Movable singularities of ODEs: a topological approach

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Abstract

Near a regular point of a complex, scalar and first-order ODE, a local system of solutions defines a trivial fibration disk $\times$ disk. The holomorphic foliation associated to the ODE is obtained by patching up all these local systems, giving a partition of the ambient space into connected Riemann surfaces called leaves (maximal solutions) and singularities of the ODE. There is no reason why this object should continue to be a locally trivial fibration near a singular point, because nothing guarantees that neighboring leaves all have the same topology.

Of course in the simplest example, where the foliation is defined by the level sets of a holomorphic submersion $H : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$, the theorem of J. Milnor ensures that H is a holomorphic fibration with total space a well-chosen complement U of the singular fiber $H^{-1}(0)$. In that case the topology of the other leaves is constant, and the natural morphism $\pi_1 \left(H^{-1}(\mathrm{cst})\right) \to \pi_1(U)$ is injective: the "holes" in the leaves can only be caused by a set of finitely many "special" leaves (separatrices). Any foliation satisfying this property is deemed *incompressible*.

D. Marín and J.F. Mattei have generalized Milnor theorem to most singular planar holomorphic foliations: under generic assumptions a germ of a foliation is incompressible. We will explain the principle of their proof and how to weaken their assumptions down to an almost sharp characterization of incompressible foliations, then use this study to exhibit examples of compressible foliations.

While we understand how to guarantee incompressibility, it is not clear how the extra topology can be accounted for in case of compressibility. We will observe on some examples that it is produced by so-called *movable singularities* (poles of the solutions which do not come from singularities of the ODE). This explanation is satisfying only in a global context, for what does it mean for a solution to "tend to ∞" when the ODE is only defined in a polydisk? In this talk we propose to use compressibility failures to represent persistent movable singularities in a local context, and we will present some (hopefully) convincing arguments in favor of such a definition.

1 Context and first examples

Everything takes place in $(\mathbb{C}^2, 0)$ in the holomorphic category. The first part of this course, regarding Marín–Mattei theorem, is a survey of the paper [Tey15] (in French).

1.1 Milnor's theorem

1.1.1 Statement

Consider a holomorphic submersion

$$f : \mathcal{B} \longrightarrow \mathbb{C}$$

with a singular fiber (maybe not irreducible) $\mathcal{S} = f^{-1}(0)$. Here f is holomorphic on a (neighborhood) of the Euclidean closed ball $\mathrm{cl}(\mathcal{B}) \subset \mathbb{C}^2$ with radius so small that the fibers of f are transverse to $\partial\mathcal{B}$.

Theorem (Milnor, [Mil68])**.** *There exists a family* ***of Milnor tubes*** $(\mathcal{T}_\eta)_{\eta>0}$ *around* $\mathcal{S}$*, that we can take as inverse images* $\mathcal{T}_\eta := f^{-1}(\eta\mathbb{D})$ *for small* $\eta > 0$*, such that for all* $z \neq 0$ *in* $\eta\mathbb{D}$*:*

$$\pi_1\left(f^{-1}(z) \cap \mathcal{T}_\eta\right) \hookrightarrow \pi_1(\mathcal{T}_\eta \backslash \mathcal{S}) \simeq \pi_1(\mathcal{B} \backslash \mathcal{S}).$$

1.1.2 Monodromy

Here we discuss briefly the example

$$f(x,y) = x^p y^q$$

for coprime positive integers $p,\ q$. In that situation the singular fiber is the union $\mathcal{S} = \{x=0\} \cup \{y=0\}$ of two lines.

1.2 Holomorphic foliations

We use the duality between vector fields and 1-form fields in the planar situation. (It is customary to also identify vector fields with derivations.)

$$X = \begin{bmatrix} A \\ B \end{bmatrix} \leftrightsquigarrow A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} \leftrightsquigarrow \omega = A\mathrm{d}y - B\mathrm{d}x$$

The identity $\omega(X) = 0$ means that the integral curves of X, which are the range of maximal trajectories solving

$$\begin{cases} \dot{x}(t) & = A(x(t), y(t)) \\ \dot{y}(t) & = B(x(t), y(t)) \end{cases}, \quad t \in (\mathbb{C}, 0) \ ,$$

coincides with the graph of a solution to the differential equation

$$A(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} = B(x,y)\,.$$

The geometric object obtained by partitioning the space into such integral curves γ is called a **holomorphic foliation**. For convenience we write $\mathcal{F}_X$ or $\mathcal{F}_\omega$ the induced foliation.

- Either $\gamma = \{\mathrm{pt}\}$ is a stationary point of X: a **singularity** of $\mathcal{F}_X$. Their set is written $\mathrm{Sing}(\mathcal{F}_X)$.
- Either γ is non-constant, it is a Riemann surface: a **leaf** of $\mathcal{F}_X$.

In the holomorphic world we can always assume that $\mathrm{codim}\,\mathrm{Sing}(\mathcal{F}_X) = 2$, so that $\mathrm{Sing}(\mathcal{F}_X)$ is discrete for planar foliations.

1.2.1 Local structure near a regular point

Because of the theorem of vector field rectification, if $X(p) \neq 0$ then there exists a holomorphic change of coordinates

$$\Psi \ : \ \left(\mathbb{C}^2, 0\right) \longrightarrow \left(\mathbb{C}^2, p\right)$$

conjugating X to, say, $\frac{\partial}{\partial x}$:

$$\Psi^* X := \mathrm{D}\Psi^{-1}(X \circ \Psi) = \frac{\partial}{\partial x}.$$

What it means is that the leaves of the foliation $\Psi^*\mathcal{F}_X$ are given by the discs

$$\{x \in \varepsilon\mathbb{D},\ y = \mathrm{cst}\}$$

or, in other words, that the function

$$f \ : \ (x,y) \longmapsto y$$

is a first-integral of $\Psi^* X$. Therefore, near a regular point the leaves of $\mathcal{F}_X$ are given by the level sets of the holomorphic submersion $\Psi_* f$.

1.2.2 Elementary singular points

Take a stationary point p of X, say $p=(0,0)$. The linear part of X can be identified with a matrix

$$(ax+by+\ldots)\frac{\partial}{\partial x}+(cx+dy+\ldots)\frac{\partial}{\partial y}\leftrightsquigarrow\left[\begin{array}{cc}a & b\\ c & d\end{array}\right]=:\mathrm{D}_0X.$$

Its spectrum $\{\lambda_1,\lambda_2\}\subset\mathbb{C}$ contains dominant data about the dynamics of X near p.

Definition.

1. When D_0X is not nilpotent, say $\lambda_2\neq0$, we define the **eigenratio** of X at p as the quotient

$$\lambda:=\frac{\lambda_1}{\lambda_2}.$$

2. In that case, we say that p is an elementary singularity of X if $\lambda\notin\mathbb{Q}_{>0}$. This covers the following cases:
   (a) **hyperbolic** singularity: $\lambda\notin\mathbb{R}$;
   (b) **node** singularity: $\lambda>0$;
   (c) **saddle** singularity: $\lambda<0$ (**quasi-resonant** when $\lambda\notin\mathbb{Q}$);
   (d) **saddle-node** singularity: $\lambda=0$. The λ_2-eigenspace is called the **strong eigendirection**.

