Quartic Liénard equations with linear damping

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Motivation

- A simplified version of Hilbert's 16th problem deals with finding the maximum number $I_{n,m}$ of limit cycles of a polynomial Liénard equation

$$\left\{\begin{array}{lcl} \dot{x} & = & y \\ \dot{y} & = & -y\sum_{j=0}^{n} a_j x^j - \sum_{j=0}^{m} b_j x^j, \end{array}\right. \tag{1}$$

where $(a_0, \ldots, a_n, b_0, \ldots, b_m) \in \mathbb{R}^{n+m+2}$ and $a_n, b_m \neq 0$.

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  where $(a_0, \ldots, a_n, b_0, \ldots, b_m) \in \mathbb{R}^{n+m+2}$ and $a_n, b_m \neq 0$.
- When $m = 1$ (resp. $m > 1$) we call (1) a classical Liénard equation (resp. a generalized Liénard equation).

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  where $(a_0, \ldots, a_n, b_0, \ldots, b_m) \in \mathbb{R}^{n+m+2}$ and $a_n, b_m \neq 0$.
- When $m = 1$ (resp. $m > 1$) we call (1) a classical Liénard equation (resp. a generalized Liénard equation).
- In the classical case, we know that $l_{1,1} = 0$, $l_{2,1} = 1$ (see [Lins, De Melo,Pugh,1977]) and $l_{3,1} = 1$ (see [Li,Llibre,2012]).

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- *The goal of our presentation is to show that* $l_{1,4} = 2$, *under condition that (1) with* $(n, m) = (1, 4)$ *is of singular type.*
- Our focus is on the quartic Liénard equation with linear damping

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -y(a_0 + x) - (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + x^4), \end{cases} \tag{2}$$

where $(a_0, b_0, b_1, b_2, b_3) \sim (0, 0, 0, 0, 0)$.

Statement of results

Theorem

There exists a small neighborhood V of the origin in the parameter space $(a_0, b_0, b_1, b_2, b_3)$ such that (2) has at most two limit cycles for each $(a_0, b_0, b_1, b_2, b_3) \in V$.

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- We also study the slow-fast version of (2): $\{\dot{x} = y, \dot{y} = -y(a_0 + x) - \epsilon(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + x^4)\}$, where $\epsilon \sim 0$, $\epsilon > 0$ and $(a_0, b_0, b_1, b_2, b_3) \sim (0, 0, 0, 0, 0)$.

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Theorem

There exists a small $\epsilon_0 > 0$ and a small neighborhood V of the origin in the parameter space $(a_0, b_0, b_1, b_2, b_3)$ such that the slow-fast system has at most two limit cycles for each $(\epsilon, a_0, b_0, b_1, b_2, b_3) \in [0, \epsilon_0] \times V$.

Proof of Theorem 1 and Theorem 2

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  2. We study the slow-fast Liénard equation in arbitrarily large compact sets in the phase space by using singular perturbation theory and the family blow-up.
  3. We study the slow-fast Liénard equation near infinity by using an appropriate Poincaré-Lyapunov compactification.

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Step 1–Bringing the Liénard equation (2) to a slow-fast system

- We may assume that $a_0 = 0$ in (2):

$$\begin{cases} \dot{x} &= y \\ \dot{y} &= -yx - (b_0 + b_1 x + b_2 x^2 + b_3 x^3 + x^4), \end{cases} \tag{3}$$

  with a new parameter $(b_0, b_1, b_2, b_3) \sim (0,0,0,0)$. We denote the system (3) by X_{b_0,b_1,b_2,b_3}.

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- Using a linear coordinate change $(x, y) = (\epsilon \bar{x}, \epsilon^2 \bar{y})$, with $\epsilon > 0$ and $\epsilon \sim 0$, we convert the system $X_{\epsilon^4 B_0, \epsilon^3 B_1, \epsilon^2 B_2, \epsilon B_3}$ to

$$\left\{ \begin{array}{rcl} \dot{\bar{x}} & = & \epsilon \bar{y} \\ \dot{\bar{y}} & = & \epsilon\big(- \bar{y}\bar{x} - \epsilon(B_0 + B_1 \bar{x} + B_2 \bar{x}^2 + B_3 \bar{x}^3 + \bar{x}^4)\big), \end{array} \right. \tag{4}$$

  where $(B_0, B_1, B_2, B_3) \in \mathbb{S}^3$.

Step 1–Bringing the Liénard equation (2) to a slow-fast system

- After dividing (4) by the positive constant ϵ, we conclude that $X_{\epsilon^4 B_0,\epsilon^3 B_1,\epsilon^2 B_2,\epsilon B_3}$ is (linearly) equivalent to

$$\left\{ \begin{array}{rcl} \dot{\bar{x}} & = & \bar{y} \\ \dot{\bar{y}} & = & -\bar{y}\bar{x} - \epsilon(B_0 + B_1\bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + \bar{x}^4), \end{array} \right. \tag{5}$$

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  where $(B_0, B_1, B_2, B_3) \in \mathbb{S}^3$.
- Thus, instead of studying system $X_{\epsilon^4 B_0, \epsilon^3 B_1, \epsilon^2 B_2, \epsilon B_3}$, with $\epsilon > 0$ and $(B_0, B_1, B_2, B_3) \in \mathbb{S}^3$, we can study system (5) which is of singular type.

Step 2–Slow-fast Liénard systems (5) at infinity in the phase space

- We can study the dynamics of (5) near infinity on the Poincaré-Lyapunov disc of type $(2, 5)$.

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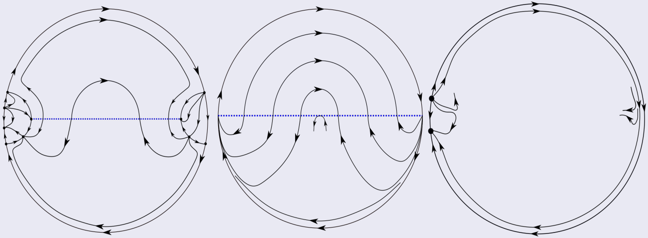

Figure: Dynamics of (5) near infinity on the Poincaré-Lyapunov disc of type $(2, 5)$.

