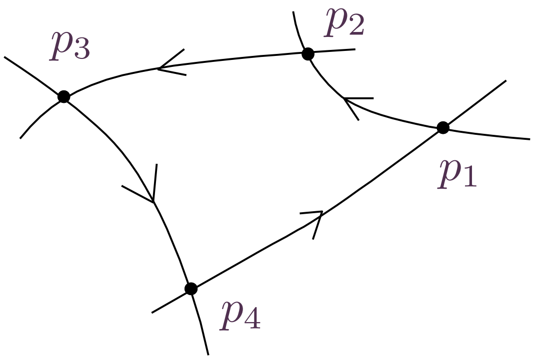
$\mathrm{PSL}(2, \mathbb{C})$ and the exponential

(Quarterly Journal of Math. v. 69, n. 1)

Motivation: 16^{th} Hilbert's Problem



Hyperbolic polycycles

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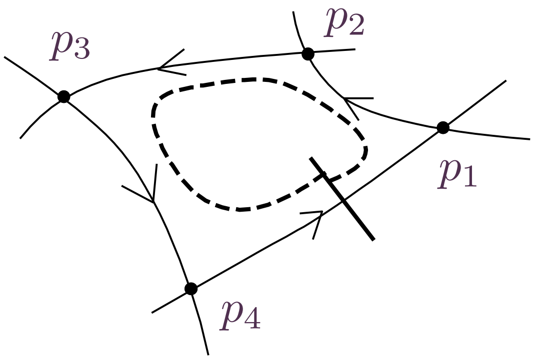


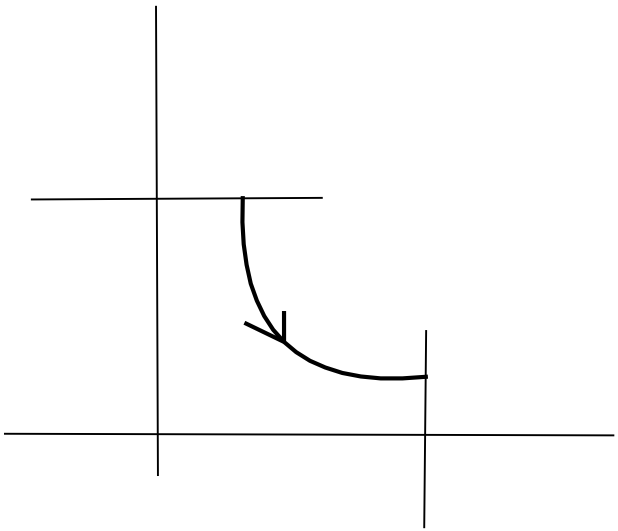
Hyperbolic polycycles

p_1, p_2, p_3, p_4 hyperbolic saddles

Local normal form at each saddle point

$$x\frac{\partial}{\partial x} - y(r + \ldots)\frac{\partial}{\partial y}$$

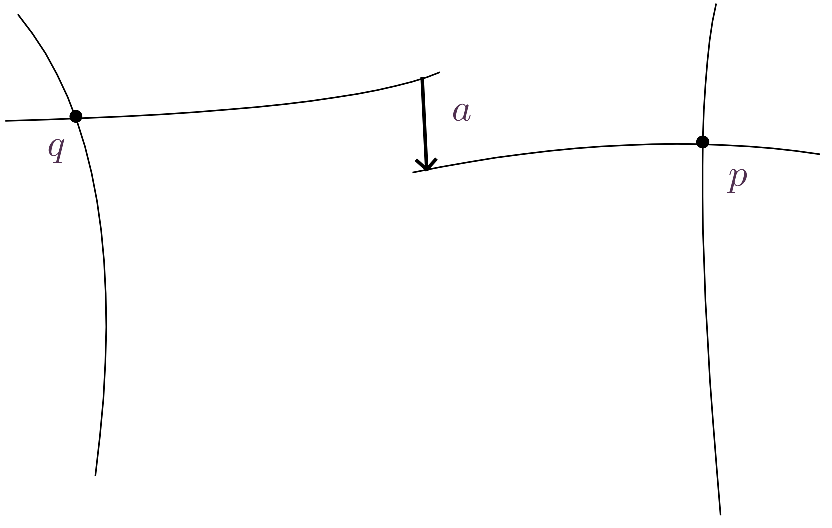
$r \in \mathbb{R}_{>0}$



Dulac map

$$D(x) = x^r(1 + \ldots)$$

Breaking parameter for saddle connections



The *essential* part of the first return map is given by

$$P(a,r,x) = (((x^{r_1} + a_1)^{r_2} + a_2)^{r_3} + \cdots)^{r_n} + a_n$$

This is a real analytic function in the domain $a \in \mathbb{R}^n, r \in \mathbb{R}^n_{>0}$ and

$$x \in I(a,r)$$

where $I(a,r) \subset \mathbb{R}$ is an open interval of the form $(b, +\infty)$ defined by the conditions

$$x > 0$$

$$x^{r_1} + a_1 > 0$$

$$(x^{r_1} + a_1)^{r_2} + a_2 > 0, \ldots.$$

Fixed point counting problem (Roussarie's question)

Given $n \geqslant 1$, we consider the function

$$P(a,r,x) = (((x^{r_1} + a_1)^{r_2} + a_2)^{r_3} + \cdots)^{r_n} + a_n$$

with parameters $a \in \mathbb{R}^n, r \in \mathbb{R}^n_{>0}$.

We want to estimate the number of isolated real solutions $x \in I(a,r)$ of

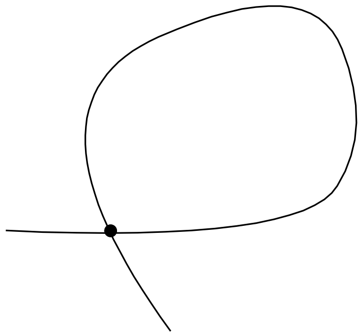
$$F(a,r,x) = P(a,r,x) - x = 0$$

UNIFORMLY on the parameters a, r.

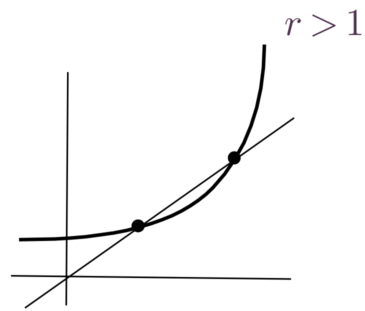
Let

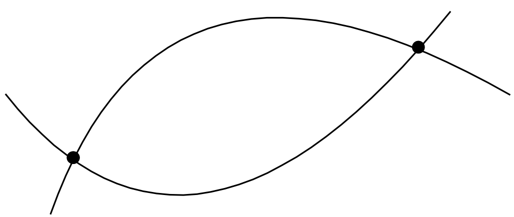
$$\mathrm{Fix}(n) = \sup_{a,r} \{\text{isolated solutions } x \in I(a,r) \text{ of } F(a,r,x) = 0\}$$

Easy cases



Fix(1)=2





$\mathrm{Fix}(2) = 3$

$-0.01516 + (0.0042592 + x^2)^{1/3} - x = 0$ (has three solutions)

For $n = 3$:

Fix(3) = 5   (This is quite hard to obtain..)