1.2.3 Reduction

Every singularity $p\in\mathbb{C}^2$ of a holomorphic foliation $\mathcal{F}$ can be "reduced" through a proper rational map (see *e.g.* A. Seidenberg algorithm [Sei68])

$$E\,:\,\mathcal{M}\to\left(\mathbb{C}^2,p\right)$$

where $\mathcal{M}$ is a conformal neighborhood of a tree $E^{-1}(0)$ of divisors $\mathbb{P}_1(\mathbb{C})$ with normal crossings (at points called **corners**). The pull-back $E^*\mathcal{F}$ only possesses reduced singularity. It may also happen that $E^*\mathcal{F}$ admit **dicritic** components: the foliation is regular and transverse to some divisor.

An important ingredient to anything pertaining to foliations $\mathcal{F}$ with an invariant curve S is the Camacho-Sad index of an elementary singular point p. Generically speaking, the index $\mathrm{CS}(\mathcal{F},S,p)$ is given by the eigenratio $\frac{\lambda_1}{\lambda_2}$ of $\mathcal{F}$ at p if S is the eigendirection of λ_2.

Theorem (Camacho-Sad index formula [CS82])**.** *If $\mathcal{F}$ is a reduced foliation on a neighborhood $\mathcal{V}$ of a compact invariant curve S, then*

$$\sum_{p\in\mathrm{Sing}(\mathcal{F})}\mathrm{CS}(\mathcal{F},S,p)=\mathrm{chern}\left(S\hookrightarrow\mathcal{V}\right).$$

1.2.4 Movable singularities in polynomial foliations: Painlevé property

Take two polynomials $P,\ Q$ and consider the 1-form $\omega:=P\mathrm{d}y-Q\mathrm{d}x$.

Example. Saddle-nodes $x\frac{\mathrm{d}y}{\mathrm{d}x}=y^{k+1}$.

Definition. We say that the polynomial differential equation $\omega=0$ has the **Painlevé property** if the analytic continuation (along some path) of any solution $x\mapsto y(x)$ may only have singularities of the following type:

- either the singularity $(x_*,y(x_*))$ is a singularity of $\mathcal{F}_\omega$ (a **fixed** singularity);
- or locally $y(x)=\frac{a}{x-x_*}+$holomorphic (a **movable** singularity which is moreover a pole of order 1).

More generally we may speak of a **ramified movable pole** or **essential movable singularity** *etc.* by providing corresponding local expressions for the analytic continuation near x_*.

Remark. Such singularities are deemed movable because the location of x_* is a non-constant function of the leaf.

Theorem. *[Lor05, Pai73]*

1. *If all solutions to $\omega = 0$ have no movable singularity then the equation is affine,* i.e. $\deg Q \leq 1$.

2. *If $\omega = 0$ has the Painlevé property then it is (at most) a Riccati equation,* i.e. $\deg Q \leq 2$.

3. *There are no essential movable singularities.*

1.3 Why go to complex domain?

The topology of leaves is richer: it encodes more information. Such as:

1. the monodromy of linear systems: Galoisian considerations;

2. periods of 1-forms: isosynchronicity problem for centers;

3. obstructions to solve (co)homological equations;

4. ...

2 Marín–Mattei's theorem: incompressible foliations

In [MM08, MM14], D. Marín and J.–F. Mattei have found sufficient conditions for a foliation to be *incompressible*.

Definition. $\mathcal{F}$ stands for a germ of a holomorphic foliation near a singular point $(0, 0)$, say on a small Euclidean ball $\mathcal{B}$. We say that $\mathcal{F}$ is **incompressible** if there exists:

- a finite union $\mathcal{S} \subset \mathcal{B}$ of $\mathcal{F}$-invariant, analytic curves containing the singularity, say $\mathcal{S} = \{f = 0\}$ for some analytic f, called **distinguished separatrices**;

- a family of Milnor tubes $(\mathcal{T}_\eta)_{0<\eta\leq\eta_0}$ for $\mathcal{S}$ (as before: $T_\eta = f^{-1}(\eta\mathbb{D})$), on which $\mathcal{F}$ is holomorphic;

- a neighborhood $\mathcal{U}$ of $(0, 0)$;

such that:

1. $\mathcal{U} \subset \mathcal{T}_{\eta_0}$ and the inclusion induces an isomorphism between fundamental groups $\pi_1(\mathcal{U}\backslash\mathcal{S}) \simeq \pi_1(\mathcal{T}\backslash\mathcal{S})$;

2. $\mathcal{T}_\eta \subset \mathcal{U}$ for all η small enough;

3. for every leaf $\mathcal{L}$ of $\mathcal{F}|_{\mathcal{U}\backslash\mathcal{S}}$ the canonical morphism induced by $\iota : \mathcal{L} \hookrightarrow \mathcal{U}\backslash\mathcal{S}$

$$\iota^* : \pi_1(\mathcal{L}) \longrightarrow \pi_1(\mathcal{U}\backslash\mathcal{S})$$

   is injective.

Remark. Condition (1) means that the topology of the ambient space must be "as simple as possible": one is not allowed to take complicated neighborhoods to compensate for the potential complicated topology of the leaves.

2.1 Motivations: nonlinear monodromy

Incompressibility ensures the existence of a *foliated* universal covering: the universal covering

$$\pi_{\mathcal{U}} : \widetilde{\mathcal{U}\backslash\mathcal{S}} \to \mathcal{U}\backslash\mathcal{S}$$

is also (by restriction) a universal covering for each leaf of $\mathcal{F}$. The automorphisms group of this covering then consists in symmetries of the foliation $\pi_{\mathcal{U}}^{*}\mathcal{F}$, and therefore naturally acts on its leaves space $\tilde{\Omega}_{\mathcal{U}}$:

$$\mathfrak{m}_{\mathcal{U}} : \mathtt{Aut}(\pi_{\mathcal{U}}) \longrightarrow \mathtt{Aut}\left(\tilde{\Omega}_{\mathcal{U}}\right)$$

which is called the **monodromy** of $(\mathcal{F},\mathcal{U},\mathcal{S})$. The quotient $\Omega_{\mathcal{U}} := \tilde{\Omega}_{\mathcal{U}}/\mathfrak{m}_{\mathcal{U}}$ can be canonically identified with the leaves space of $\mathcal{F}$.

Heuristically, the analytic structure of the space of leaves up to diffeomorphism is in bijection with the local analytic class of the singularity. D. Marín and J.-F. Mattei proved that the "germification" of $(\mathcal{F},\mathcal{U},\mathcal{S})$ as $\eta_0 \to 0$ is a complete modulus of local classification for generic foliations [MM14].

If one wishes to endow $\Omega_{\mathcal{U}}$ with a structure of a (non Hausdorff) analytic variety, then one needs the existence of a curve $\mathcal{C}$ (not necessarily irreducible) which fulfills the following definition.

Definition. A germ of analytic curve $\mathcal{C}$ (not necessarily irreducible) is a **completely connected transverse** of $\mathcal{F}$ if there exists a pair $(\mathcal{U},\mathcal{S})$ in which $\mathcal{F}$ is incompressible and such that:

1. $\mathcal{C}\backslash\mathcal{S}$ is a smooth analytic curve transverse to the leaves of $\mathcal{F}$;
2. $\mathrm{Sat}_{\mathcal{F}}(\mathcal{U}\cap\mathcal{C}\backslash\mathcal{S}) = \mathcal{U}\backslash\mathcal{S}$,
3. each leaf of $\pi^{*}\mathcal{F}$ intersects at most once each connected component of $\pi^{-1}(\mathcal{C})$.