Step 2–Transformation of (5) in the negative $\bar{x}$-direction

- We define the coordinate change

$$(\bar{x}, \bar{y}) = (\frac{-1}{\rho^2}, \frac{Y}{\rho^5}),$$

where $\rho > 0$, $\rho \sim 0$ and Y is kept in a large compact set. In the coordinates (ρ, Y), after multiplication by the positive factor ρ^3, system (5) can be written as:

$$\left\{ \begin{array}{rcl} \dot{\rho} & = & \frac{1}{2}\rho Y \\ \dot{Y} & = & \frac{5}{2}Y^2 + \rho Y - \epsilon\big(B_0\rho^8 - B_1\rho^6 + B_2\rho^4 - B_3\rho^2 + 1\big). \end{array} \right. \tag{6}$$

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- When $\rho = \epsilon = 0$, the singularity at $Y = 0$ of (6) is linearly zero. We blow up the origin $(\rho, Y, \epsilon) = (0, 0, 0)$ using

$$(\rho, Y, \epsilon) = (v\bar{\rho}, v\bar{Y}, v^2\bar{\epsilon}),\ v \geq 0,\ \bar{\epsilon} \geq 0,\ \bar{\rho} \geq 0,\ (\bar{\rho}, \bar{Y}, \bar{\epsilon}) \in \mathbb{S}^2.$$

Step 2–Transformation of (5) in the negative $\bar{x}$-direction

- *The family chart* $\{\bar{\epsilon} = 1\}$. System (6) changes, after dividing by v, into

$$\begin{cases} \dot{\bar{\rho}} &= \frac{1}{2}\bar{\rho}\bar{Y} \\ \dot{\bar{Y}} &= \frac{5}{2}\bar{Y}^2 + \bar{\rho}\bar{Y} - 1 + O(v^2) \end{cases} \tag{7}$$

where $\bar{\rho} \geq 0$ and $(\bar{\rho}, \bar{Y})$ is kept in a large compact set. When $v = 0$, system (7) has one hyperbolic and attracting node at $(\bar{\rho}, \bar{Y}) = (0, -\sqrt{\frac{2}{5}})$ and one hyperbolic and repelling node at $(\bar{\rho}, \bar{Y}) = (0, \sqrt{\frac{2}{5}})$.

Step 2–Transformation of (5) in the negative $\bar{x}$-direction

- *The phase directional chart* $\{\bar{\rho}=1\}$. In the chart $\{\bar{\rho}=1\}$ system (6) becomes, after dividing by v,

$$\left\{\begin{array}{rcl} \dot{v} & = & \frac{1}{2}v\bar{Y} \\ \dot{\bar{\epsilon}} & = & -\bar{\epsilon}\bar{Y} \\ \dot{\bar{Y}} & = & \bar{Y}+2\bar{Y}^2-\bar{\epsilon}\big(1+O(v^2)\big), \end{array}\right. \tag{8}$$

where $(v,\bar{\epsilon})\sim(0,0)$, $v\geq 0$, $\bar{\epsilon}\geq 0$ and $\bar{Y}$ is kept in a large compact set. If $v=\bar{\epsilon}=0$, then system (8) has a hyperbolic saddle at $\bar{Y}=-\frac{1}{2}$ with eigenvalues $(-\frac{1}{4},\frac{1}{2},-1)$ and a semi-hyperbolic singularity at $\bar{Y}=0$ with the $\bar{Y}$-axis as the unstable manifold and a two dimensional center manifold transverse to the unstable manifold. Center manifolds can be written as $\bar{Y}=\bar{\epsilon}\big(1+O(v,\bar{\epsilon})\big)$, with the following dynamics $\{\dot{v}=\frac{1}{2}v\bar{\epsilon}\big(1+O(v,\bar{\epsilon})\big),\dot{\bar{\epsilon}}=-\bar{\epsilon}^2\big(1+O(v,\bar{\epsilon})\big)\}$.

Step 2–Transformation of (5) in the negative $\bar{x}$-direction

- *The phase directional chart* $\{\bar{Y}=1\}$. System (6) changes, after dividing by v, into

$$\left\{\begin{array}{rcl} \dot{v} & = & v\left(\frac{5}{2}+\bar{\rho}-\bar{\epsilon}(1+O(v^2))\right) \\ \dot{\bar{\epsilon}} & = & -2\bar{\epsilon}\left(\frac{5}{2}+\bar{\rho}-\bar{\epsilon}(1+O(v^2))\right) \\ \dot{\bar{\rho}} & = & \bar{\rho}\left(-2-\bar{\rho}+\bar{\epsilon}(1+O(v^2))\right), \end{array}\right. \tag{9}$$

where $(v,\bar{\epsilon})\sim(0,0)$, $v\geq 0$, $\bar{\epsilon}\geq 0$ and $\bar{\rho}\geq 0$ is kept in a large compact set. System (9) has a hyperbolic saddle at $(v,\bar{\epsilon},\bar{\rho})=(0,0,0)$ with eigenvalues $(\frac{5}{2},-5,-2)$.

Step 2–Transformation of (5) in the negative $\bar{x}$-direction

- *The phase directional chart* $\{\bar{Y} = -1\}$. System (6) changes, after dividing by v, into

$$\left\{\begin{array}{lcl} \dot{v} & = & v\big(-\frac{5}{2} + \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\big) \\ \dot{\bar{\epsilon}} & = & -2\bar{\epsilon}\big(-\frac{5}{2} + \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\big) \\ \dot{\bar{\rho}} & = & \bar{\rho}\big(2 - \bar{\rho} - \bar{\epsilon}(1 + O(v^2))\big), \end{array}\right. \qquad (10)$$

  where $(v, \bar{\epsilon}) \sim (0,0)$, $v \geq 0$, $\bar{\epsilon} \geq 0$ and $\bar{\rho} \geq 0$ is kept in a large compact set. Besides the hyperbolic saddle found in the chart $\{\bar{\rho} = 1\}$, we find an extra hyperbolic saddle at $(v, \bar{\epsilon}, \bar{\rho}) = (0,0,0)$ of (10) with eigenvalues $(-\frac{5}{2}, 5, 2)$.

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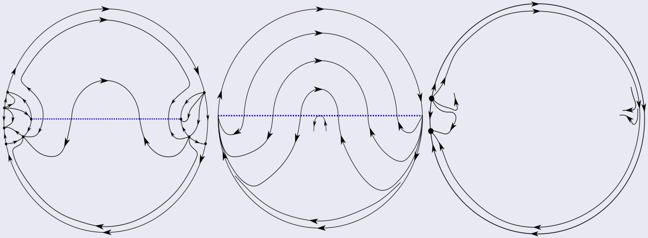

Figure: Dynamics of (5) near infinity on the Poincaré-Lyapunov disc of type $(2, 5)$.