Upper bound for Fix(n):

Khovanskii *Fewnomials* Theory:

$$\mathrm{Fix}(n) \leqslant 2^{n(2n-1)}(n+1)^{2n} \qquad \text{(Gabrielov)}$$

Quite "unrealistic":   It gives $\mathrm{Fix}(3) \leqslant 1.3 \times 10^7$

The function $F(x, a, r)$ lies on the o-minimal structure $\mathbb{R}_{\mathrm{an,exp}}$

Derivation-division algorithm

A variant of Descartes' rule of signs (based on Rolle's theorem)

for $n = 1$

$$a + x^r - x$$

$$\Downarrow \partial$$

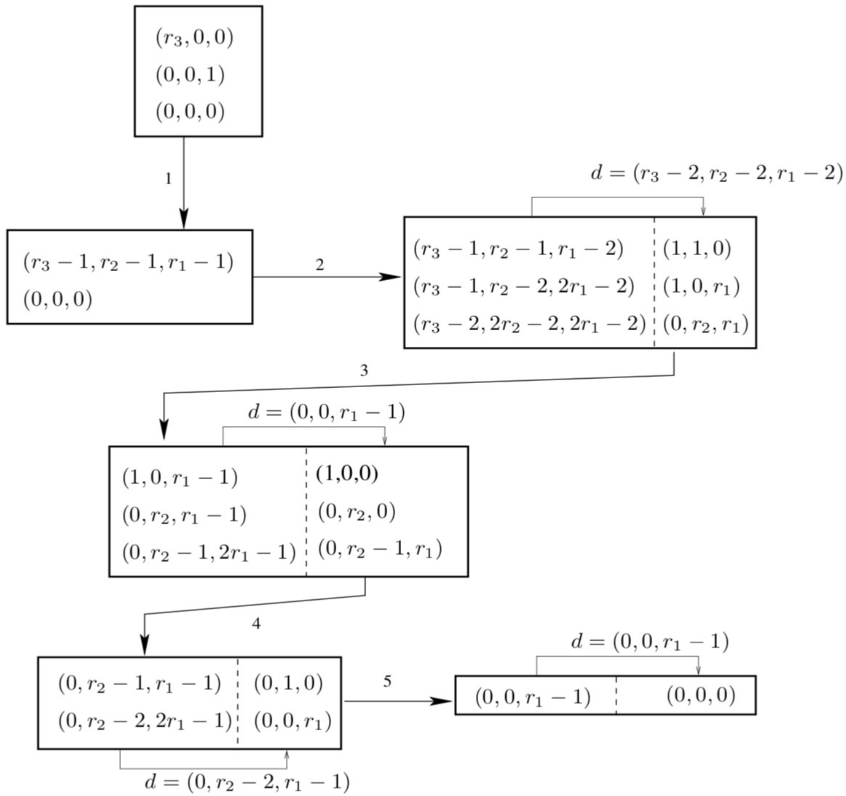
$$r x^{r-1} - 1$$

$$\Downarrow \partial$$

$$r(r-1)x^{r-2} \overset{\text{div}}{\Rightarrow} r(r-1) \qquad \text{is non-vanishing if } r \neq 1$$

There can be an infinite number of solutions, precisely when $a = 0$ and $r = 1$ (Center conditions)

Derivation-division algorithm for $n = 3$ (5 steps)



Quite good results for $n \leqslant 4$....

n	$\mathrm{dd}(n)$	$\mathrm{Fewnom}(n)$	Exact value of $\mathrm{Fix}(n)$
1	2	8	2
2	3	5184	3
3	5	$\approx 1.3 \times 10^7$	5
4	13	$\approx 1.0 \times 10^{14}$	?
5	≈ 65000	$\approx 2.0 \times 10^{21}$	?

Unfortunately.....

the derivation-division algorithm gives a bound that grows like....

The Ackermann Function !!!

(thanks to the Online Encyclopedia of integer sequences) $\mathrm{dd}(n) = A(n+1, 1)$

$$\mathrm{dd}(6) \approx 2^{2^{2^{2^{\cdot^{\cdot^{\cdot}}}}}} \quad (65000\ \mathrm{times})$$

In fact, it would be interesting to obtain good

Lower bounds

for Fix(n).

Why? This “toy models” lie very far from the “algebraic world” (zeros of Abelian integrals, perturbation of centers, etc), and perhaps they can give better lower bounds for Hilb(n).

(Using a very easy inductive argment, we can show that $\mathrm{Fix}(n)$ grows at least linearly with n)

Application: A generalized Chebishev system whose span contains $F(a,r,x)$:

$F(a,r,x)$ is a solution of a $Lu=0$

$$L=\frac{d}{dx}\frac{1}{\rho_s}\frac{d}{dx}\cdots\frac{d}{dx}\frac{1}{\rho_1}$$

where $\rho_1, \ldots, \rho_r$ are positive functions on the appropriate interval I.

A basis of solutions is given by fixing an $x_0 \in I$ (say $x_0=1$) and inverting L

$$m_1=\rho_1, \quad m_2=\rho_2\int_{x_0}\rho_2, \ldots., \quad m_s=\rho_1\int_{x_0}\rho_2\int_{x_0}\cdots\int_{x_0}\rho_s$$

For instance, for $F=ax^r+bx$ we obtain

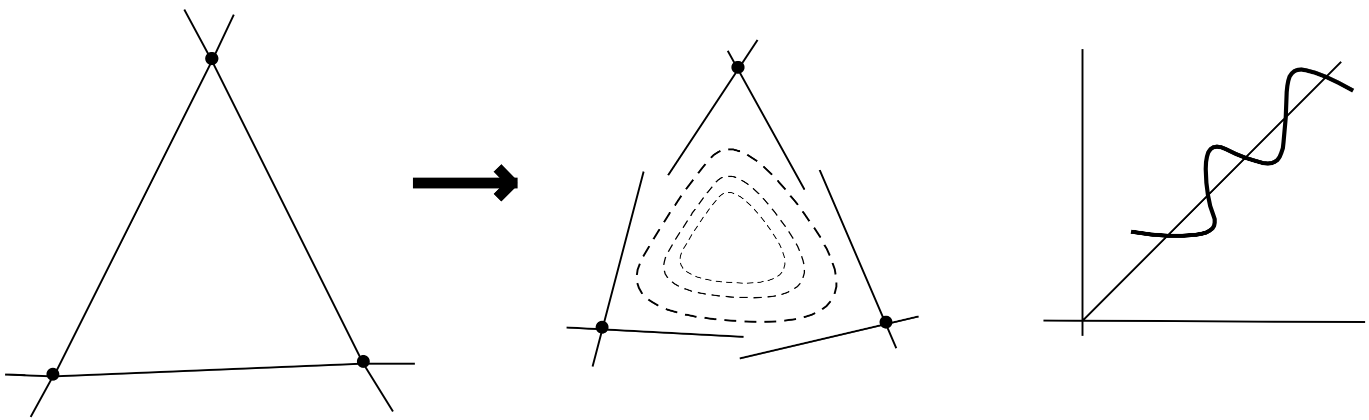
$$m_1=x, \qquad m_2=x\int x^{r-2}=\begin{cases} x\,(x^{r-1}-1)/(r-1) & ,r\neq 1 \\ x\ln(x) & ,r=1 \end{cases}$$

And $F=c_1m_1+c_2m_2$, where $c_1=F(1)=a+b$, $c_2=\frac{1}{x^{r-2}}\frac{d}{dx}\frac{1}{x}F\,(1)=a(r-1)$

$$ax^r + bx = (a+b)\underbrace{x}_{m_1} + (r-1)a\underbrace{x\frac{x^{r-1}-1}{r-1}}_{m_2}$$

We can compute the bifurcation diagram for small values of n (Mardesic: topologically the same as the usual catastrophes).