Observe that there always exists a curve $\mathcal{C}$ satisfying (1) and (2) [Lor10, p161]. A corollary of the theorem of Marín–Mattei is the existence of such a curve $\mathcal{C}$ [MM08, Théorème 6.1.1, p900]

Remark. As for the linear context, it is expected that the non-linear monodromy carries Galoisian information about solvability by quadratures of the underlying ODE.

2.2 Statement

Theorem (Marín–Mattei)**.** *[MM08, MM14] Every foliation whose singular reduction does not contain any saddle-node or quasi-resonant, non-linearizable saddle [+ technical assumption about "dead branches"] is incompressible.*

Remark.

1. It is possible to describe explicitly a set $\mathcal{S}$: it is the union of the closure of the separatrices of $E^{*}\mathcal{F}$ crossing non-dicritic components of the exceptional divisor, plus a leaf per dicritic component.
2. The hypothesis about the local type of singularities is generic once the combinatorial data of the reduction tree and the number of final elementary singularities is fixed.

End of first course

The principal aim of this course is to establish the following three theorems.

Theorem A. *A germ of a saddle-node or a quasi-resonant saddle is incompressible.*

One may then hope that Marín–Mattei theorem holds true in all generality. Unfortunately this is not the case.

Theorem B. *There exists singular, non-dicritic foliations which are compressible.*

It is possible, though, to weaken the assumptions in Marín–Mattei theorem: incompressibility and existence of a completely connected transverse will depend crucially on how the saddle-nodes are placed and oriented in the reduction.

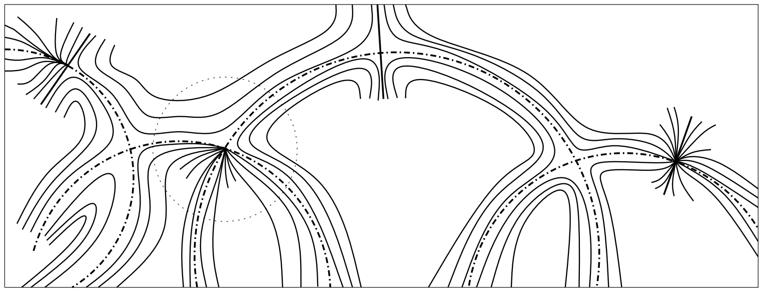

Figure 1: A saddle-node in a corner: not strongly presentable

Definition.

1. A foliation $\mathcal{F}$ is **presentable** if it is incompressible and admits a completely connected transverse.
2. $\mathcal{F}$ is **strongly presentable** if the strong eigendirection of any saddle-node in a (minimal) reduction of $\mathcal{F}$ never coincides with a divisor.

Theorem C. *Every germ of a strongly presentable foliation is presentable.*

Although this is not a criterion, failure to be strongly presentable should bring suspicion about presentability. Completely characterizing presentable foliations seems to be a hard task, for these are global constraints although strongly presentability only deals with local properties of the singularities.

2.3 Proof

We work in the reduced foliation $E^*\mathcal{F}$ which, for simplicity reasons, we just write $\mathcal{F}$.

The construction consists in gluing together finitely many pairwise distinct local blocks $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$, each one containing an elementary singularity, with blocks covering the regular part. A result *à la* Van Kampen allows to localize in the blocks $\mathcal{B}_\alpha$ all the difficulty.

These blocks cannot be arbitrary: they must comply with the requirements we give below. Additionally, they must possess one degree of freedom, related to their "size", allowing to control the resulting neighborhood of the special set $\mathcal{S}$.

Blocks assembly is done by induction, by picking a component of the exceptional divisor then by browsing through the reduction tree. We get to another component D_{j+1} by passing through a corner, and we arrive there with a outbound size specified by the component D_j, and we assemble the blocks living on D_{j+1} to this one by adjusting their size.

All this is only partially correct, since some special regular blocks (those containing the so-called "dead branches", see below) do not possess a degree of freedom. A technical obstruction therefore appears when D_{j+1} is attached to at least two dead branches and does not have any other singularity save for those shared with D_j. Therefore, the construction must start from D_{j+1}. Obviously, we should not want more than one such component.

In the following sections we sketch the ingredients that ensures that Theorems A and C are true.

2.3.1 1-connectedness and localization of the proof

Definition. [MM08, Définition 1.2.2, p861]Let a foliation $\mathcal{F}$ be given on a domain $\mathcal{U}$ and let $A \subset B$ be subsets of $\mathcal{U}$. We say that A is **1–connected** in B (relatively to $\mathcal{F}$) if for each leaf $\mathcal{L}$ of $\mathcal{F}$ and all paths α from A and β of $B \cap \mathcal{L}$ which are homotopic in B, there exists a path in $A \cap \mathcal{L}$ which is homotopic to both α in A and β in $B \cap \mathcal{L}$.

Definition. [MM08, Definition 2.1.1, p864]The notation ∂A stands for the closed set $\mathrm{cl}\,(A\backslash\mathrm{int}\,(A))$.

1. We say that $\mathcal{B}_\alpha$ is an **adapted foliated block** if:

   (FB1) each connected component of $\partial\mathcal{B}_\alpha$ is incompressible in $\mathcal{B}_\alpha$;

(FB2) $\mathcal{F}$ is transverse to $\partial\mathcal{B}_\alpha$;

(FB3) $\mathcal{F}$ is incompressible in $\mathcal{B}_\alpha$;

(FB4) each connected component of $\partial\mathcal{B}_\alpha$ is 1–connected in $\mathcal{B}_\alpha$.

2. We will say that the collection of adapted foliated blocks $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is an **adjusted assembly** if:
   - for every α, $\beta \in \mathcal{A}$ the intersection $\mathcal{B}_\alpha \cap \mathcal{B}_\beta$ is either empty or a connected component of $\partial\mathcal{B}_\alpha$ and of $\partial\mathcal{B}_\beta$;
   - $E\left(\bigcup_{\alpha\in\mathcal{A}}\mathcal{B}_\alpha\right)$ is a neighborhood $\mathcal{U}$ of the singularity with a deleted special set $\mathcal{S}$.

Here is the localization theorem.

Theorem. [MM08, Théorème 2.1.2, p864] *If $(\mathcal{B}_\alpha)_{\alpha\in\mathcal{A}}$ is an adjusted assembly relatively to $\mathcal{F}$ then $\mathcal{F}$ is incompressible in $(\mathcal{U},\mathcal{S})$.*

2.3.2 Dead branches and initial component

Definition.

1. A **dead branch** B of $\mathcal{F}$ is a maximal union of neighboring components of the exceptional divisor, each one being of one the following types:
   - there are at most two corners and no other singularity of $\mathcal{F}$ (a **link** of B);
   - there is exactly one corner and no other singularity of $\mathcal{F}$ (the **end** of B),
   - there is exactly one corner and one singularity of $\mathcal{F}$ (the **anchoring component** of B),

   in such a way that B has exactly one end and one anchoring component. The incidence graph of B is therefore a tree having the combinatorial structure of a chain.
2. An **initial component** of $\mathcal{F}$ is a non-dicritic component $\mathcal{C}$ of $E^{-1}(0)$ to which is anchored at least two dead branches and having a single additional singularity of $\mathcal{F}$.