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- We introduce the coordinate change

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$$\begin{cases} \dot{\rho} &= -\frac{1}{2}\rho Y \\ \dot{Y} &= -\frac{5}{2}Y^2 - \rho Y - \epsilon\left(B_0\rho^8 + B_1\rho^6 + B_2\rho^4 + B_3\rho^2 + 1\right). \end{cases} \tag{11}$$

- When $\rho = 0$ and $\epsilon > 0$, system (11) has no singularities. When $\rho = \epsilon = 0$, the singularity at $Y = 0$ of (11) is linearly zero.

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- When $\rho = 0$ and $\epsilon > 0$, system (11) has no singularities. When $\rho = \epsilon = 0$, the singularity at $Y = 0$ of (11) is linearly zero. To desingularize (11) we use the following blow-up at the origin in (ρ, Y, ϵ)-space: $(\rho, Y, \epsilon) = (v\bar{\rho}, v\bar{Y}, v^2\bar{\epsilon})$, $v \geq 0$, $v \sim 0$, $\bar{\epsilon} \geq 0$, $\bar{\rho} \geq 0$, $(\bar{\rho}, \bar{Y}, \bar{\epsilon}) \in \mathbb{S}^2$.

Step 2–Transformation of (5) in the positive $\bar{x}$-direction

- *The family chart* $\{\bar{\epsilon} = 1\}$. System (11) changes, after dividing by v, into

$$\left\{ \begin{array}{rcl} \dot{\bar{\rho}} & = & -\frac{1}{2}\bar{\rho}\bar{Y} \\ \dot{\bar{Y}} & = & -\frac{5}{2}\bar{Y}^2 - \bar{\rho}\bar{Y} - 1 + O(v^2) \end{array} \right. \qquad (12)$$

where $\bar{\rho} \geq 0$ and $(\bar{\rho}, \bar{Y})$ is kept in a large compact set.

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where $\bar{\rho} \geq 0$ and $(\bar{\rho}, \bar{Y})$ is kept in a large compact set. When $v = 0$, system (12) has no singularities.

The phase directional chart $\{\bar{\rho} = 1\}$. In the chart $\{\bar{\rho} = 1\}$ system (11) becomes, after dividing by v,

$$\begin{cases} \dot{v} & = & -\frac{1}{2}v\bar{Y} \\ \dot{\bar{\epsilon}} & = & \bar{\epsilon}\bar{Y} \\ \dot{\bar{Y}} & = & -\bar{Y} - 2\bar{Y}^2 - \bar{\epsilon}\big(1 + O(v^2)\big), \end{cases} \qquad (13)$$

where $(v, \bar{\epsilon}) \sim (0, 0)$, $v \geq 0$, $\bar{\epsilon} \geq 0$ and $\bar{Y}$ is kept in a large compact set.

Step 2–Transformation of (5) in the positive $\bar{x}$-direction

- When $v = \bar{\epsilon} = 0$, system (13) has a hyperbolic saddle at $\bar{Y} = -\frac{1}{2}$ with eigenvalues $(\frac{1}{4}, -\frac{1}{2}, 1)$ and a semi-hyperbolic singularity at $\bar{Y} = 0$ with the stable manifold $\{v = \bar{\epsilon} = 0\}$ and a two dimensional center manifold transverse to the stable manifold. Center manifolds are given by $\bar{Y} = -\bar{\epsilon}(1 + O(v, \bar{\epsilon}))$ and the dynamics inside the center manifolds is given by $\{\dot{v} = \frac{1}{2} v\bar{\epsilon}(1 + O(v, \bar{\epsilon})), \dot{\bar{\epsilon}} = -\bar{\epsilon}^2(1 + O(v, \bar{\epsilon}))\}$.
- *The phase directional chart* $\{\bar{Y} = 1\}$. In this chart, system (11) changes, after dividing by v, into

$$\begin{cases} \dot{v} &= v\big(-\frac{5}{2} - \bar{\rho} - \bar{\epsilon}(1 + O(v^2))\big) \\ \dot{\bar{\epsilon}} &= -2\bar{\epsilon}\big(-\frac{5}{2} - \bar{\rho} - \bar{\epsilon}(1 + O(v^2))\big) \\ \dot{\bar{\rho}} &= \bar{\rho}\big(2 + \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\big), \end{cases} \tag{14}$$

where $(v, \bar{\epsilon}) \sim (0, 0)$, $v \geq 0$, $\bar{\epsilon} \geq 0$ and $\bar{\rho} \geq 0$ is kept in a large compact set. When $v = \bar{\epsilon} = 0$, system (14) has a hyperbolic saddle at $\bar{\rho} = 0$ with eigenvalues $(-\frac{5}{2}, 5, 2)$.

Step 2–Transformation of (5) in the positive $\bar{x}$-direction

- *The phase directional chart* $\{\bar{Y} = -1\}$. In this phase directional chart, system (11) changes, after dividing by v, into

$$\begin{cases} \dot{v} &= v\big(\frac{5}{2} - \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\big) \\ \dot{\bar{\epsilon}} &= -2\bar{\epsilon}\big(\frac{5}{2} - \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\big) \\ \dot{\bar{\rho}} &= \bar{\rho}\big(- 2 + \bar{\rho} - \bar{\epsilon}(1 + O(v^2))\big), \end{cases} \tag{15}$$

where $(v, \bar{\epsilon}) \sim (0, 0)$, $v \geq 0$, $\bar{\epsilon} \geq 0$ and $\bar{\rho} \geq 0$ is kept in a large compact set.

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$$\begin{cases} \dot{v} &= v\left(\frac{5}{2} - \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\right) \\ \dot{\bar{\epsilon}} &= -2\bar{\epsilon}\left(\frac{5}{2} - \bar{\rho} + \bar{\epsilon}(1 + O(v^2))\right) \\ \dot{\bar{\rho}} &= \bar{\rho}\left(-2 + \bar{\rho} - \bar{\epsilon}(1 + O(v^2))\right), \end{cases} \tag{15}$$

  where $(v, \bar{\epsilon}) \sim (0, 0)$, $v \geq 0$, $\bar{\epsilon} \geq 0$ and $\bar{\rho} \geq 0$ is kept in a large compact set. Besides the hyperbolic saddle found in the chart $\{\bar{\rho} = 1\}$, system (15) has an extra hyperbolic saddle at $(v, \bar{\epsilon}, \bar{\rho}) = (0, 0, 0)$ with eigenvalues $(\frac{5}{2}, -5, -2)$.

Step 2–Transformation of (5) in the positive $\bar{x}$-direction

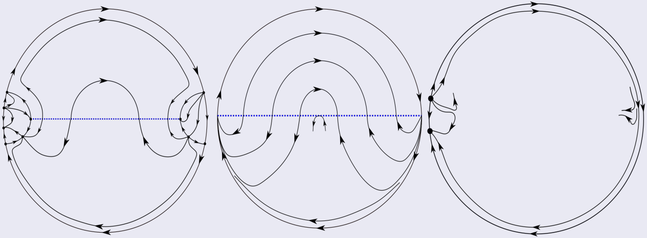

Figure: Dynamics of (5) near infinity on the Poincaré-Lyapunov disc of type $(2, 5)$.