Interesting: For $n \geqslant 3$, there is not enough freedom to get the full catastrophe .



(The ciclicity of the origin is 3 (Mourtada), which smaller than the maximal number of possible global fixed points $\mathrm{Fix}(3) = 5$).

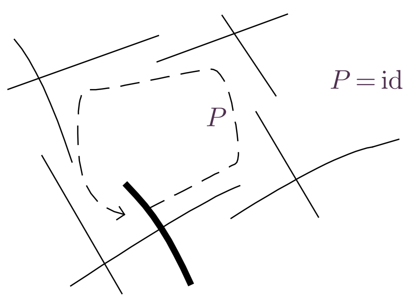
$$ax^r + bx = (a+b)\underbrace{x}_{m_1} + (r-1)a \underbrace{x\frac{x^{r-1}-1}{r-1}}_{m_2}$$

Center conditions: $(a+b)=0$ and $(r-1)a=0 \iff (r=1 \wedge a=-b) \vee (a=0, b=0)$

More generally, if we expand

$$F = (((x^{r_1}+a_1)^{r_2}+a_2)^{r_3}+\cdots)^{r_n}+a_n - x$$

as $F=\sum c_k m_k$, the coefficients $c_1,\dots,c_s$ are polynomials in $\mathbb{Z}[a,r]$ which give the center conditions.



Center problem: "Compute" the algebraic variety $V_n = \mathcal{V}(c_1,\dots,c_s) \subset \mathbb{R}^{2n}_{a,r}$

Structure for the (pseudo) group of power-translations

(Harvey Friedman question - 70')

Consider the following subgroups of $\mathrm{Homeo}(\mathbb{R}, +\infty)$

$$T_{\mathbb{R}} = \{t_a : x \to x + a, a \in \mathbb{R}\}$$

$$P_{\mathbb{Q}_{>0}} = \{p_r : x \to x^r, r \in \mathbb{Q}_{>0}\} \quad \text{(positive rational exponents)}$$

and let $G = \langle T_{\mathbb{R}}, P_{\mathbb{Q}_{>0}} \rangle$ be the group generated by finite compositions of elements of T and P

$$w = \underbrace{t_{a_n} p_{r_n} \cdots t_{a_2} p_{r_2} t_{a_1} p_{r_1}}_{\text{basic word}} : \quad x \to \qquad (\cdots(x^{r_1} + a_1)^{r_2} + a_2)^{r_3} \cdots)^{r_n} + a_n$$

Basic simplifications rules: $\quad p_r p_s = p_{rs}, \qquad t_a t_b = t_{a+b} \qquad p_1 = t_0 = \mathrm{id}$

Is it G isomorphic to the free product $T_{\mathbb{R}} \star P_{\mathbb{Q}_{>0}}$?

In other words, if $r_1, \ldots, r_n \neq 1$ and $a_1, \ldots, a_n \neq 0$ then w is not the identity homeomorphism.

This is much harder than it seems.... (some incomplete proofs where published)

(The original question of Friedman was for $P_{3^{\mathbb{Z}}} = \{x \to x^3, x \to x^{1/3}, \ldots\}$)

White (1988)

Yes for integer powers of a fixed prime number $p > 2$.

Adeleke, Glass, Morley (1992)

Yes for T and $P_{\mathrm{odd},>0} = \{x \to x^{p/q}, p \text{ and } q \text{ odd positive coprime integers}\}$

The case of power maps with even exponents $x \to x^2$ turns out to be much harder...

Cohen (1994)

Yes! $G = \langle T_{\mathbb{R}}, P_{\mathbb{Q}_{>0}} \rangle$ is isomorphic to the free product $T_{\mathbb{R}} \star P_{\mathbb{Q}_{>0}}$.

(45 pages of difficult Galois' theory).

Higman question

The same question for positive real powers

$$P_{\mathbb{R}_{>0}} = \{x \to x^r, r \in \mathbb{R}_{>0}\}$$

Do $T_{\mathbb{R}}$ and $P_{\mathbb{R}_{>0}}$ generate their free product?

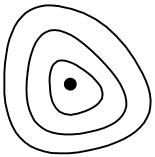
Motivation for group theorists: It not so easy to obtain “natural” explicit examples of free groups.

Nice application (for holomorphic foliations):

For any positive integers $p \neq q$, the subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ generated by

$$z \to \frac{z}{(1 - z^p)^{1/p}} \quad \text{and} \qquad z \to \frac{z}{(1 - z^q)^{1/q}}$$

is free. This is used to generate wild centers for analytic vector fields

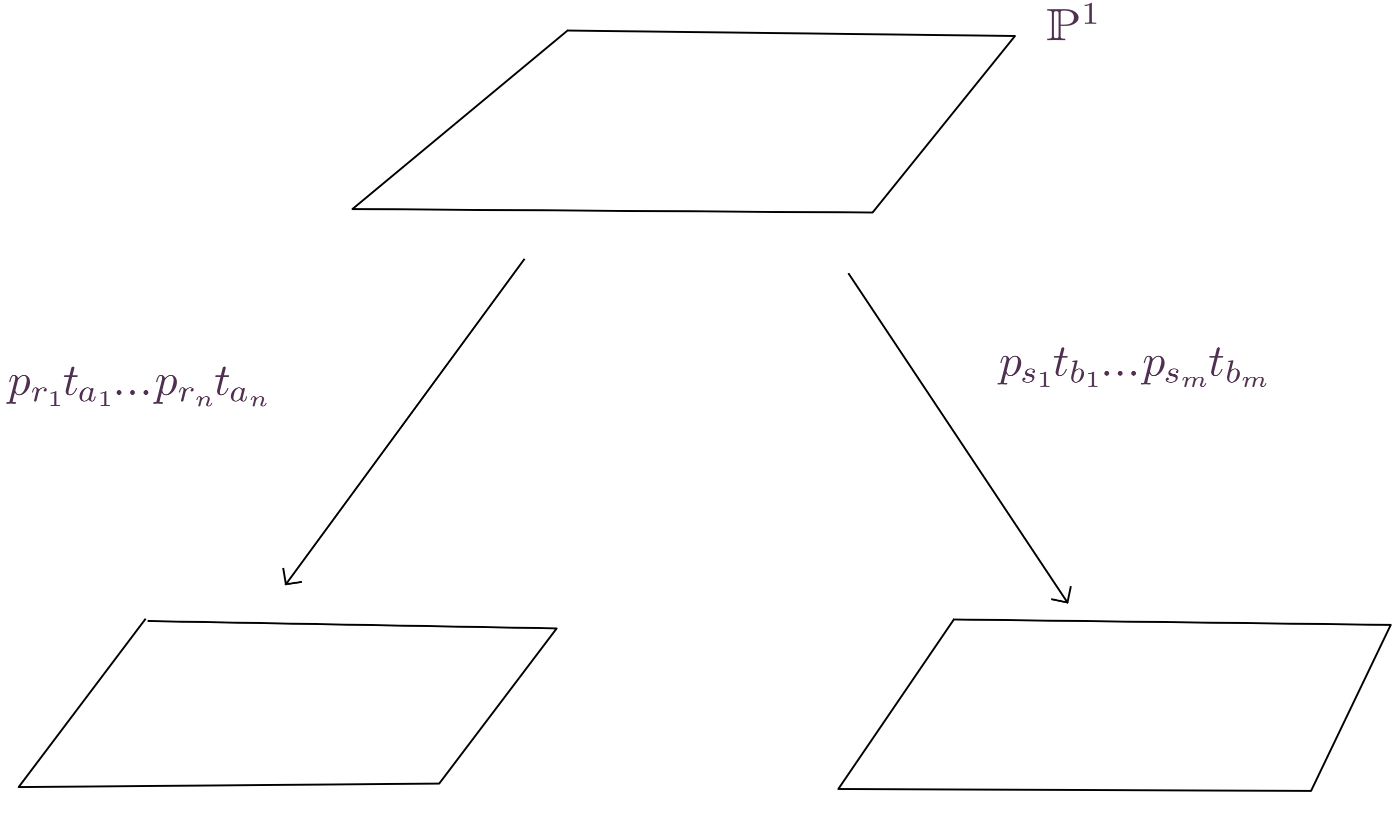


(no “reasonable” first integral can exist).

