Phi(u, u')$ are two smooth functions defined on a rectangle $R = [\bar{U}_1, \tilde{U}_1] \times [\bar{U}_2, \tilde{U}_2]$ and let us suppose that $\frac{\partial \Psi}{\partial u}, \frac{\partial \Psi}{\partial u'}, \frac{\partial \Phi}{\partial u}$ and $\frac{\partial \Phi}{\partial u'}$ are nonzero for all $(u, u') \in R$.

- We further assume that the equation $\{\det(\Psi, \Phi)(u, u') = 0\}$ for contact points is equivalent on $R$ to an equation $\{E(u, u') = 0\}$, where $E$ is a smooth function on $R$, and where $\frac{\partial E}{\partial u}$ and $\frac{\partial E}{\partial u'}$ are nonzero for all $(u, u') \in R$. (Equivalent means $\det(\Psi, \Phi) = F\cdot E$, where the factor $F$ is a smooth nowhere zero function on $R$.)

- Now we can define a regular pair of foliations $(\tilde{\Psi}, \tilde{\Phi})$ on $R$ as follows: the curves $\{\Psi(u, u') = \alpha\}$ (resp. $\{\Phi(u, u') = \beta\}$) are the leaves of foliation $\tilde{\Psi}$ (resp. $\tilde{\Phi}$).
Proof of Theorem 5

Let us suppose that \( \Psi(u, u') \) and \( \Phi(u, u') \) are two smooth functions defined on a rectangle \( R = [\bar{U}_1, \tilde{U}_1] \times [\bar{U}_2, \tilde{U}_2] \) and let us suppose that \( \frac{\partial \Psi}{\partial u}, \frac{\partial \Psi}{\partial u'}, \frac{\partial \Phi}{\partial u} \) and \( \frac{\partial \Phi}{\partial u'} \) are nonzero for all \( (u, u') \in R \).

We further assume that the equation \( \{ \text{det}(\Psi, \Phi)(u, u') = 0 \} \) for contact points is equivalent on \( R \) to an equation \( \{ E(u, u') = 0 \} \), where \( E \) is a smooth function on \( R \), and where \( \frac{\partial E}{\partial u} \) and \( \frac{\partial E}{\partial u'} \) are nonzero for all \( (u, u') \in R \). (Equivalent means \( \text{det}(\Psi, \Phi) = F \cdot E \), where the factor \( F \) is a smooth nowhere zero function on \( R \).)

Now we can define a regular pair of foliations \( (\tilde{\Psi}, \tilde{\Phi}) \) on \( R \) as follows: the curves \( \{ \Psi(u, u') = \alpha \} \) (resp. \( \{ \Phi(u, u') = \beta \} \)) are the leaves of foliation \( \tilde{\Psi} \) (resp. \( \tilde{\Phi} \)).

Each leaf and the curve \( \{ E(u, u') = 0 \} \) are simple connected...
Proof of Theorem 5

We relate the number of intersection points of two leaves \( \{\Psi(u, u') = \alpha\} \) and \( \{\Phi(u, u') = \beta\} \) in \( R \) with the number of intersection points of the curve \( \{E(u, u') = 0\} \) and one of these two leaves in \( R \).

Lemma

Let \( (\widetilde{\Psi}, \widetilde{\Phi}) \) be a regular pair of foliations on \( R \) as defined above and let \( \alpha, \beta \in \mathbb{R} \) be arbitrary but fixed. Let \( N(\alpha, \beta) \) be the number of intersection points of \( \{\Psi(u, u') = \alpha\} \) with \( \{\Phi(u, u') = \beta\} \) in \( R \), counting multiplicity, and let \( N(\beta) \) be the number of intersection points of \( \{E(u, u') = 0\} \) with \( \{\Phi(u, u') = \beta\} \) in \( R \), counting multiplicity. If \( N(\beta) \) is finite, then

\[
N(\alpha, \beta) \leq N(\beta) + 1. \quad (13)
\]
The slow divergence integral on a Möbius band

Proof of Theorem 5

To find at most 3 solutions of the system in $[\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$, for each $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$, we use the lemma twice. The system (9) is a special case of the more general system

$$
\begin{align*}
\exp \left( \frac{\bar{I}_-(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) &= \alpha \\
\exp \left( \frac{\bar{I}_-(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) &= \beta
\end{align*}
$$

where $\alpha, \beta \in \mathbb{R}$, and it suffices to prove that (14) has at most 3 solutions in $[\bar{u}, \tilde{u}] \times [\bar{u}, \tilde{u}]$, for each fixed $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, with $\bar{\epsilon} > 0$, and $\alpha, \beta \in \mathbb{R}$. 