Step 2–Transformation of (5) in the positive and negative $\bar{y}$-direction

- We have

$$\left\{\begin{array}{rcl} \dot{\rho} & = & \frac{1}{5}\rho^2 X + \frac{\epsilon}{5}\left(B_0\rho^9 + B_1 X\rho^7 + B_2 X^2\rho^5 + B_3 X^3\rho^3 + X^4\rho\right) \\ \dot{X} & = & \frac{2}{5}X\Big(\rho X + \epsilon\big(B_0\rho^8 + B_1 X\rho^6 + B_2 X^2\rho^4 \\ & & +B_3 X^3\rho^2 + X^4\big)\Big) + 1. \end{array}\right. \tag{16}$$

and

$$\left\{\begin{array}{rcl} \dot{\rho} & = & \frac{1}{5}\rho^2 X - \frac{\epsilon}{5}\left(B_0\rho^9 + B_1 X\rho^7 + B_2 X^2\rho^5 + B_3 X^3\rho^3 + X^4\rho\right) \\ \dot{X} & = & \frac{2}{5}X\Big(\rho X - \epsilon\big(B_0\rho^8 + B_1 X\rho^6 + B_2 X^2\rho^4 \\ & & +B_3 X^3\rho^2 + X^4\big)\Big) - 1. \end{array}\right. \tag{17}$$

Quartic Liénard equations with linear damping

Step 2–Transformation of (5) in the positive $\bar{x}$-direction

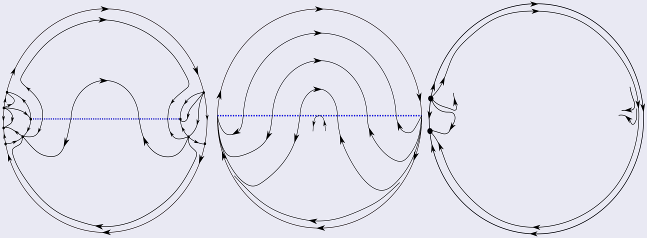

Figure: Dynamics of (5) near infinity on the Poincaré-Lyapunov disc of type $(2, 5)$.

Step 3–Slow-fast Liénard systems (5) in compact sets in the phase space

- Suppose that K is any compact set in the $(\bar{x}, \bar{y})$-plane and fix it. *We prove that system (5) has at most two limit cycles in* K*, for each fixed* $\epsilon \geq 0$, $\epsilon \sim 0$ *and* $(B_0, B_1, B_2, B_3) \in \mathbb{S}^3$.

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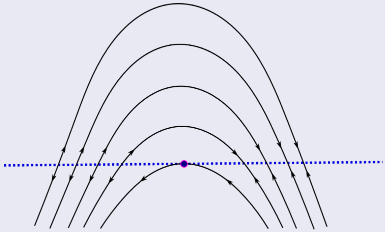
- When $\epsilon = 0$:



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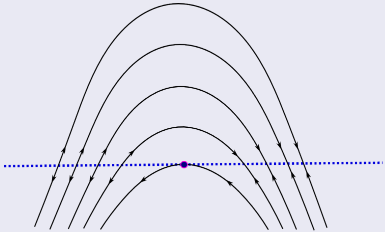

- Our problem is equivalent with the following problem: *Prove that slow-fast and regular codimension 4 saddle-node bifurcations can produce at most 2 small-amplitude limit cycles.*

Regular and slow-fast codimension 4 saddle-node bifurcations

- We consider:

$$X_{\epsilon,b,\lambda} : \begin{cases} \dot{x} = y \\ \dot{y} = -xy + \epsilon\Big(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + x^4 + x^5 G(x,\lambda) \\ \qquad +y^2 H(x,y,\lambda)\Big). \end{cases}$$

Quartic Liénard equations with linear damping

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- When $b = 0$ and $\epsilon > 0$, then the origin $(x, y) = (0, 0)$ is a nilpotent singularity of saddle-node type.

Quartic Liénard equations with linear damping

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Blow-up in the parameter space

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Blow-up in the parameter space

- We first reparametrize the b-parameters, by introducing weighted spherical coordinates: $(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3 B_1, r^2 B_2, r B_3), \quad r \geq 0, \;\; B = (B_0, B_1, B_2, B_3) \in \mathbb{S}^3.$

Quartic Liénard equations with linear damping

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- Instead of coordinates on the sphere, we use one of the 8 charts of the sphere:

Quartic Liénard equations with linear damping

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- Instead of coordinates on the sphere, we use one of the 8 charts of the sphere:
  a) Jump region: $(b_0, b_1, b_2, b_3) = (\pm r^4, r^3 B_1, r^2 B_2, r B_3)$

Blow-up in the parameter space

b) Slow-fast Hopf region:

$(b_0, b_1, b_2, b_3) = (r^4 B_0, -r^3, r^2 B_2, rB_3)$

Blow-up in the parameter space

b) Slow-fast Hopf region:
$(b_0, b_1, b_2, b_3) = (r^4 B_0, -r^3, r^2 B_2, rB_3)$
c) Saddle region: $(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3, r^2 B_2, rB_3)$

Blow-up in the parameter space

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e) Slow-fast codimension 3 saddle region:
$(b_0, b_1, b_2, b_3) = (r^4 B_0, r^3 B_1, r^2 B_2, r)$.

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f) Slow-fast codimension 3 elliptic region:
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- We obtain

$$X_{\epsilon,B,r,\lambda} : \begin{cases} \dot{x} = y \\ \dot{y} = -xy + \epsilon\Big(r^4 B_0 + r^3 B_1 x + r^2 B_2 x^2 + rB_3 x^3 + x^4 + x^5 G(x,\lambda) \\ \qquad +y^2 H(x,y,\lambda)\Big). \end{cases}$$

Blow-up of the origin $(x, y, r) = (0, 0, 0)$ (primary blow-up)

- We blow up the origin using the blow up transformation

$$(x, y, r) = (u\bar{x}, u^2\bar{y}, u\bar{r}), \; u \geq 0, \; \bar{r} \geq 0, \; (\bar{x}, \bar{y}, \bar{r}) \in S^2. \quad (18)$$

The study of the dynamics in the blown-up coordinates will be done in different charts:

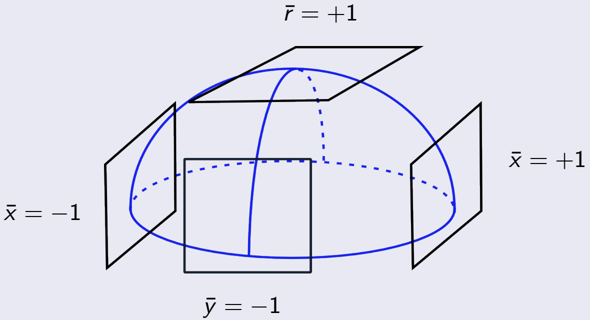

Figure: Different charts.