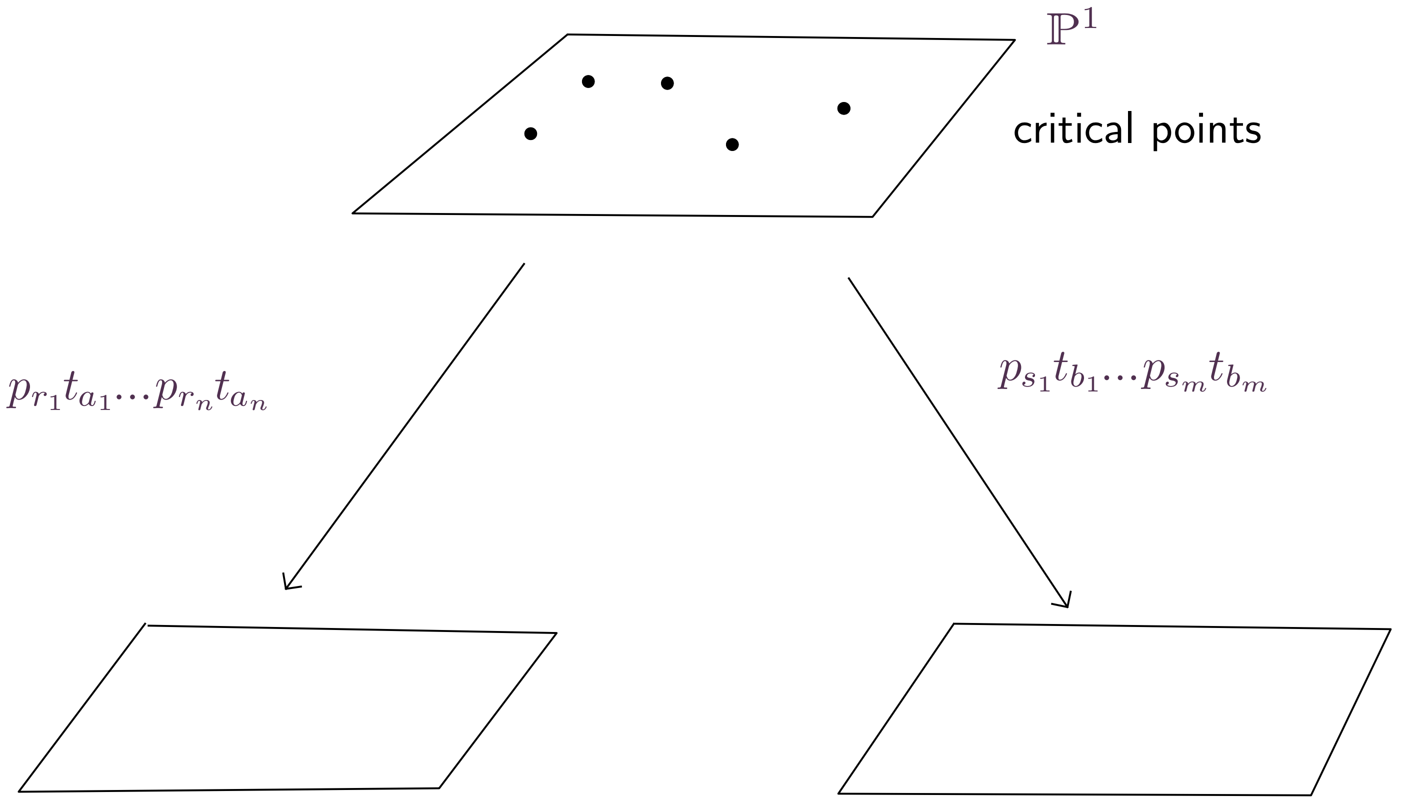
(Very) basic idea of the proof for $\langle P_{\mathbb{Q}_{>0}}, T_{\mathbb{R}} \rangle$:

Is there two different to express a ramified covering using powers and translations?

Assume $r_1, ..., r_n, s_1, ..., s_m$ are positive integers



Compare the ramification points



The procedure is similar to a classical Theorem of Ritt $\longrightarrow$

The structure of the monoid defined by univariate polynomial maps, where the operation is the composition.

A polynomial P is *indecomposable* if $\deg(P) \geqslant 2$ and P cannot be written

$$P = R \circ S$$

with $\deg(R), \deg(S) \geqslant 2$.

Ritt: Given indecomposable polynomials P_i, Q_j, suppose that the identity

$$P_1 \circ \cdots \circ P_k = Q_1 \circ \cdots \circ Q_l$$

holds. Then $k = l$ and we can go from one decomposition to the other through a sequence of identities

$$z^n \circ z^r g(z^n) = z^r g(z)^n \circ z^n, \qquad T_m \circ T_n = T_n \circ T_m$$

T_n is the n^{th} Tchebyshev polynomial $T((z+1/z)/2) = (z^n + 1/z^n)/2$.

For $n=2$: $T_2(z)=2z^2-1$ is "expressible" in $\langle P_{\mathbb{Q}_{>0}}, T_{\mathbb{R}}\rangle$ and we have