It is well-known [MM08, p866] that for a non-dicritic **generalized curve** (a foliation reduced by the desingularization morphism of its separatrix set) there is at most a single initial component, which allows for the inductive proof to work. Moreover this initial component has exactly two dead branches anchored to it, one of which has an end which is the divisor being created by the first blow-up.

This property can be restated word for word in the case of strongly presentable foliations, which allows to get rid of the [technical assumption about "dead branches"] in Marín–Mattei theorem.

Theorem. *Let $\mathcal{F}$ be a germ of strongly presentable foliation, reduced by a minimal morphism $E : \mathcal{M} \to (\mathbb{C}^2, 0)$. Then $E^{-1}(0)$ contains at most one initial component.*

When it exists, this component has exactly two dead branches anchored to it. Each corner singularity are linearizable rational saddles. The end of one of these branches is the first divisor created by the reduction.

Let us present examples of non-strongly presentable foliations having a lot of dead branches.

Example. Let us start with $\mathcal{F}_{\omega_1}$ (a resonant node):

$$\omega_1(x,y) \quad := \quad (x-y)\,\mathrm{d}x + x\mathrm{d}y\,.$$

The foliation is reduced after one blow-up and its reduction contains a single singularity: a saddle-node in formal normal form $t^2\mathrm{d}y - y(1-t)\,\mathrm{d}t$.

More generally one easily checks that for every $n \in \mathbb{N}_{>0}$ the 1–form

$$\omega_n(x,y) \quad := \quad \left(\frac{x}{n} - y^n\right)\mathrm{d}x + nxy^{n-1}\mathrm{d}y$$

is reduced after n blow-ups and its reduction is a dead branch with n components. When $n > 1$, the last blow-up creates a divisor having a corner and a saddle-node in normal form

$$\hat{\omega}_n(t,y) \quad := \quad t^2\mathrm{d}y + y\left(1 - \frac{1}{n}t\right)\mathrm{d}t$$

through which passes the strict transform of the unique separatrix $\{x=0\}$ of $\mathcal{F}_{\omega_n}$.

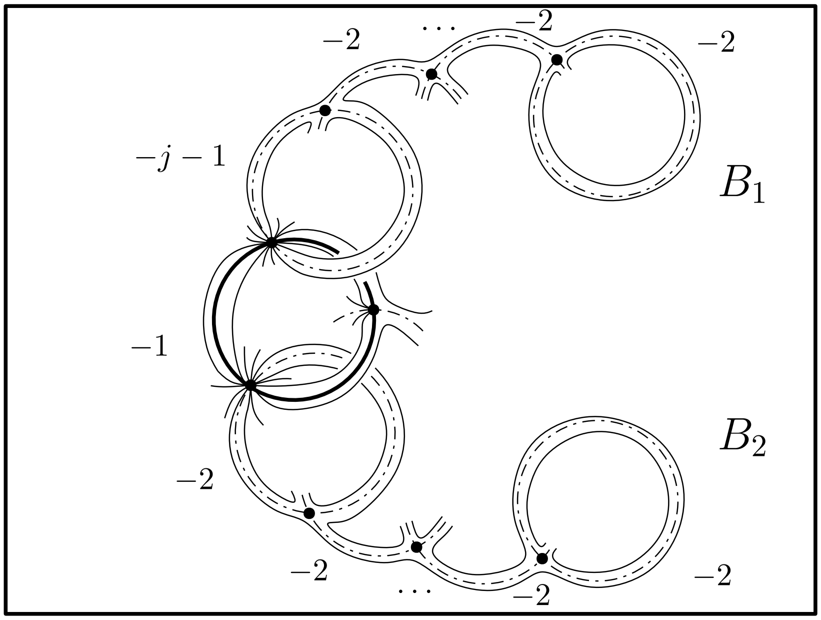

Figure 2: An initial component. Numbers indicate the Chern class of the components.

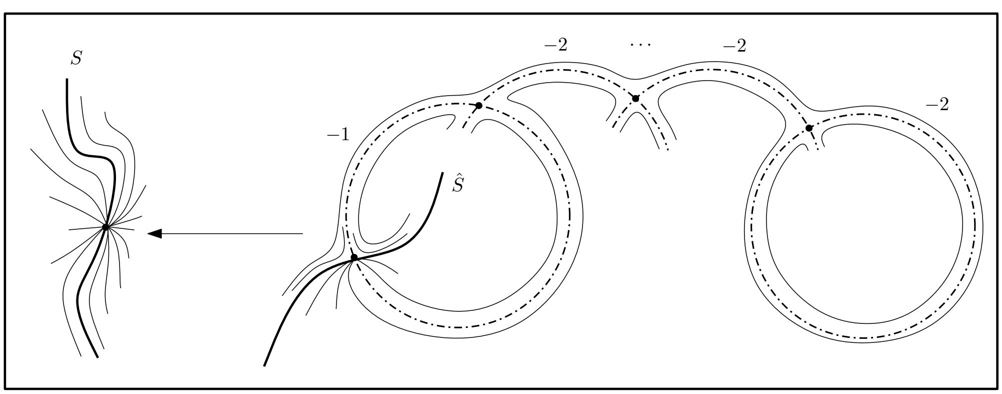

Figure 3: A dead branch obtained by reducing a single curve (bold).

Example. Let us now explain briefly how to build a foliation $\mathcal{F}$ having an initial component with as many dead branches as we wish. We consider a neighborhood of a divisor D with Chern class -1 containing $m \in \mathbb{N}_{>0}$ singularities $(p_j)_{1\leq j\leq m}$, each one of which is locally conjugate to $\mathcal{F}_{\omega_j}$ (given in the previous example) and a resonant-saddle p_0 which we describe below in more details. The divisor D will be the first one created when reducing the foliation $\mathcal{F}$.

After reduction of each p_j, exactly m dead branches are anchored to D and its Chern class has become

$$\mathfrak{c} \quad := \quad -1 - \sum_{j=1}^{m} n_j \in \mathbb{Z}_{<-1}\,.$$

The resulting anchoring singularity of the dead branch at p_j is a saddle-node whose strong direction coincides with D: its Camacho-Sad index with respect to D is 0. Let us write Δ_j its strong holonomy (tangent-to-identity).

At p_0 we put a resonant-saddle with linear part $\mathfrak{c}x\mathrm{d}y - y\mathrm{d}x$ in a local chart where $D = \{x = 0\}$, whose holonomy along D

$$\Delta_0\,(h) \quad = \quad \exp\left(2\mathrm{i}\pi\mathfrak{c}\right)h + \cdots = h + \cdots$$

is the inverse of $\bigcirc_{j=1}^{m}\Delta_j$. It is possible to find such a foliation near p_0 because of the realization part of Martinet–Ramis theorem [MR83].

The fact that all these local ingredients can be glued together to form the reduction of a foliation in $\left(\mathbb{C}^2, 0\right)$ follows from a slightly more general discussion than the one presented further down in Section 2.4, but is nonetheless folklorally true (we refer to [Lor10] for details).