The family chart $\bar{r} = +1$

- In this family chart, the vector field $X_{\epsilon,B,r,\lambda}$ yields, after division by the positive factor u,

$$X^F_{\epsilon,B,u,\lambda} : \begin{cases} \dot{\bar{x}} = \bar{y} \\ \dot{\bar{y}} = -\bar{x}\bar{y} + \epsilon u\Big(B_0 + B_1\bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x},\lambda) \\ \qquad +\bar{y}^2 H(u\bar{x}, u^2\bar{y},\lambda)\Big). \end{cases}$$

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- We write $\bar{\epsilon} = \epsilon u \sim 0$ ($\epsilon \in [0, M]$, $u \sim 0$):

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  where $\bar{\epsilon} \sim 0$ is a singular perturbation parameter.

Quartic Liénard equations with linear damping

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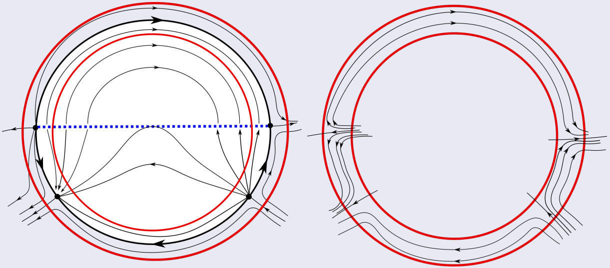
Quartic Liénard equations with linear damping

The phase-directional charts $\{\bar{x} = \pm 1, \bar{y} = \pm 1\}$



The slow-fast Hopf region

- We consider

$$\begin{cases} \dot{x} = y \\ \dot{y} = -xy + \epsilon\Big(r^4 B_0 - r^3 x + r^2 B_2 x^2 + rB_3 x^3 + x^4 + x^5 G(x,\lambda) \\ \qquad +y^2 H(x,y,\lambda)\Big). \end{cases}$$

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Theorem

Let $B_1 = -1$. Given any $B_i^1 > 0$, $i = 2,3$. There exist a neighborhood V of $(x,y) = (0,0)$, $r_0 > 0$ and $B_0^1 > 0$ such that $X_{\epsilon,B,r,\lambda}$ has at most 2 limit cycles in V for each $(\epsilon, B_0, B_2, B_3, r, \lambda) \in [0,M] \times [-B_0^1, B_0^1] \times [-B_2^1, B_2^1] \times [-B_3^1, B_3^1] \times [0, r_0] \times \Lambda$.

The slow-fast Hopf region

- It is sufficient to consider the following singular perturbation system:

$$\begin{cases} \dot{\bar{x}} = \bar{y} \\ \dot{\bar{y}} = -\bar{x}\bar{y} + \delta^2\Big(\delta\bar{B}_0 - \bar{x} + B_2\bar{x}^2 + B_3\bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x},\lambda) \\ \qquad +\bar{y}^2 H(u\bar{x}, u^2\bar{y},\lambda)\Big), \end{cases}$$

where $(\epsilon, B_0) = (\delta^2, \delta\bar{B}_0)$, with $\delta \sim 0$ and $\bar{B}_0 \sim 0$ ($\bar{B}_0$ is the regular breaking parameter).

Quartic Liénard equations with linear damping

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- We have two types of limit periodic sets: the contact point and detectable canard limit periodic sets

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- From the following theorem it follows that the limit cycles may bifurcate from the contact point $(\bar{x}, \bar{y}) = (0, 0)$.

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Theorem

(i) *Let $B_2^0 > 0$ be any arbitrarily small fixed number and let $K := \mathcal{B} \cap \{|B_2| \geq B_2^0\}$. There exist small $\delta_0 > 0$, $\bar{B}_0^0 > 0$, $u_0 > 0$ and a neighborhood U of $(\bar{x}, \bar{y}) = (0, 0)$ such that the following statements are true.*

1. *The family $X^F_{\delta^2,(\delta\bar{B}_0,-1,B_2,B_3),u,\lambda}$ has at most 1 (hyperbolic) limit cycle in U for each $(\delta, \bar{B}_0, B_2, B_3, u, \lambda) \in [0, \delta_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times K \times [0, u_0] \times \Lambda$.*

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2. *When we fix $(\delta, B_2, B_3, u, \lambda) \in]0, \delta_0] \times K \times [0, u_0] \times \Lambda$, the $\bar{B}_0$-family $X^F_{\delta^2,(\delta\bar{B}_0,-1,B_2,B_3),u,\lambda}$ undergoes, in U and at $\bar{B}_0 = 0$, a Hopf bifurcation of codimension 1. Assume $(B_2, B_3) \in K$ and $B_2 > 0$. When $\bar{B}_0$ increases there is in U an attracting hyperbolic focus and no limit cycle; when $\bar{B}_0$ decreases there is in U a repelling hyperbolic focus and an attracting limit cycle of which the size monotonically grows as $\bar{B}_0$ decreases. Assume $(B_2, B_3) \in K$ and*

The slow-fast Hopf region

Theorem

(i) 2 *Assume* $(B_2, B_3) \in K$ *and* $B_2 < 0$. *When* $\bar{B}_0$ *decreases there is in* U *a repelling hyperbolic focus and no limit cycle; when* $\bar{B}_0$ *increases there is in* U *an attracting hyperbolic focus and a repelling limit cycle of which the size monotonically grows as* $\bar{B}_0$ *increases.*

The slow-fast Hopf region

Theorem

(i) 2 *Assume* $(B_2, B_3) \in K$ *and* $B_2 < 0$. *When* $\bar{B}_0$ *decreases there is in* U *a repelling hyperbolic focus and no limit cycle; when* $\bar{B}_0$ *increases there is in* U *an attracting hyperbolic focus and a repelling limit cycle of which the size monotonically grows as* $\bar{B}_0$ *increases.*

(ii) *There exist small* $\delta_0 > 0$, $\bar{B}_0^0 > 0$, $B_2^0 > 0$ *and* $u_0 > 0$ *and a neighborhood* U *of* $(\bar{x}, \bar{y}) = (0, 0)$ *such that the family* $X^F_{\delta^2,(\delta\bar{B}_0,-1,B_2,B_3),u,\lambda}$ *has at most 2 limit cycles in* U *for each* $(\delta, \bar{B}_0, B_2, B_3, u, \lambda) \in [0, \delta_0] \times [-\bar{B}_0^0, \bar{B}_0^0] \times \mathcal{B} \cap \{|B_2| \leq B_2^0\} \times [0, u_0] \times \Lambda$.