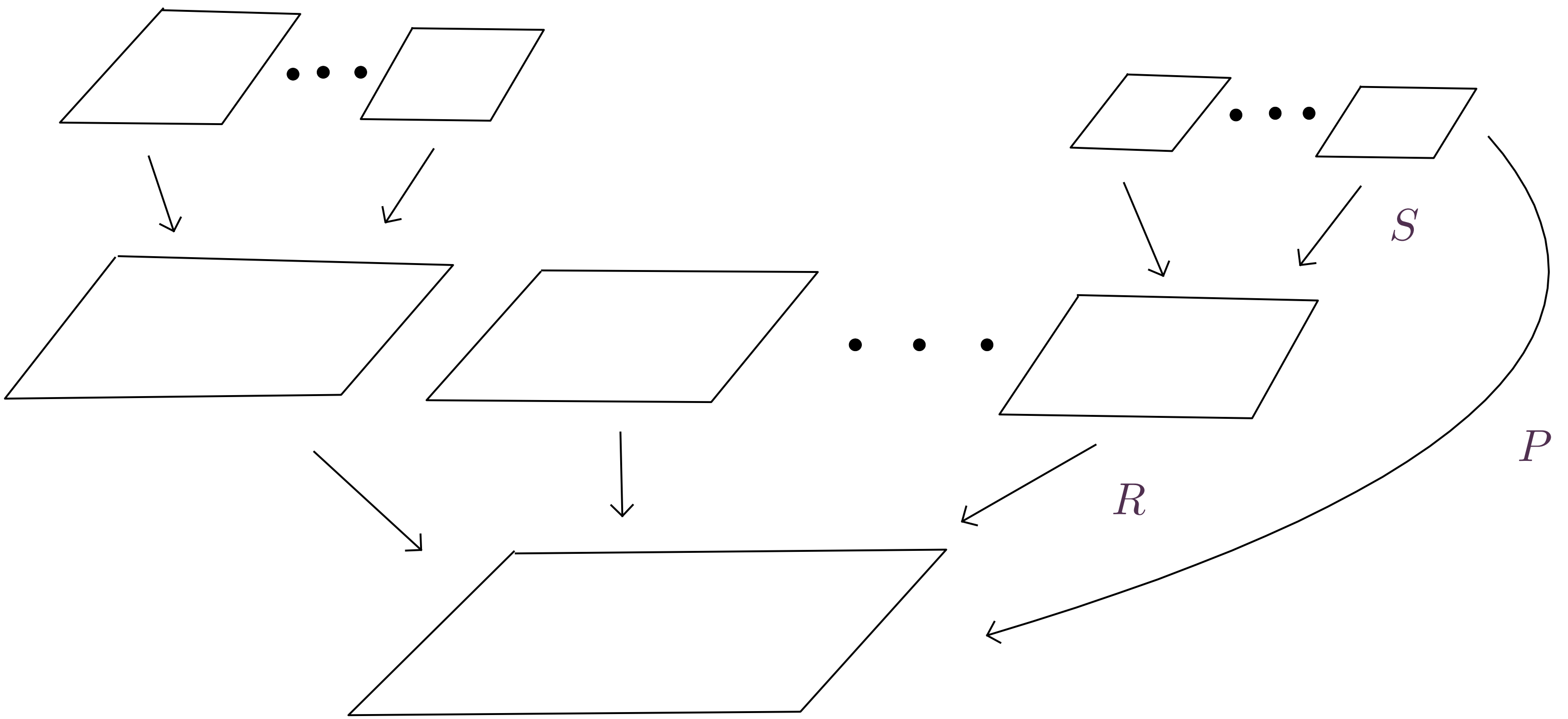
$$T_2\left(z+\frac{1}{z}\right)=z^2+\frac{1}{z^2}$$

i.e. there are two different ways to express the same degree 4 covering $\mathbb{P}^1\to\mathbb{P}^1$.

(This is one of the subtle parts in the Cohen's proof).

Both Ritt's and Cohen's proofs use the *imprimitivity structure* of the monodromy group of these coverings.

$P = R \circ S$



$$\mathrm{mon}(P) \subset \mathrm{mon}(R) \wr \mathrm{mon}(S)$$

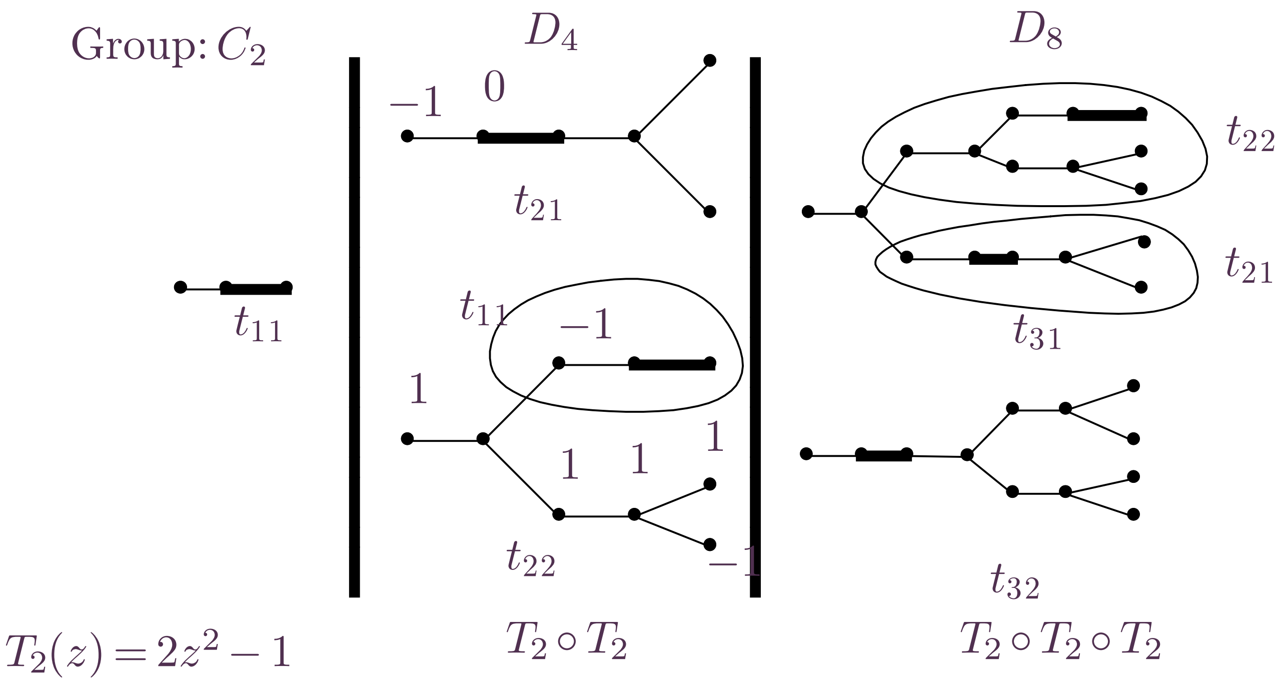
the monodromy of P is a subgroup of the wreath product of the monodromy of R and S

In general, the monodromy of the covering map associated to a word

$$p_{r_1}t_{a_1}\dots p_{r_n}t_{a_n}$$

with $r_1,\dots,r_n\in\mathbb{Z}_{>0}$ is a subgroup of the wreath product $C_n\wr C_{n-1}\wr\cdots\wr C_1$, where each C_i is a cyclic subgroup.

It is better to see this as the group of automorphisms of a rooted tree.



In the above drawing, one realizes D_2, D_4 and D_8 as subgroups of C_2, $C_2\wr C_2$ and $C_2\wr C_2\wr C_2$, respectively, using the monodromy of the Chebyshev polynomials

De la Harpe book's problem: Proof Cohen's theorem using some "Ping-Pong" type argument

In fact, we can see Cohen's proof as a sort of Ping-Pong in field extensions:

Consider an "irreducible" word:

$$w = p_{r_1} t_{a_1} \dots p_{r_n} t_{a_n}$$

We want to show that $w(x)$ is not the identity. This is obvious if all r_i are integers.

Assume that at least one of the r_i is not an integer.

Then, Cohen proves that the algebraic field extension

$$F_0 = \mathbb{C}(x) \subset \mathbb{C}(x, w(x)) = F_w$$

is strict. (very ingenious inductive argument on the length of w)

We call F_w the Cohen extension defined by w.

Naïve approach using asymptotic developpements....

citation from Ritt's 1922 paper

* An idea which presents itself naturally is to consider this problem as one in undetermined coefficients. One might hope, for instance, with a judicious use of linear transformations, to show without actually determining the coefficients of the polynomials, that aside from Cases (*b*) and (*c*) mentioned in the introduction there is only one possibility, which would, of course, have to be Case (*a*). A study of the equations for the coefficients convinces me that such a plan would not be easy to carry out, and that the function-theoretic methods used here are not far-fetched.