2.3.3 Adapted foliated block for non-degenerate elementary singularities

We will not go into full details, and content ourselves with proving that a solitary non-degenerate elementary singularity is incompressible. The construction of a full-fledged foliated adapted block containing this singular point can become technical, and can be safely ignored for these singularities. The main point is that for any leaf $\mathcal{L}$ the trace of $\mathrm{cl}\,(\mathcal{L})$ on the boundary of a block is connected, which guarantees 1-connectedness (FB4).

Up to change the local variables, we may assume that a foliation $\mathcal{F}$ with non-zero eigenratio λ is induced by

$$\omega_R \quad := \quad \lambda x\mathrm{d}y - y\,(1 + R)\,\mathrm{d}x$$

where $R \in x\mathbb{C}\,\{x, y\}$. Let $\mathcal{V}$ be a polydisk $\rho_0\mathbb{D} \times r_0\mathbb{D}$ so small that:

- R is holomorphic on a neighborhood of $\mathrm{cl}\,(\mathcal{V})$;
- $||R||_{\mathcal{V}} < 1$.

We consider the domain

$$\mathcal{V}^* \quad := \quad \mathcal{V}\backslash\,\{x = 0\}\ \ ,$$

on which $\mathcal{F}$ is everywhere transverse to the fibers of $\Pi\ :\ (x, y) \longmapsto x$, and go the universal covering

$$\begin{aligned} \mathcal{E}\,:\,\tilde{\mathcal{V}} \quad &\longrightarrow \quad \mathcal{V}^* \\ (z, y) \quad &\longmapsto \quad (\exp z, y)\ . \end{aligned}$$

Objects hatted with a "~" will indicate pulled-back objects in the universal covering, in particular we use the projection

$$\tilde{\Pi}\,:\,(z, y) \quad \longmapsto \quad z\,.$$

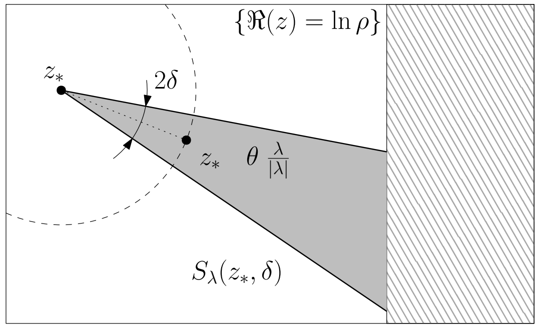

Figure 4: A stability beam (grayed-out region).

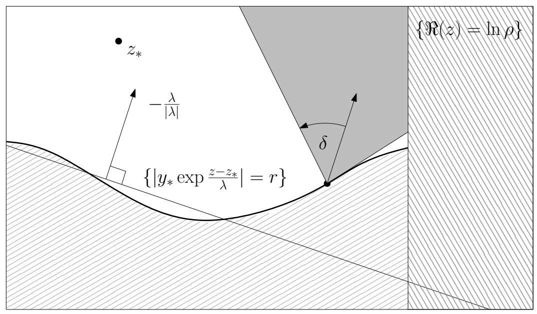

Figure 5: The universal covering of a leaf passing through $p_* = (z_*, y_*) \in \tilde{\mathcal{U}}$ (complement of hatched regions). The real line $\left\{\left|y_* \exp \frac{z-z_*}{\lambda}\right| = r\right\}$ stands for the trace on the boundary $\tilde{\Pi}\left(\tilde{\mathcal{U}}\right) \times r\mathbb{S}^1$ of the corresponding leaf for the linear foliation $\tilde{\omega}_0$.

Proposition. *For each* $0 < \rho \leq \rho_0$ *and* $0 < r \leq r_0$ *we set* $\mathcal{U}(\rho, r) := \rho\mathbb{D} \times r\mathbb{D}$. *Every leaf of* $\tilde{\mathcal{F}} := \mathcal{E}^*\mathcal{F}|_{\mathcal{U}(\rho,r)}$ *is simply connected.*

Here the local separatrix set $\mathcal{S}$ of $\mathcal{F}$ coincides with $\mathcal{B} \cap \{xy = 0\}$, so that the Milnor tubes $\mathcal{T}_\eta$ are simple to describe and they obviously satisfy the properties required by the incompressibility definition. Therefore it only remains to prove the proposition for $\mathcal{V} := \mathcal{U}(\rho, r)$.

Definition. Being given $z_* \in \widetilde{\Pi}\left(\tilde{\mathcal{V}}\right)$ we call **stability beam** of vertex z_* and opening $\frac{\pi}{2} > \delta > 0$ the region of $\tilde{\mathcal{V}}$ given by

$$S_\lambda(z_*, \delta) := \left\{z_* - t\theta\frac{\lambda}{|\lambda|} \, : \, |\arg\theta| < \delta \, , \, t \geq 0\right\} \cap \tilde{\Pi}\left(\tilde{\mathcal{V}}\right) \ni z_* \quad .$$

The name of "stability beam" is justified by the following lemma.

Lemma. *Take* $\rho > 0$ *and* $r > 0$ *such that*

$$M := \sup_{(x,y)\in\mathcal{U}(\rho,r)} |R(x,y)| < 1$$

and let $\delta := \arccos(M)$. *Then for all* $p_* = (z_*, y_*) \in \tilde{\mathcal{U}}$, *every path* γ *based at* z_* *and included in* $S_\lambda(z_*, \delta)$ *lifts through* $\widetilde{\Pi}$ *in the leaf of* $\tilde{\mathcal{F}}$ *containing* p_*.

Proof. It is a simple variational argument: because of the transversality between $\widetilde{\Pi}$ and $\widetilde{\mathcal{F}}$, it is sufficient to ensure that a local solution $z \mapsto y(z)$ never escapes $r\mathbb{D}$. But this is clear since in a stability beam the modulus of y decreases. □

Remark. Existence of stability beams impose a really strong condition on the shape of the boundary of a leaf: it cannot be too irregular (conic convexity). Therefore the universal covering of a typical leaf is very much alike Figure 5.

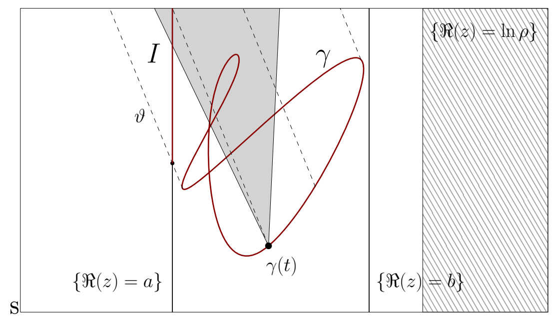

Figure 6: Homotopy within stability beams from γ to I.