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- R. Huzak, *Canard Explosion Near Non-Liénard Type Slow-Fast Hopf Point, 2018*

The slow-fast Hopf region

- On the other hand, the slow dynamics of our system, which is given by

$$\bar{x}' = -1 + B_2\bar{x} + B_3\bar{x}^2 + \bar{x}^3 + u\bar{x}^4 G(u\bar{x}, \lambda), \tag{19}$$

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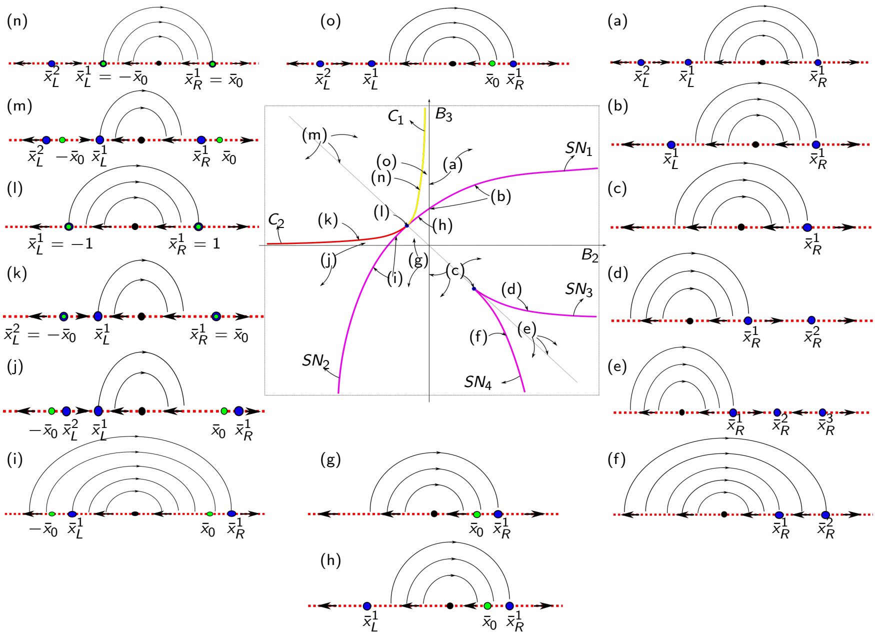
- The discriminant

$$J = B_2^2 B_3^2 - 4B_2^3 + 4B_3^3 - 18B_2B_3 - 27$$

  of the cubic $\bar{x}$-polynomial $-1 + B_2\bar{x} + B_3\bar{x}^2 + \bar{x}^3$ can be used to find out how many real zeros the slow dynamics has.

Quartic Liénard equations with linear damping



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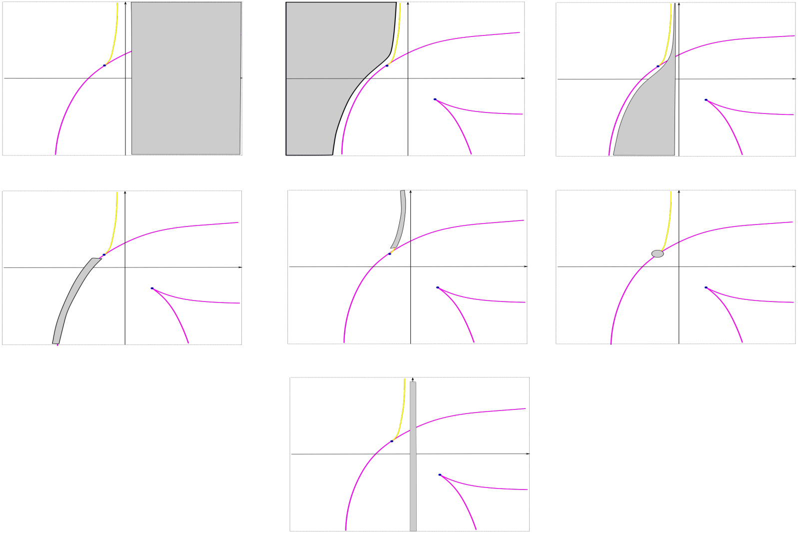

The slow-fast Hopf region

- We study zeros of the slow divergence integral along the slow curve if the slow dynamics is regular:

$$I(y, B_2, B_3, u, \lambda) = \int_{-\sqrt{2y}}^{\sqrt{2y}} \frac{\bar{x}\, d\bar{x}}{-1 + B_2\bar{x} + B_3\bar{x}^2 + \bar{x}^3 + u\bar{x}^4 G(u\bar{x}, \lambda)}.$$

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The slow-fast Hopf region

- If we write $\tau = (\delta, \bar{B}_0, B_2, B_3, u, \lambda)$, then we have

$$\frac{\partial \Delta}{\partial \bar{y}}(\bar{y}, \tau) = -\frac{1}{\delta^4} L_+(\bar{y}, \tau) \exp \mathcal{I}_+(\bar{y}, \tau) - \Big(-\frac{1}{\delta^4} L_-(\bar{y}, \tau) \exp \mathcal{I}_-(\bar{y}, \tau) \Big)$$

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- If we introduce the analytic function $A(\alpha,\beta) = \frac{\exp\alpha - \exp\beta}{\alpha-\beta} > 0$ if $\alpha \neq \beta$ and $A(\alpha,\alpha) = \exp\alpha$, and if we write $\mathcal{I} = \mathcal{I}_+ - \mathcal{I}_-$, then we have:

The slow-fast Hopf region

$$\frac{\partial \Delta}{\partial \bar{y}}(\bar{y},\tau) = -\frac{1}{\delta^6} A(\alpha,\beta)\left(\delta^2 \mathcal{I}(\bar{y},\tau) + O(\delta^2)\right)$$

with $\alpha = \mathcal{I}_+(\bar{y},\tau) + \ln(L_+(\bar{y},\tau))$ and $\beta = \mathcal{I}_-(\bar{y},\tau) + \ln(L_-(\bar{y},\tau))$

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- We don't specify the $O(\delta^2)$-term since it is not the leading order part in the expression $\delta^2\mathcal{I} + O(\delta^2)$.
- Using Rolle's theorem, it can be shown that the number of periodic orbits of $X^F_{\delta^2,(\delta\bar{B}_0,-1,B_2,B_3),u,\lambda}$ near the set $\cup_{\bar{y}\in[\mu,\eta]}\Gamma_{\bar{y}}$, at the τ-level, is bounded by $1+$ the number of zeros (counting multiplicity) of $\delta^2\mathcal{I}$ w.r.t. $\bar{y}\in[\mu,\eta]$.