Proof of Theorem 5

We denote by $\Psi_{B,\lambda,\bar{\epsilon}}(u, u')$, $\Phi_{B,\lambda,\bar{\epsilon}}(u, u')$ the functions on the left-hand side of (14). We have

$$\det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) = \exp\left(\frac{I_-(u) + I_-(u') + o(1)}{\bar{\epsilon}^2}\right)$$

$$- \exp\left(\frac{I_+(u) + I_+(u') + o(1)}{\bar{\epsilon}^2}\right).$$
Proof of Theorem 5

We denote by $\Psi_{B,\lambda,\bar{\epsilon}}(u, u')$, $\Phi_{B,\lambda,\bar{\epsilon}}(u, u')$ the functions on the left-hand side of (14). We have

$$\det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) = \exp \left( \frac{l_-(u) + l_-(u') + o(1)}{\bar{\epsilon}^2} \right) - \exp \left( \frac{l_+(u) + l_+(u') + o(1)}{\bar{\epsilon}^2} \right).$$

This implies that the set $\{\det(\Psi_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}})(u, u') = 0\}$ of the contact points between the two foliations $\Psi_{B,\lambda,\bar{\epsilon}}$ and $\Phi_{B,\lambda,\bar{\epsilon}}$ is equivalent for $\bar{\epsilon} > 0$ to $\{E_{B,\lambda,\bar{\epsilon}}(u, u') = 0\}$ with

$$E_{B,\lambda,\bar{\epsilon}}(u, u') = l_-(u) - l_+(u') + l_-(u') - l_+(u) + o(1).$$
Proof of Theorem 5

We define the following system:

\[
\begin{cases}
E_{B,\lambda,\bar{\epsilon}}(u, u') = I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0 \\
\Phi_{B,\lambda,\bar{\epsilon}}(u, u') = \exp \left( \frac{I_-(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{I_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) = \beta.
\end{cases}
\] (15)
Proof of Theorem 5

We define the following system:

\[
\begin{align*}
E_{B,\lambda,\bar{\epsilon}}(u, u') &= I_-(u) - I_+(u') + I_-(u') - I_+(u) + o(1) = 0 \\
\Phi_{B,\lambda,\bar{\epsilon}}(u, u') &= \exp \left( \frac{\bar{I}_-(u', B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) + \exp \left( \frac{\bar{I}_+(u, B, \lambda, \bar{\epsilon})}{\bar{\epsilon}^2} \right) = \beta.
\end{align*}
\]

(15)

Following Lemma 13, if we denote by \( \mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha, \beta) \) (resp. \( \mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta) \)) the number of solutions of (14) (resp. (15)), counting multiplicity, in \([\bar{u}, \tilde{u}] \times [\tilde{u}, \bar{u}]\), then

\[
\mathcal{N}_{B,\lambda,\bar{\epsilon}}(\alpha, \beta) \leq 1 + \mathcal{N}_{B,\lambda,\bar{\epsilon}}(\beta).
\]
Proof of Theorem 5

We have

$$\det(E_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) =$$

$$\left( \frac{\partial I_-}{\partial u} (u) - \frac{\partial I_+}{\partial u} (u) + o(1) \right) \exp \left( \frac{I_-(u') + o(1)}{\bar{\epsilon}^2} \right)$$

$$- \left( \frac{\partial I_-}{\partial u} (u') - \frac{\partial I_+}{\partial u} (u') + o(1) \right) \exp \left( \frac{I_+(u) + o(1)}{\bar{\epsilon}^2} \right).$$
Proof of Theorem 5

We have

$$\det(E_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}}) =$$

$$\left(\frac{\partial I_-}{\partial u} (u) - \frac{\partial I_+}{\partial u} (u) + o(1)\right) \exp \left(\frac{I_-(u') + o(1)}{\bar{\epsilon}^2}\right)$$

$$- \left(\frac{\partial I_-}{\partial u} (u') - \frac{\partial I_+}{\partial u} (u') + o(1)\right) \exp \left(\frac{I_+(u) + o(1)}{\bar{\epsilon}^2}\right).$$