Let us prove now the proposition (see Figure 6). For a loop $\tilde{\gamma}$ within a leaf $\tilde{\mathcal{L}}$ of $\tilde{\mathcal{F}}$ we set $\gamma:=\tilde{\Pi}\circ\tilde{\gamma}$ its projection whose range is a compact contained in a strip $\{a\leq\Re(z)\leq b\}$ for two real numbers $a\leq b<\ln\rho$. Any stability beam $S_{\lambda}(\gamma(t),\delta)$ intersects one or the other line bounding the strip, say $\{\Re(z)=a\}$ to fix ideas, by following a line of direction $\vartheta\in\mathbb{S}^{1}$. For $t\in[0,1]$ we consider a parameterization $s\in[0,1]\mapsto h_{s}(t)$ of the segment linking $\gamma(t)$ to $\{\Re(z)=a\}$ following the direction $\vartheta\in\mathbb{S}^{1}$. Then $(h_{s})_{s\in[0,1]}$ is a free homotopy between γ and a path whose image is a segment I. Since the homotopy takes place in the union of stability beams it lifts in the leaf $\widetilde{\mathcal{L}}$ as an homotopy between $\tilde{\gamma}$ and a path tangent to $\tilde{\mathcal{F}}|_{\tilde{\Pi}^{-1}(I)}$. But the latter is a smooth real 1-dimensional foliation, transverse to the projection $\tilde{\Pi}$: since I is contractible its leaves also are, so that $\tilde{\gamma}$ is homotopically trivial in $\tilde{\mathcal{L}}$.

2.3.4 Adapted foliated block for (convergent) saddle-nodes

We repeat the same procedure as before but for saddle-nodes. Up to change the local variables, we may assume that the foliation $\mathcal{F}$ is induced by

$$\omega_{R}\quad:=\quad x^{k+1}\mathrm{d}y-y(1+R)\,\mathrm{d}x$$

where $R\in x\mathbb{C}\{x,y\}$ (we do not consider the case of a *divergent* saddle-node here).

Fix a point $p_{*}:=(z_{*},y_{*})\in\tilde{\mathcal{V}}$. For $\theta\in\mathbb{S}^{1}$ we build the path $z_{\theta}\,:\,t\geq0\mapsto z_{\theta}(t)$ solution of

$$\dot{z}_{\theta}(t)\quad=\quad-\theta\exp(kz_{\theta}(t))$$

with initial value $z_{\theta}(0)=z_{*}$. Implicitly:

$$\exp(kz_{\theta}(t))\quad=\quad\frac{\exp(kz_{*})}{1+k\theta t\exp(kz_{*})}\,.$$

Also:

$$\begin{cases}\Re(z_{\theta}(t)) & \sim_{t\to+\infty}-\frac{1}{k}\ln t\\ \Im(z_{\theta}(t)) & \sim_{t\to+\infty}\Im(z_{*})-\frac{1}{k}\arg\theta\end{cases}\,.$$

Yet it may happen that before going to $-\infty$ the real part of z_{θ} exceed $\ln\rho$ (it is particularly the case when $\theta\exp(kz_{*})<0$ since then $\exp(kz_{\theta})$ admits a pole).

Definition. We call **stability beam** of vertex z_{*} and opening $\frac{\pi}{2}>\delta>0$ the region of $\tilde{\mathcal{U}}(\rho,r)$ containing z_{*} and given by

$$S_{0}(z_{*},\delta)\quad:=\quad\left\{z_{\theta}(t)\,:\,|\arg\theta|<\delta\,,\,t\geq0\,,\,(\forall\tau\leq t)\;z_{\theta}(\tau)\in\tilde{\mathcal{U}}\right\}\,.$$

As before we can lift in a leaf any path included in a stability beam.

Remark. Here again the boundary of a leaf has a conic convexity property. As a consequence the universal covering of a typical leaf looks like what happens in Figure 9. The presence of «tongues» with infinite extent on the left comes from the "saddle" part. Indeed in the sectors $\left\{\left|\arg\left(x^{k}\right)-\pi\right|<\frac{\pi}{2}\right\}$, bounded by dashed lines in the figure, the y-coordinate of the leaf is of order $\exp\left(\frac{-1}{kx^{k}}\right)$ and tends rapidly to infinity when x nears the singularity. On the contrary the leaves over node sectors $\left\{\left|\arg\left(x^{k}\right)\right|<\frac{\pi}{2}\right\}$ tend flatly to 0 (see *e.g.* [Tey04]).

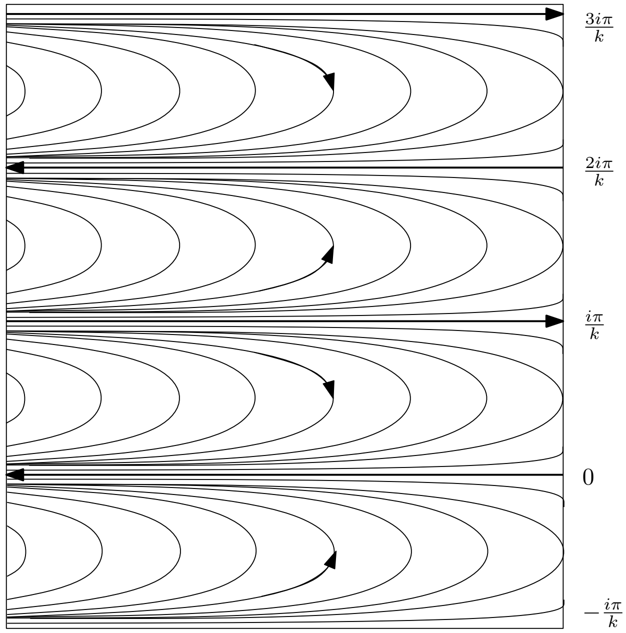

Figure 7: Integral curves z_1. We obtain z_θ by translating z_1 by $-\mathrm{i}\frac{\arg\theta}{k}$

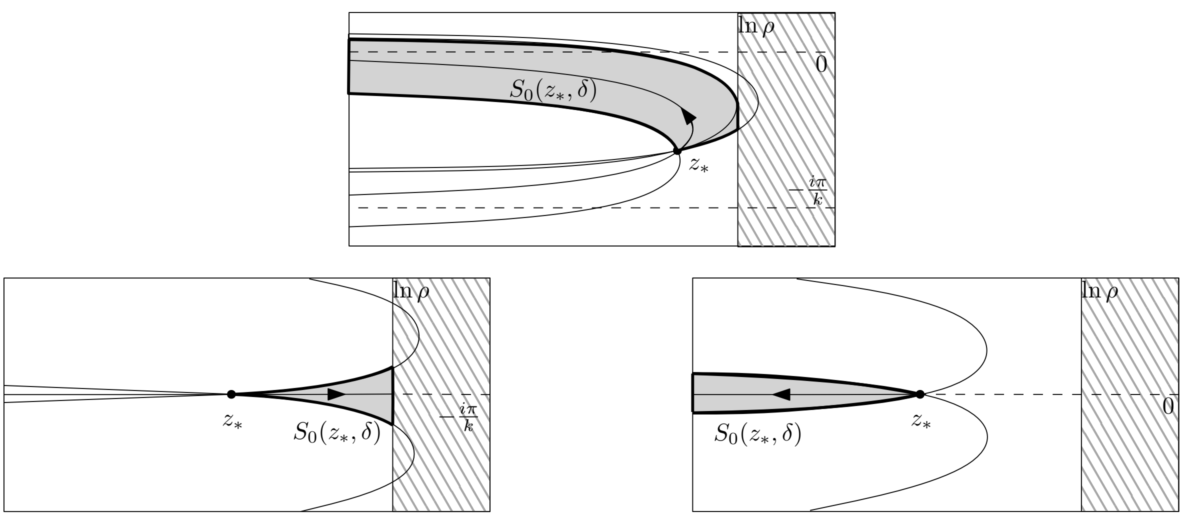

Figure 8: Some stability beams (uniformly grayed-out region).