The slow-fast Hopf region

- For $u = 0$ and $(B_2, B_3) = (-1, 1)$, the slow dynamics has a singularity of multiplicity 1 at $\bar{x} = 1$ and a singularity of multiplicity 2 at $\bar{x} = -1$.

Quartic Liénard equations with linear damping

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- For $u = 0$ and $(B_2, B_3) = (-1, 1)$, the slow dynamics has a singularity of multiplicity 1 at $\bar{x} = 1$ and a singularity of multiplicity 2 at $\bar{x} = -1$.
- At $\bar{x} = 1$ and near the parameter value $(\bar{B}_0, B_2, B_3, u) = (0, -1, 1, 0)$ we use a C^k-normal form (see Takens or Bonckaert):

$$\begin{cases} \dot{v}_1 & = & \delta^2 \nu_1 v_1 \\ \dot{v}_2 & = & -v_2, \end{cases} \tag{20}$$

where $-\delta^2\nu_1$, $\nu_1 > 0$, is ratio of eigenvalues of a persistent hyperbolic saddle of $X^F_{\delta^2,(\delta\bar{B}_0,-1,B_2,B_3),u,\lambda}$ near $(\bar{x}, \bar{y}) = (1, 0)$.

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- Following [De Maesschalck,Dumortier,2008] or [De Maesschalck,Dumortier, Huzak,2013], we have:

The slow-fast Hopf region

$$\delta^2\mathcal{I}_+(\bar{y},\tau) = \int_{\alpha_+(\bar{y},\tau)}^{1} \frac{-1+\nu_1\delta^2}{\nu_1 v_1} dv_1 + O(1),$$

near $\bar{y} = \frac{1}{2}$.

- The integral in the expression for $\delta^2\mathcal{I}_+$ is the divergence integral (multiplied by δ^2) calculated in the normal form coordinates from $\{v_2 = 1\}$ to $\{v_1 = 1\}$ where we assume that the orbit $\mathcal{O}^+(\bar{y},\tau)$ intersects the plane $\{v_2 = 1\}$ in a point with $v_1 = \alpha_+(\bar{y},\tau)$. Clearly, α_+ is a C^k-diffeomorphism with $\alpha_+(\frac{1}{2}, 0, 0, -1, 1, 0, \lambda) = 0$ and $\frac{\partial\alpha_+}{\partial\bar{y}} < 0$.

Quartic Liénard equations with linear damping

The slow-fast Hopf region

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- The $O(1)$-term is δ-regularly C^k in $(\bar{y},\bar{B}_0,B_2,B_3,u,\lambda)$, i.e. $O(1)$ and all its derivatives up to order k w.r.t. $(\bar{y},\bar{B}_0,B_2,B_3,u,\lambda)$ are continuous including at $\delta = 0$.

The slow-fast Hopf region

- At $\bar{x} = -1$ and near the parameter value $(\bar{B}_0, B_2, B_3, u) = (0, -1, 1, 0)$ we have a C^k-normal form (see Takens or Bonckaert):

$$\left\{ \begin{array}{lcl} \dot{v}_1 & = & -\delta^2 h(v_1, \tau) \\ \dot{v}_2 & = & v_2, \end{array} \right. \tag{21}$$

  where $h(v_1, 0, 0, -1, 1, 0, \lambda)$ has a zero of multiplicity 2 at $v_1 = 0$.

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where $h(v_1, 0, 0, -1, 1, 0, \lambda)$ has a zero of multiplicity 2 at $v_1 = 0$. Similarly, we find that

$$\delta^2 \mathcal{I}_-(\bar{y}, \tau) = \int_{\alpha_-(\bar{y},\tau)}^{1} \frac{-1 + \delta^2 \frac{\partial h}{\partial v_1}(v_1, \tau)}{h(v_1, \tau)} dv_1 + O(1),$$

near $\bar{y} = \frac{1}{2}$. The orbit $\mathcal{O}^-(\bar{y}, \tau)$ intersects the plane $\{v_2 = 1\}$ in a point with $v_1 = \alpha_-(\bar{y}, \tau)$. The function α_- and the $O(1)$-term have the same properties like α_+ and the $O(1)$-term in the expression for $\delta^2 \mathcal{I}_+$.

The slow-fast Hopf region

- Thus, we get

$$\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y},\tau) = \frac{(1-\nu_1\delta^2)\frac{\partial \alpha_+}{\partial \bar{y}}(\bar{y},\tau)}{\nu_1\alpha_+(\bar{y},\tau)} - \frac{\left(1-\delta^2\frac{\partial h}{\partial v_1}(\alpha_-(\bar{y},\tau),\tau)\right)\frac{\partial \alpha_-}{\partial \bar{y}}(\bar{y},\tau)}{h(\alpha_-(\bar{y},\tau),\tau)} + O(1),$$

near $\bar{y}=\frac{1}{2}$. Using the above expression and the properties of $\alpha_\pm$ and h, we finally get

$$\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}(\bar{y},\tau) = \frac{\beta_0+\beta_1(\bar{y}-\frac{1}{2})+O\left((\bar{y}-\frac{1}{2})^2\right)}{\nu_1\alpha_+(\bar{y},\tau)h(\alpha_-(\bar{y},\tau),\tau)} \tag{22}$$

where $\beta_0 = O(\delta, \bar{B}_0, B_2+1, B_3-1, u)$ and

The slow-fast Hopf region

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$$\beta_1 = -\nu_1 \frac{\partial \alpha_+}{\partial \bar{y}}(\frac{1}{2}, 0, 0, -1, 1, 0, \lambda) \frac{\partial \alpha_-}{\partial \bar{y}}(\frac{1}{2}, 0, 0, -1, 1, 0, \lambda) + O(\delta, \bar{B}_0, B_2 + 1, B_3 - 1, u).$$

The slow-fast Hopf region

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- Clearly, the coefficient β_1 is strictly negative. From this together with (22) and Rolle's theorem we conclude that $\frac{\partial \mathcal{I}}{\partial \bar{y}}$ has at most 1 zero (counting multiplicity) near $\bar{y} = \frac{1}{2}$.