What about allowing *negative exponents*?

$$\langle P_{\mathbb{Q}}, T_{\mathbb{C}} \rangle$$

Has no-longer a free product structure.

For instance, the following identity holds:

$$i + (i + (i + x^{-1})^{-1})^{-1} \equiv x$$

If we want to add the inversion $z \to 1/z$, it is more natural to work with $\mathrm{PSL}(2, \mathbb{C})$.

A normal form result for the groupoid $\mathcal{G}$ of germs generated by

$$\mathrm{PSL}(2,\mathbb{C}) \quad \text{and} \quad \mathrm{Exp}$$

(we need to consider groupoids since the normal form depends on the choice of branch of the logarithm)

Exp is generated by $\{e, l\}$ (with its multiple branches)

$\mathrm{PSL}(2,\mathbb{C})$ is generated by the subgroups

$$\begin{aligned} W &= \{w : z \mapsto 1/z\}, && \text{(inversion)} \\ T &= \{t_a : z \to z + a \quad : \quad a \in \mathbb{C}\} && \text{(translations)} \\ S &= \{s_\alpha : z \to \alpha z \quad : \quad \alpha \in \mathbb{C}^\star\} && \text{(scalings)} \end{aligned}$$

We consider the subgroups

$$H_0 = T \rtimes \{s_{-1}\}, \qquad H_1 = S \rtimes \{w\}$$

and choose right transversals T_0, T_1 for H_0, H_1, respectively (i.e. a collection of unique representants for the right cosets which contains the identity).

For instance.....

$$T_0 = \{\boldsymbol{s}_\rho : \rho \in \Omega\} \cup \{\boldsymbol{s}_\rho \boldsymbol{w} \boldsymbol{t}_b : \rho \in \Omega, b \in \mathbb{C}\}$$
$$T_1 = \{\boldsymbol{t}_b : b \in \mathbb{C}\} \cup \{\boldsymbol{t}_a \boldsymbol{w} \boldsymbol{t}_b : a \in \Omega, b \in \mathbb{C}^* \setminus \{-1/a\}\} \cup \{\boldsymbol{t}_c \boldsymbol{w} : c \in \mathbb{C}^*\}$$

where $\Omega = \{\alpha : \mathrm{Re}(\alpha) > 0\} \cup \{\alpha : \mathrm{Re}(\alpha) = 0, \mathrm{Im}(\alpha) > 0\}$ is the region shown in figure 1.[1]

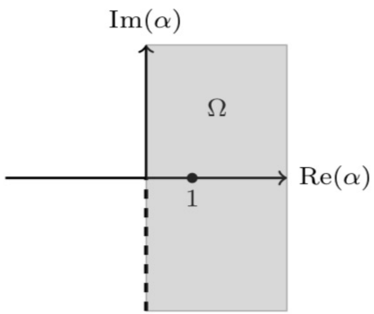


Figure 1: The region Ω

we use the (well-known) presentation of $\mathrm{PSL}(2,\mathbb{C})$, with relations such as

$$\frac{1}{a+\frac{1}{z}} = -\frac{1}{a^2}\left(-a + \frac{1}{z+\frac{1}{a}}\right), \quad \forall a \in \mathbb{C}\setminus\{0\}, z \in \mathbb{C},$$

Normal forms: Think that we have a HUGE alphabet where each letter corresponds to a germ of the exponential, log, translation, scaling or inversion....

DEFINITION: A (T_0, T_1) normal form in $\mathcal{G}$ is a word (more precisely, a path) of the form

$$g = g_0\, h_1\, g_1 \cdots h_n\, g_n, \qquad n > 0$$

where

i. g_0 lies in $\mathrm{PSL}(2,\mathbb{C})$

ii. each h_i is either e (exponential) or l (logarithm)

iii. if $h_i = e$ then $g_i \in T_0$

iv. if $h_i = l$ then $g_i \in T_1$

v. There are no subwords of the form $e\,l$ or $l\,e$

$\mathrm{NF}_{(T_0,T_1)}$ is the set of such normal forms.

There is an obvious map

$$\varphi\colon \mathrm{NF}_{(T_0,T_1)} \longrightarrow \mathcal{G}$$

(realization of a path as a germ in the groupoid)

It is easy to prove that this map is surjective, using the obvious relations in $\mathcal{G}$

For example: for $\theta \in \mathrm{PSL}(2,\mathbb{C})$, if we write $\theta = H_0\, h$, with $h \in T_0$ (right transversal)

$$e\, t_a = s_{\exp(a)} e$$

$$e\, s_{-1} = w\, e$$

$$\cdots e\, H_0\, h \cdots = \cdots H_1 e\, h \cdots$$

(we notice that this is very similar to the normal form for the HNN extensions...)

$$\text{Further relation:} \qquad l\, e = t_{2\pi i k}$$

(k depends on the choice of branch of logarithm)

Unfortunately, φ is not injective

For instance, recall that the following identity holds

$$T_2\left(\frac{z+\frac{1}{z}}{2}\right)=\frac{z^2+\frac{1}{z^2}}{2}$$

where $T_2(z)=2z^2-1$ is the second Chebyshev polynomial.

This implies that the following normal form

$$s_{-1}\,t_2\,wt_{-1/4}p_2\,t_{-1/2}\,wt_1p_2t_{-1}wt_{1/2}p_{1/2}t_{1/4}wt_{-2}p_{1/2}t_2$$

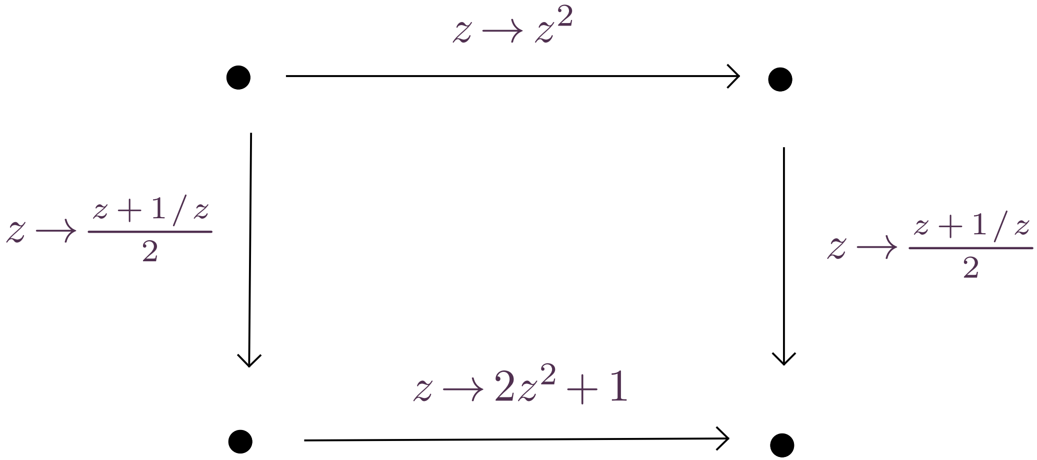
is mapped to the identity germ by φ. (but perhaps this is the only type of counterexamples...)

We use the notation:

$$p_\alpha=e\,s_\alpha l,\quad \alpha\in\mathbb{C}^\star$$

which represents the power map $z\to z^\alpha$

Two ways of writing the second Chebyshev polynomial



A relation involving double coverings $\mathbb{P}^1 \to \mathbb{P}^1$

Theorem (Normal Form): φ is injective when restricted to TAME normal forms.

What is TAME?