Clearly, the equation \(\{\det(E_{B,\lambda,\bar{\epsilon}}, \Phi_{B,\lambda,\bar{\epsilon}})(u, u') = 0\}\) is equivalent for \(\bar{\epsilon} > 0\) to \(\{\bar{E}_{B,\lambda,\bar{\epsilon}}(u, u') = 0\}\) where

$$\bar{E}_{B,\lambda,\bar{\epsilon}}(u, u') = I_-(u') - I_+(u) + o(1).$$
Proof of Theorem 5

Lemma 13 implies that

\[ \mathcal{N}_{B, \lambda, \bar{\epsilon}}(\beta) \leq 1 + \mathcal{N}_{B, \lambda, \bar{\epsilon}}, \]

where \( \mathcal{N}_{B, \lambda, \bar{\epsilon}} \) is the number of solutions (counting multiplicity) of the system \( \{ I_- (u) - I_+ (u') + I_- (u') - I_+ (u) + o(1) = 0, I_- (u') - I_+ (u) + o(1) = 0 \} \), or equivalently the system

\[
\begin{align*}
  I_- (u) - I_+ (u') + o(1) &= 0 \\
  I_- (u') - I_+ (u) + o(1) &= 0.
\end{align*}
\] (16)

Thus, we have proved that

\[ \mathcal{N}_{B, \lambda, \bar{\epsilon}}(\alpha, \beta) \leq 2 + \mathcal{N}_{B, \lambda, \bar{\epsilon}}, \]

for each \( (B, \lambda, \bar{\epsilon}) \sim (0, 0, 0) \), with \( \bar{\epsilon} > 0 \), and \( \alpha, \beta \in \mathbb{R} \).
Proof of Theorem 6

Assume that $f(\omega(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)]$. 
Proof of Theorem 6

Assume that $f(\omega(u_0), 0) = 0$, $\frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0$ and $f(x, 0) < 0$ for all $x \in [\alpha(u_0), \omega(u_0)]$.

First, let us prove that $\text{Cycl}(X_{\epsilon, \mu}, L_{u_0}) \leq 1$, i.e. there are no 2-periodic orbits Hausdorff close to $L_{u_0}$. 
Proof of Theorem 6

Assume that \( f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0 \) and \( f(x, 0) < 0 \) for all \( x \in [\alpha(u_0), \omega(u_0)] \).

First, let us prove that \( \text{Cycl}(X_{\epsilon, \mu}, L_{u_0}) \leq 1 \), i.e. there are no 2-periodic orbits Hausdorff close to \( L_{u_0} \).

Suppose, on the contrary, that for \((B, \lambda, \bar{\epsilon}) \sim (0, 0, 0), \bar{\epsilon} > 0\), \( X_{\bar{\epsilon}^2, \bar{\epsilon} B, \lambda} \) has a 2–periodic orbit intersecting \( \Sigma_+ \) in two points \( \bar{u} \sim u_0 \) and \( \tilde{u} \sim u_0 \), with \( u(B, \lambda, \bar{\epsilon}) < \bar{u} < \tilde{u} \).
Proof of Theorem 6

Assume that \( f(\omega(u_0), 0) = 0, \frac{\partial f}{\partial x}(\omega(u_0), 0) \neq 0 \) and \( f(x, 0) < 0 \) for all \( x \in [\alpha(u_0), \omega(u_0)] \).

First, let us prove that \( \text{Cycl}(X_{\epsilon, \mu}, L_{u_0}) \leq 1 \), i.e. there are no 2-periodic orbits Hausdorff close to \( L_{u_0} \).

Suppose, on the contrary, that for \( (B, \lambda, \bar{\epsilon}) \sim (0, 0, 0), \bar{\epsilon} > 0 \), \( X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda} \) has a 2–periodic orbit intersecting \( \Sigma_+ \) in two points \( \bar{u} \sim u_0 \) and \( \bar{u} \sim u_0 \), with \( u(B, \lambda, \bar{\epsilon}) < \bar{u} < \tilde{u} \).