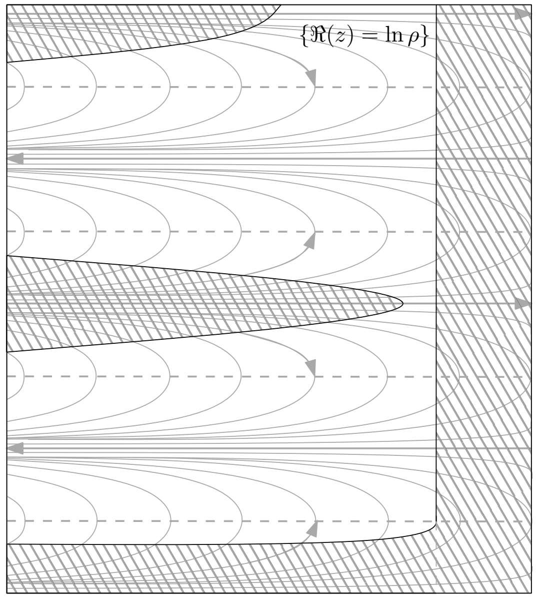

Figure 9: Universal covering of a typical leaf (complement of hatched regions).

We prove now that a leaf of $\mathcal{F}_{\omega_R}$ is simply connected. The strategy is similar to that of non-degenerate singularities, but the fact that the "direction" of a stability beam is not constant brings in more subtlety. It is sufficient to build a free homotopy $(h_s)_{s\in[0,1]}$ between the projection $\gamma := \tilde{\Pi}\circ\tilde{\gamma}$ of a tangent cycle $\tilde{\gamma}$ and a path bounding a region with empty interior, such that for all fixed t the path $s\mapsto h_s(t)$ lies in $S_\lambda(h_0(t),\delta)$.

The process is depicted in Figure 10 and is done in two steps.

- Parts of the range of γ contained in the "saddle" strips $\{\cos(k\Im(z))\leq 0\}$ are sent within $\{\cos(kz)=0\}\cup\{\Re(z)=\ln\rho\}$ by following paths z_1.
- Parts of the range of γ contained within the "node" strips $\{\cos(k\Im(z))\geq 0\}$ are sent within $\{\Re(z)=a\}\cup\Gamma$ following paths z_1, where Γ is the union of the ranges of trajectories z_1 emanating from the points $\ln\rho+\mathrm{i}^{\pi}/_{2k}+\mathrm{i}^{\pi}/_{k}\mathbb{Z}$.

2.3.5 Stubborn paths

We explain now why a saddle-node cannot be oriented arbitrarily in the reduction tree in order to invoke the localization theorem. It is because the "horizontal" component $\partial\mathcal{U}(\rho,r)\cap\{|y|=r\}$ is *not* 1-connected in $\mathcal{U}(\rho,r)$.

Definition. A tangent path Γ having endpoints in a single transverse $\{y=\mathrm{cst}\}$ and whose lift $\tilde{\Gamma}$ in the universal covering links two distinct components of $\mathrm{cl}\left(\tilde{\mathcal{L}}\right)\cap\{|y|=r\}$, will be called a **stubborn path**. Because it will not let itself be pushed out of the block.

2.4 An example of a compressible foliation

2.4.1 Local construction

This example is based on the model saddle-node

$$\omega_0(x,y) \quad := \quad x^2\mathrm{d}y - y\mathrm{d}x\,,$$

but the construction generalizes straightforwardly for higher codimension $x^{k+1}\mathrm{d}y-y\mathrm{d}x$. We exploit the fact that there exists stubborn paths (see previous Section), that is, paths which cannot be pushed to the boundary of leaves $\mathcal{L}$ because $\partial\mathcal{L}\cap\partial\mathcal{B}$ is not connected.

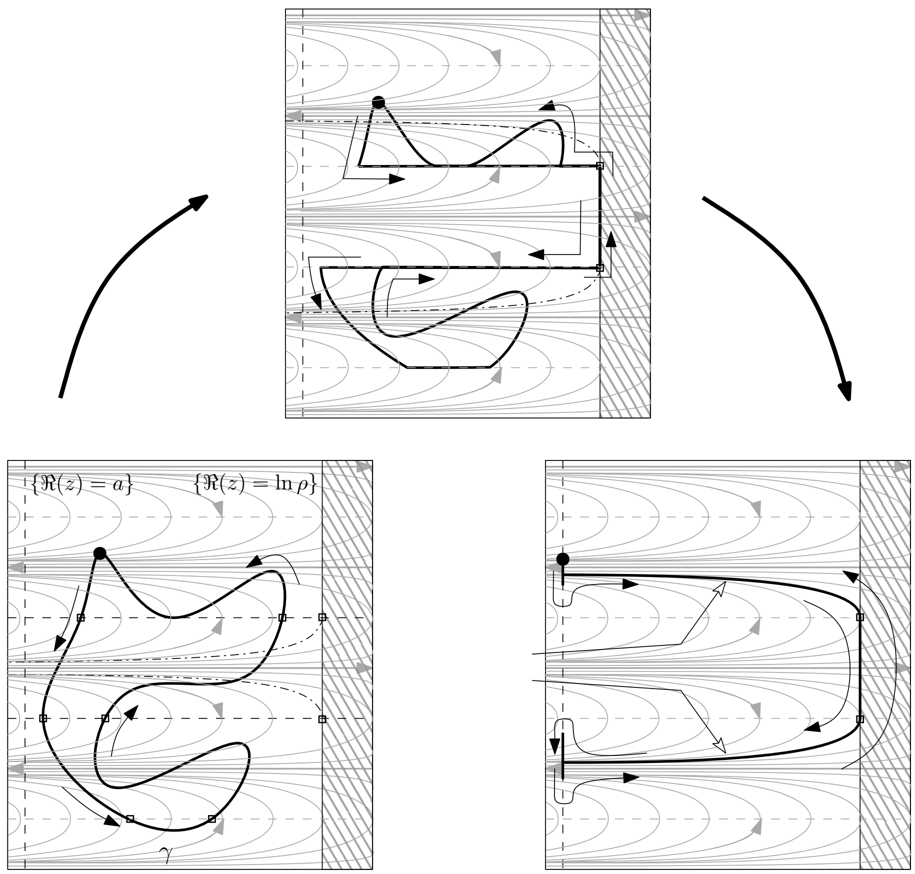

Figure 10: Homotopy within stability beams from γ to Γ.