A NF is algebraic if it has the form

$$g = \theta_0 p_1 \theta_1 p_1 \cdots \theta_n p_n$$

where p_i are power maps with rational exponents (lying in $\Omega \cap \mathbb{Q}$) and $\theta_i \in T_1$ (transversal to H_1)

Further, g is affine if each θ_i is an map

Algebro-transcendental decomposition

Now, each normal form g can be uniquely decomposed as

$$g = \quad a_0 \quad \gamma_1 \quad a_1 \quad \cdots \quad \gamma_m \quad a_m$$

where: the a_i are algebraic subpaths and $\gamma_i = e, l$ or p_α with $\alpha \in \Omega \setminus \mathbb{Q}$.

(to guarantee uniqueness, we require that there are no subpaths of the form $e\, s_\alpha\, l$...)

m is called the height of g.

Def. g is tame if either height(g)=0 and g is affine or else for each algebraic subpath bounded by

$$[e, l], \quad [l, e], \quad [l, p], \quad [p, e], \quad [p, p]$$

is of affine type.

Main theorem: if g is tame (and not the identity normal form) then $\varphi(g)$ is not the identity germ.

Idea of the proof of the main theorem: For $\mathrm{height}(g) = 0$ this is Cohen's theorem.

For $m = \mathrm{height}(g) \geqslant 1$, we consider the chain of field extensions (starting from $F_0 = \mathbb{C}(x)$) obtained by adding successive germs

$$g = \underbrace{a_0}_{\mathrm{Cohen type}} \quad \underbrace{\gamma_1}_{\mathrm{transcendental type}} \quad \underbrace{a_1}_{\mathrm{Cohen type}} \quad \cdots \quad \underbrace{\gamma_m}_{\mathrm{transcendental type}} \quad \underbrace{a_m}_{\mathrm{Cohen type}}$$

And we prove (inductively) that $\mathrm{trans.deg}\, F_g / F_0 = m$

For this last part, we use in an essential way Ax's theorem:

Let $k \subset K$ be fields of char 0:

Theorem(Ax)

Suppose that K_0 is a field such that $k \subset K_0 \subset K$ and $\mathrm{trans.deg}_k(K_0) = n$ for some $n \geq 1$. Let $\delta \in \mathrm{Der}_k(K)$ be a derivation such that $\mathrm{Const}(\delta) = k$, and suppose that

$$\omega_1, \ldots, \omega_n \in \Omega^1_k(K)$$

are closed 1-forms defined over K_0 satisfying $\omega_i(\delta) = 0$, for $i = 1, \ldots, n$. Then $\omega_1, \ldots, \omega_n$ are linearly dependent over k.

We apply this result in the case where the 1-forms are

$$\omega = \frac{dx}{x} - dy \quad (x = e^y) \quad \text{or} \qquad \omega = \frac{dx}{x} - \alpha\frac{dy}{y} \quad (x = y^\alpha), \quad \alpha \notin \mathbb{Q}$$

Geometric interpretation: The germ of associated to g is is a local leaf of a 1-foliation in some $(\mathbb{P}^1)^m$, defined by some derivation ∂ in the kernel of these ω. We compute the dimension of the Zariski closure of such a leaf.

Assume that k is algebraically closed.

Theorem(Ax) Suppose that there exists nonzero $x_1, ..., x_m \in K$ and elements $e_1, ..., e_m \in k$ not all zero such that the differential form

$$\sum_{i=1}^{m} e_i \frac{d x_i}{x_i}$$

is *exact*. Then, $x_1, ..., x_m$ satisfy a power resonance relation over k.

i.e. there exists integers (not all zero) $k_1, ..., k_m$ such that

$$x_1^{k_1} .... x_m^{k_m} = c, \quad \text{for some } c \in k$$

Geometrically, this means that the leaf of the foliation (defined by g) lies in this monomial hypersurface. This contradicts some inductive hypothesis based on the Cohen extensions...

In fact, we can consider more general foliations...

$$\log - \text{Lambert function}$$

$$z \to z + \ln(z)$$

corresponds to a leaf of

$$\left(1 + \frac{1}{z}\right) dz = dw$$

For instance, this allows us to include the flow map of one-dimensional the one-vector fields

$$\boldsymbol{w_{k,\lambda} = \frac{x^{k+1}}{1 + \lambda x^k} \frac{\partial}{\partial x}}, \qquad k \in \mathbb{Z}, \quad \lambda \in \mathbb{C}$$

Consequence: Suppose that a diffeo tangent to the identity is expressed as

$$x \to \exp(t_1 w_{k_1,\lambda_1}) \cdots \exp(t_n w_{k_n,\lambda_n})(x) \quad \text{(finite composition of time } t \text{ flow maps)}$$

then it cannot be the identity except if there some obvious inner simplifications (e.g. $t_i = 0$)

General formulation (?) Compare the automorphism group of certain foliations with the automorphism group of one of its (transcendental) leafs.

E.g. The function $x = e^y$ is a leaf E of

$$\omega = \frac{dx}{x} - dy$$

and the only relations (in $\mathcal{G}$)

$$e\,s_{-1} = w\,e \qquad \text{and} \qquad e\,t_b = s_{e^b}\,e$$

correponds to maps from E to E which comes from automorphisms of $\mathcal{F}_\omega$.

First application

Consider the following subgroups of $\mathrm{Homeo}(\mathbb{R}, +\infty)$

$$T = \{x \to x + a, \quad a \in \mathbb{R}\}$$

$$S^+ = \{x \to \alpha x, \quad \alpha \in \mathbb{R}_{>0}\}$$

$$\mathrm{Exp} = \{x \to \exp(x), \quad x \to \ln(x)\}$$

Let $\mathrm{Aff}^+ = T \rtimes S^+$ be the subgroup of positive affine maps. Let $G_{\mathrm{Aff}^+, \mathrm{Exp}}$ be the subgroup generated by Aff^+ and Exp.

The conjugation by the exponential defines an isomorphism

$$\theta : T \longrightarrow S^+$$

(as subgroups of Aff^+)...

HNN extensions

Given a group G and an isomorphism $\theta\colon H \to K$ between two subgroups $H, K \subset G$ which is not inner.

there exists a group G' containing G and an element $s \in G'$ such that

$$\theta(a) = s\,a\,s^{-1}, \quad \forall a \in H$$

(i.e. the homomorphism θ becomes inner in G'). s is the stable letter.

Notation: $G\star_\theta$

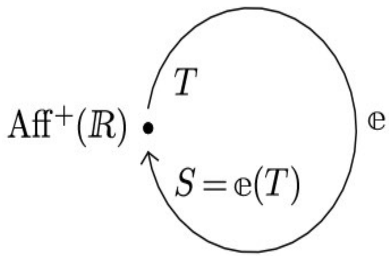
In our case: There is a surjective morphism

$$\phi\colon \mathrm{Aff}^+\star_\theta \to G_{\mathrm{Aff}^+,\mathrm{Exp}}$$

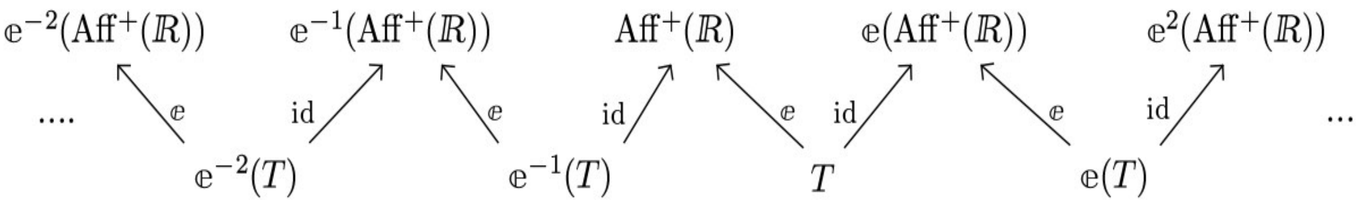
mapping the stable letter to the exponential map.