We have for \( u \in [\bar{u}, \tilde{u}] \):

\[
\Delta'_{B, \lambda, \bar{\epsilon}}(u) = - \exp \left( \frac{\mathcal{I}_-(u, B, \lambda, \bar{\epsilon}) - l_+(P_{B, \lambda, \bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2} \right)
+ \exp \left( \frac{l_+(u) - \mathcal{I}_-(P_{B, \lambda, \bar{\epsilon}}^{-1}(u), B, \lambda, \bar{\epsilon}) + o(1)}{\bar{\epsilon}^2} \right).
\]
Proof of Theorem 7

Let \( u_0, u_1 \in \Sigma_+ \), with \( u_0 < u_1 \), be arbitrary but fixed, and let us suppose that \( T(u_0, u_1, 0) \neq 0 \), where \( T \) is the total slow divergence integral.
Proof of Theorem 7

Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed, and let us suppose that $T(u_0, u_1, 0) \neq 0$, where $T$ is the total slow divergence integral.

Suppose, on the contrary, that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has two 2-periodic orbits, one intersecting $\Sigma_+$ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_1$, and the other in $\tilde{u} \sim u_0$ and $\tilde{\tilde{u}} \sim u_1$. 
Proof of Theorem 7

- Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed, and let us suppose that $T(u_0, u_1, 0) \neq 0$, where $T$ is the total slow divergence integral.

- Suppose, on the contrary, that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0), \bar{\epsilon} > 0,$ $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has two 2–periodic orbits, one intersecting $\Sigma_+$ in two points $\tilde{u} \sim u_0$ and $\tilde{\bar{u}} \sim u_1$, and the other in $\tilde{u} \sim u_0$ and $\tilde{\bar{u}} \sim u_1$.

- Then we have $\bar{u} < \tilde{\bar{u}} < \tilde{u} < \bar{u}$ or $\tilde{\bar{u}} < \bar{u} < \tilde{u} < \tilde{\bar{u}}.$
The slow divergence integral on a Möbius band

Proof of Theorem 7

Let $u_0, u_1 \in \Sigma_+$, with $u_0 < u_1$, be arbitrary but fixed, and let us suppose that $T(u_0, u_1, 0) \neq 0$, where $T$ is the total slow divergence integral.

Suppose, on the contrary, that for $(B, \lambda, \bar{\epsilon}) \sim (0, 0, 0)$, $\bar{\epsilon} > 0$, $X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda}$ has two 2–periodic orbits, one intersecting $\Sigma_+$ in two points $\bar{u} \sim u_0$ and $\tilde{u} \sim u_1$, and the other in $\bar{u} \sim u_0$ and $\tilde{u} \sim u_1$.

Then we have $\bar{u} < \bar{u} < \tilde{u} < \tilde{u}$ or $\bar{u} < \bar{u} < \tilde{u} < \tilde{u}$.

Suppose without loss of generality that $\bar{u} < \bar{u} < \tilde{u} < \tilde{u}$.
Proof of Theorem 7

- Let \( u_0, u_1 \in \Sigma_+ \), with \( u_0 < u_1 \), be arbitrary but fixed, and let us suppose that \( T(u_0, u_1, 0) \neq 0 \), where \( T \) is the total slow divergence integral.

- Suppose, on the contrary, that for \( (B, \lambda, \bar{\epsilon}) \sim (0, 0, 0), \bar{\epsilon} > 0 \), \( X_{\bar{\epsilon}^2, \bar{\epsilon}B, \lambda} \) has two 2–periodic orbits, one intersecting \( \Sigma_+ \) in two points \( \bar{u} \sim u_0 \) and \( \tilde{u} \sim u_1 \), and the other in \( \tilde{u} \sim u_0 \) and \( \tilde{\tilde{u}} \sim u_1 \).

- Then we have \( \bar{u} < \tilde{u} < \tilde{\tilde{u}} < \tilde{\tilde{u}} \) or \( \bar{u} < \bar{u} < \tilde{\tilde{u}} < \tilde{\tilde{u}} \).

- Suppose without loss of generality that \( \bar{u} < \bar{u} < \tilde{\tilde{u}} < \tilde{\tilde{u}} \). Then
  \[
  \Delta_{B, \lambda, \bar{\epsilon}}(\bar{u}) = \Delta_{B, \lambda, \bar{\epsilon}}(\tilde{u}) = 0, \quad P_{B, \lambda, \bar{\epsilon}}(\bar{u}) = \tilde{u}, \quad P_{B, \lambda, \bar{\epsilon}}(\tilde{u}) = \tilde{\tilde{u}},
  \]
  \[
  P_{B, \lambda, \bar{\epsilon}}([\bar{u}, \tilde{u}]) = [\tilde{u}, \tilde{\tilde{u}}] \quad \text{and} \quad P_{B, \lambda, \bar{\epsilon}}^{-1}([\tilde{u}, \tilde{\tilde{u}}]) = [\tilde{u}, \tilde{\tilde{u}}].
  \]
Proof of Theorem 7