The slow-fast codimension 3 saddle/elliptic region

- We consider

$$\begin{cases} \dot{x} = y \\ \dot{y} = -xy + \epsilon\Big(r^4 B_0 + r^3 B_1 x + r^2 B_2 x^2 \pm r x^3 + x^4 + x^5 G(x, \lambda) \\ \qquad\quad + y^2 H(x, y, \lambda)\Big). \end{cases}$$

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Theorem

Let $B_3 = +1$ or $B_3 = -1$. There exist a neighborhood V of $(x,y) = (0,0)$, $r_0 > 0$ and a (B_0, B_1, B_2)-neighborhood U_3 of the origin such that $X_{\epsilon,B,r,\lambda}$ has at most 2 limit cycles in V for each $(\epsilon, B_0, B_1, B_2, r, \lambda) \in [0, M] \times U_3 \times [0, r_0] \times \Lambda$.

The slow-fast codimension 3 saddle/elliptic region

- It is sufficient to consider the following singular perturbation system:

$$\begin{cases} \dot{\bar{x}} = \bar{y} \\ \dot{\bar{y}} = -\bar{x}\bar{y} + \bar{\epsilon}\Big(B_0 + B_1\bar{x} + B_2\bar{x}^2 \pm \bar{x}^3 + \bar{x}^4 + u\bar{x}^5 G(u\bar{x}, \lambda) \\ \qquad +\bar{y}^2 H(u\bar{x}, u^2\bar{y}, \lambda)\Big) \end{cases}$$

Quartic Liénard equations with linear damping

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- The contact point $(\bar{x}, \bar{y}) = (0, 0)$ can produce at most 2 limit cycles (the coefficient in front of $\bar{x}^4$ is $\neq 0$).

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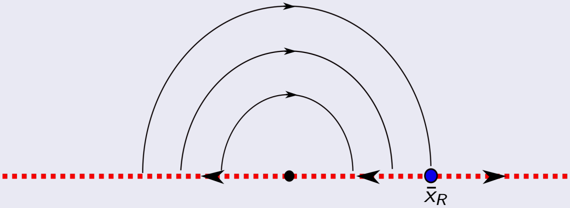
- The contact point $(\bar{x},\bar{y}) = (0,0)$ can produce at most 2 limit cycles (the coefficient in front of $\bar{x}^4$ is $\neq 0$).
- In the saddle case, there are no detectable canard limit cycles $\left(\bar{x}' = \bar{x}^2(1 + O(\bar{x}))\right)$
- In the elliptic case, the slow dynamics is given by:

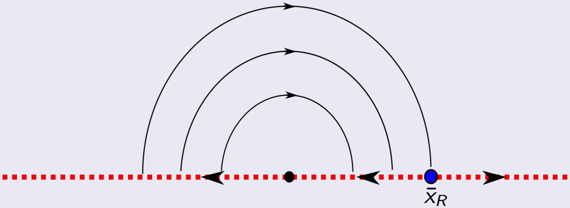
$$\bar{x}' = B_1 + B_2\bar{x} + \bar{x}^2(-1 + \bar{x} + u\bar{x}^2 G(u\bar{x},\lambda))$$

The slow-fast codimension 3 elliptic region



- There are 3 types of limit periodic sets:

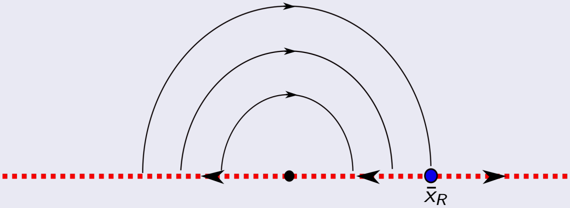
The slow-fast codimension 3 elliptic region



- There are 3 types of limit periodic sets:
  1. the contact point: at most 2 limit cycles

Quartic Liénard equations with linear damping

The slow-fast codimension 3 elliptic region



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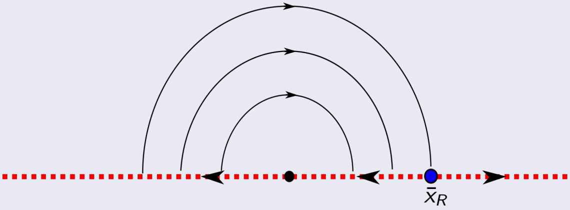

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- The slow divergence integral

$$I(\bar{y}, B_1, B_2, u, \lambda) = \int_{-\sqrt{2\bar{y}}}^{\sqrt{2\bar{y}}} \frac{wdw}{d(w, B_1, B_2, u, \lambda)}, \ \bar{y} \in]0, \frac{1}{2}\bar{x}_R^2[,$$

becomes $\infty - \infty$ as $(B_1, B_2) \to (0, 0)$. The function $d(\bar{x}, B_1, B_2, u, \lambda)$ is the right hand side of the slow dynamics.

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- As shown in [De Maesschalck, Dumortier, 2010], it is better to deal with $\delta^2 \frac{\partial \mathcal{I}}{\partial \bar{y}}$ which is well approximated by the (well defined) derivative of the slow divergence integral

$$\frac{\partial I}{\partial \bar{y}}(\bar{y}, B_1, B_2, u, \lambda) = \frac{-2\sqrt{2\bar{y}}\left(B_2 + 2\bar{y} + uO((\sqrt{2\bar{y}})^3)\right)}{d(\sqrt{2\bar{y}}, B_1, B_2, u, \lambda).d(-\sqrt{2\bar{y}}, B_1, B_2, u, \lambda)}.$$

The slow-fast codimension 3 elliptic region

- Clearly, for any fixed small $\mu > 0$ we have that $\frac{\partial I}{\partial \bar{y}}(\bar{y}, B_1, B_2, u, \lambda)$ is strictly negative for all $\bar{y} \in [\mu, \frac{1}{2}\bar{x}_R^2 - \mu]$ and $\lambda \in \Lambda$, by taking the parameter (B_1, B_2, u) sufficiently small.

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- We have to study separately the cyclicity of $\Gamma_{\frac{1}{2}\bar{x}_R^2}$ (Takens normal forms).

Future Research

Slow-fast and regular codimension ℓ bifurcations, $\ell \geq 5$

- Find the number of limit cycles in

$$\begin{cases} \dot{x} = y \\ \dot{y} = -(a_0 + x)y + \epsilon\Big(b_0 + b_1 x + \ldots + b_{\ell-1}x^{\ell-1} \pm x^{\ell}\Big), \end{cases}$$

where $(a_0, b_0, \ldots, b_{\ell-1}) \sim (0, 0, \ldots, 0)$.

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Thank you for your attention!