Theorem: ϕ is an isomorphism.

Graph of groups:



$= R \rtimes \langle x \mapsto \exp(x) \rangle$



Second application

A "finitary" version of Lemme 1:

We consider the following groups

$$\tilde{G} = \left\{ f(x) = x + \sum_{k \geqslant 1} a_k x^{-k} \colon a_k \in \mathbb{C} \right\}$$

(formal diffeomorphisms tangent to identity at ∞ with no translation term), and

$$\tilde{H} = \exp \tilde{G} \exp^{-1} \qquad (\exp^{-1} = \log)$$

(this makes sense as a transseries e.g. for $f(x) = x + a\,x^{-k}$ we have

$$\exp f \log(x) = \exp(\ln(x)) \exp(1 + a \ln(x)^{-k})$$

We can see $\tilde{G}$ and $\tilde{H}$ as subgroups of the group $\mathbb{T}_{>1}$ of *infinite large transseries* (Ecalle)

(we have the right to compose such transseries, take inverses, etc).

Question: Consider the subgroup

$$\langle \tilde{G}, \tilde{H} \rangle \subset \mathbb{T}_{>1}$$

Is it isomorphic to the free product $\tilde{G} \star \tilde{H}$?

Equivalently, for arbitrary $f_1, ..., f_n \in \tilde{G} \setminus \{\mathrm{id}\}$,

$$f_1 (\exp f_2 \exp^{-1})\, f_3 (\exp f_4 \exp^{-1}) ... (\exp f_n \exp^{-1}) \neq \mathrm{id}$$

Special case: $G_{\mathrm{finit}} \subset \tilde{G}$ (finitary subgroup)

$$G_{\mathrm{finit}} = \left\{ f(x) = \mathrm{Exp}\left(a_1 x^{-k_1} x \frac{\partial}{\partial x} \right) ... \mathrm{Exp}\left(a_n x^{-k_n} x \frac{\partial}{\partial x} \right)(x) : k_i \in \mathbb{Z}_{\geqslant 2}, a_i \in \mathbb{C} \right\}$$

$\left(\mathrm{Exp}\left(t x^{-k} x \frac{\partial}{\partial x} \right)(x) = x(1 + t x^{-k})^{1/k}\right.$ - tangent to identity to order $-k+1$). We define

$H_{\mathrm{finit}} = \exp G_{\mathrm{finit}} \exp^{-1}$

Theorem: $\langle \tilde{G}_{\mathrm{finit}}, \tilde{H}_{\mathrm{finit}} \rangle$ is isomorphic to their product

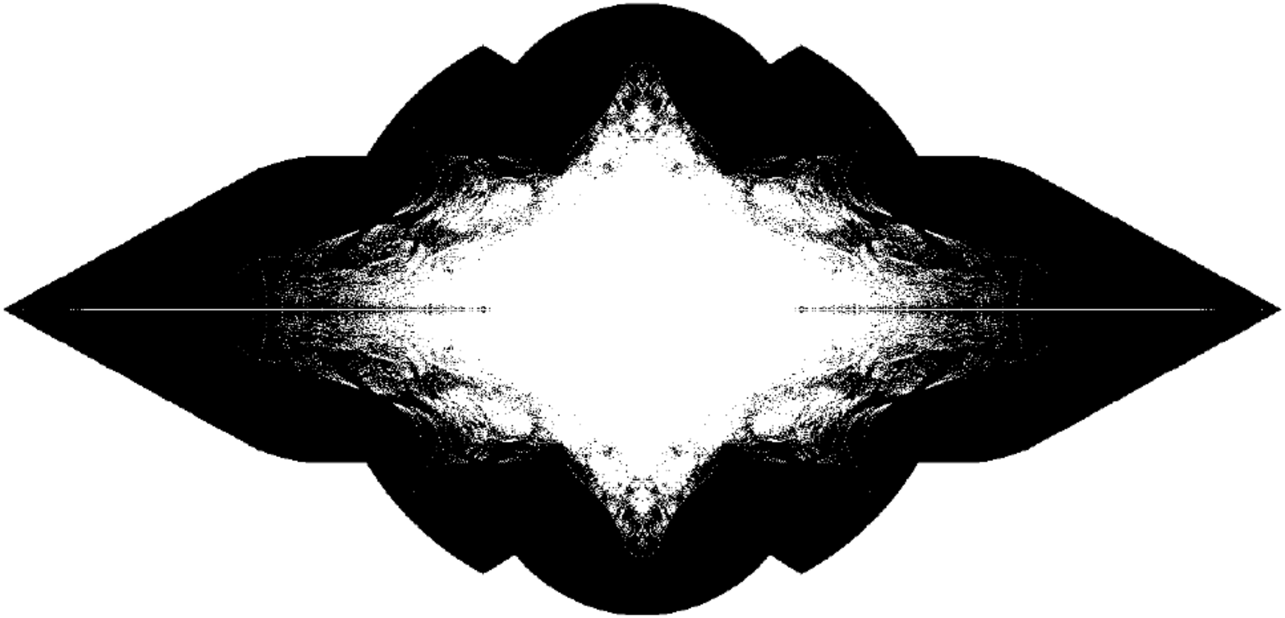
We can ask similar questions replacing $\exp$ by $x \to x^\lambda$ or Lambert maps....

Testing for freenes in $\mathrm{PSL}(2,\mathbb{C})$: The "particular case" of 2-parabolic subgroups

Any (non-commutative) subgroup with two parabolic generators is conjugated to

$$\boldsymbol{P}_a = \left\langle z \to z + a, z \to \frac{z}{1 + a z} \right\rangle, \qquad a \in \mathbb{C}^\star$$

We say that a is a *free point* if $\boldsymbol{P}_a$ is a free group.



Set of free points in the $\mathbb{C}_\lambda$ plane, $\lambda = a^2/2$ (dark colored) (picture by J. Gilman)

Unifying setting (Lattes maps and more...)

Finite quotient of affine maps (Milnor)

$$\begin{array}{ccc} \mathbb{C}/\Lambda & \xrightarrow{\varphi(x)=ax+b} & \mathbb{C}/\Lambda \\ \theta \downarrow & & \downarrow \theta \\ \mathbb{P}^1\setminus\mathcal{E} & \xrightarrow{f} & \mathbb{P}^1\setminus\mathcal{E} \end{array}$$

θ finite covering

Example: $\Lambda = 2\pi\mathbb{Z}$, $\varphi(x) = nx$, $\theta(x) = \exp(ix)$, and $f(z) = z^n$ (power maps)
$\Lambda = 2\pi i\mathbb{Z}$, $\varphi(x) = nx$, $\theta(x) = 2\cos(x)$, and $f(x) = T_n(x)$ (Chebyshev pols)

Example: $\Lambda = \mathbb{Z} + i\mathbb{Z}$, $\varphi(x) = (1+i)z$, $\theta = \wp_\Lambda$ and $f = \theta\varphi\theta^{-1}(z) = \frac{(z+1/z)}{2i}$
(Λ of rank 2 $\longrightarrow$ Lattes maps)

A new relation involving $\wp$-function