We get

\[ \Delta'_{B, \lambda, \bar{\epsilon}}(u) = - \exp \left( \frac{I_-(u) - I_+(P_{B, \lambda, \bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2} \right) \]

\[ + \exp \left( \frac{I_+(u) - I_-(P^{-1}_{B, \lambda, \bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2} \right), \]

where \( u \in [\bar{u}, \bar{u}] \).
Proof of Theorem 7

- We get

$$
\Delta'_{B, \lambda, \bar{\epsilon}}(u) = - \exp \left( \frac{l_-(u) - l_+(P_{B, \lambda, \bar{\epsilon}}(u)) + o(1)}{\bar{\epsilon}^2} \right) + \exp \left( \frac{l_+(u) - l_-(P_{B, \lambda, \bar{\epsilon}}^{-1}(u)) + o(1)}{\bar{\epsilon}^2} \right),
$$

where $u \in [\bar{u}, \bar{u}]$.

- The equation $\{\Delta'_{B, \lambda, \bar{\epsilon}}(u) = 0\}$ is equivalent for $\bar{\epsilon} > 0$ and $u \in [\bar{u}, \bar{u}]$ to an equation given in (10). Since $T(u_0, u_1, 0) \neq 0$, $u \sim u_0$, $P_{B, \lambda, \bar{\epsilon}}(u)$, $P_{B, \lambda, \bar{\epsilon}}^{-1}(u) \sim u_1$ for all $u \in [\bar{u}, \bar{u}]$, (10) has no solutions w.r.t. $u \in [\bar{u}, \bar{u}]$. 

Renato Huzak

The slow divergence integral on a Möbius band
Proof of Theorem 7

We get

\[ \Delta_B', \lambda, \bar{\epsilon}(u) = - \exp \left( \frac{l_-(u) - l_+(P_B, \lambda, \bar{\epsilon}(u)) + o(1)}{\bar{\epsilon}^2} \right) \]

\[ + \exp \left( \frac{l_+(u) - l_-(P_B^{-1}, \lambda, \bar{\epsilon}(u)) + o(1)}{\bar{\epsilon}^2} \right), \]

where \( u \in [\bar{u}, \bar{\bar{u}}] \).

The equation \( \{\Delta_B', \lambda, \bar{\epsilon}(u) = 0\} \) is equivalent for \( \bar{\epsilon} > 0 \) and \( u \in [\bar{u}, \bar{\bar{u}}] \) to an equation given in (10). Since \( T(u_0, u_1, 0) \neq 0, u \sim u_0, P_B, \lambda, \bar{\epsilon}(u), P_B^{-1}, \lambda, \bar{\epsilon}(u) \sim u_1 \) for all \( u \in [\bar{u}, \bar{\bar{u}}] \), (10) has no solutions w.r.t. \( u \in [\bar{u}, \bar{\bar{u}}] \).

This is a contradiction with \( \Delta_B', \lambda, \bar{\epsilon}(u') = 0 \). Thus, \( \text{Cycl}(X_{\epsilon, \mu}, L_{u_0, u_1}) \leq 1 \).
Future research

- The cyclicity of 1- and 2-canard cycles in the case of higher multiplicity zero in the slow divergence integral
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- The cyclicity of 1– and 2-canard cycles in the case of higher multiplicity zero in the slow divergence integral
- The cyclicity of 1– and 2-canard cycles if the slow dynamics has a hyperbolic saddle or any singularity at both corner points
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- The cyclicity of 1– and 2-canard cycles if the slow dynamics has singularities between the corner points, located away from the contact point (a generic contact point), [De Maesschalck, Dumortier, 2008]
Future research

- The cyclicity of 1– and 2-canard cycles in the case of higher multiplicity zero in the slow divergence integral.
- The cyclicity of 1– and 2-canard cycles if the slow dynamics has a hyperbolic saddle or any singularity at both corner points.
- The cyclicity of 1– and 2-canard cycles if the slow dynamics has singularities between the corner points, located away from the contact point (a generic contact point), [De Maesschalck, Dumortier, 2008].
- The cyclicity of 1– and 2-canard cycles if the slow dynamics has singularities including at the contact point (non-generic contact point).