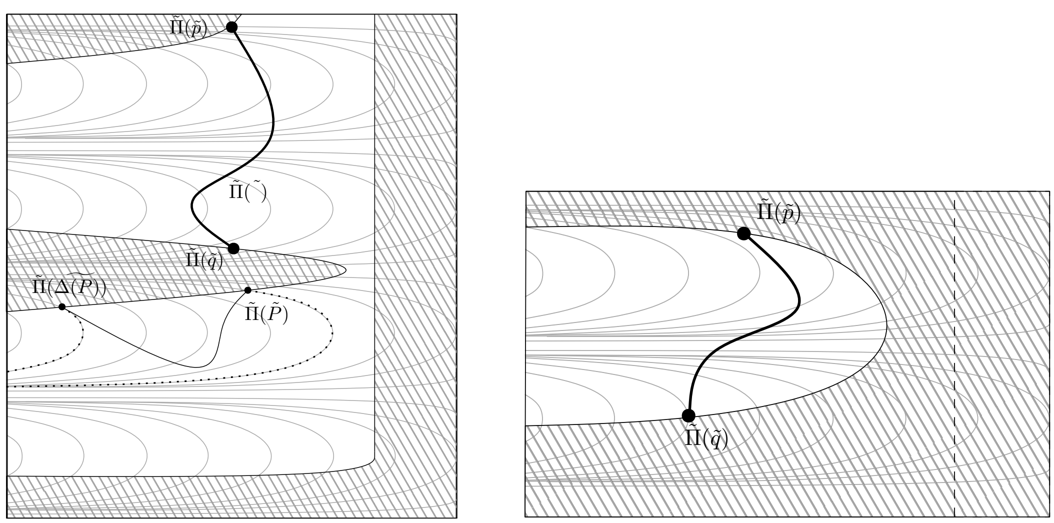

Figure 11: Projection (in logarithmic coordinates) of a stubborn path Γ (bold path at the top of the left-hand picture), to be compared with the projection of a tangent path carrying the strong holonomy Δ (bottom). The gray curves correspond to iso–argument curves of the y-coordinate of the leaf, dashed ones indicating differences of 2π. On the right-hand picture is depicted a path which is not stubborn: for $x_*^k > 0$ small enough the boundary of the leaf is connected.

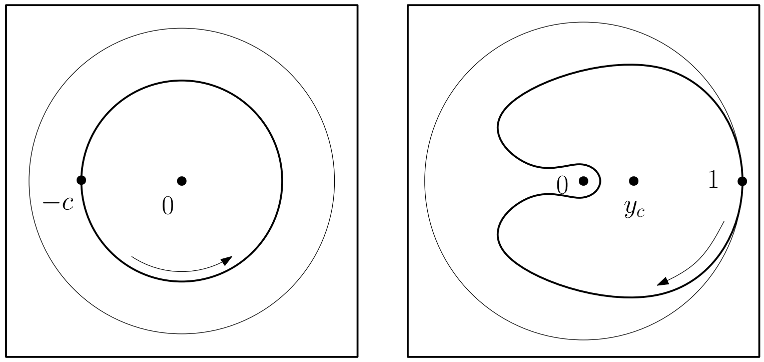

Figure 12: Projection of Γ_c on $\{y=0\}$ (left-hand picture) and on $\{x=0\}$ (right-hand picture).

The foliation $\mathcal{F}_{\omega_0}$ admits a family $(\Gamma_c)_{c\in]0,1[}$ of stubborn paths

$$\begin{array}{rcl}\Gamma_c\,:\,[-\pi,\pi] & \longrightarrow & \mathrm{cl}\,(\mathbb{D}\times\mathbb{D})\\ t & \longmapsto & \left(c\exp(\mathrm{i}t)\,,\,\exp\left(-\frac{1}{c}\left(1+\exp(-\mathrm{i}t)\right)\right)\right).\end{array}$$

The endpoint $(-c\,,\,1)$ belongs to the same transverse disk $\{y=1\}$. The geometrical explanation for these stubborn paths is the presence in the equation

$$y\frac{\mathrm{d}x}{\mathrm{d}y} = x^2$$

of a movable pole

$$x(y) = \frac{-c}{c\log y+1}.$$

Their projection on the $\{x=0\}$ line is shaped like a bean (say, if $c>\frac{1}{\pi}$), with 0 winding number around $\{y=0\}$ but with winding number 1 around the movable pole $y_c := \exp{}^{-1}/c$ (see Figure 12). These cycles do not offer candidates for incompressibility failure since each Γ_c is not homotopically trivial in $\mathrm{cl}\,(\mathbb{D}\times\mathbb{D})\setminus\{x=0\}$. By adjoining to them another singularity, though, we will be able to produce the expected cycles.

Let us pull-back ω_0 by the degree-2 mapping

$$\psi\,:\,(x,y) \longmapsto (x,1-y^2)$$

which brings ω_0 to

$$\psi^*\omega_0 = \left(y^2-1\right)\mathrm{d}x - 2yx^2\mathrm{d}y\,.$$

This foliation has three separatrices: $\{x=0\}$ and $\{y=\pm1\}$. The path Γ_c lifts through ψ as the pair of paths

$$\begin{array}{rcl}\Gamma_c^{\pm}\,:\,[-\pi,\pi] & \longrightarrow & \mathbb{D}\times\mathbb{C}\\ t & \longmapsto & \left(c\exp(\mathrm{i}t)\,,\,\pm\sqrt{1-\exp\left(-\frac{1}{c}\left(1+\exp(-\mathrm{i}t)\right)\right)}\right).\end{array}$$

Since the image of Γ_c is contained in $\mathrm{cl}\,(\mathbb{D}\times\mathbb{D})$ we can consider a fixed determination of the square-root on the cut $\mathbb{C}\backslash\mathbb{R}_{<0}$. Consider next the loop

$$\begin{array}{rcl}\gamma_c\,:\,[-2\pi,2\pi] & \longrightarrow & \mathbb{D}\times\mathbb{C}\\ t\leq 0 & \longmapsto & \Gamma_c^-(\pi+t)\\ t\geq 0 & \longmapsto & \Gamma_c^+(\pi-t)\,.\end{array}$$

By construction this loop does not wind around any branch of $\left\{x\left(y^2-1\right)=0\right\}$ (see Figure 13). Being the concatenation of two "distant" stubborn paths it cannot be trivial in the leaf of $\mathcal{F}$. Since c can be arbitrarily close to 0, the foliation $\mathcal{F}_{\psi^*\omega_0}$ is not incompressible in any neighborhood of $\mathcal{S}:=\{x=0\}$.

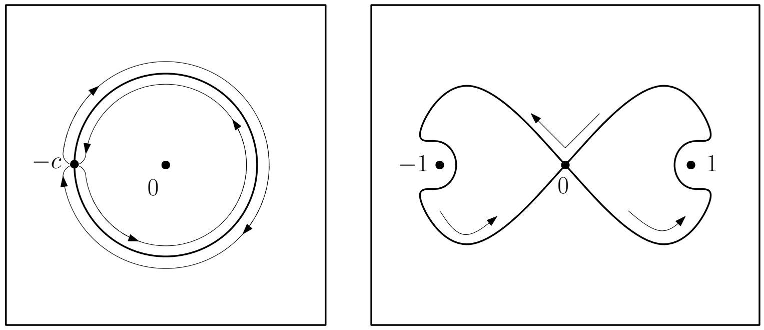

Figure 13: Projection of γ_c on $\{y=0\}$ (left) and on $\{x=0\}$ (right).

2.4.2 Embedding the local model into a singularity reduction

Let us explain now how to glue together these two blocks around $p_0=(0,-1)$ and $p_1=(0,1)$ so that $\mathcal{F}_{\psi^*\omega_0}$ embeds in the reduction of a germ of a singular foliation near $(0,0)$. Obviously we need to conform to Camacho-Sad index formula, therefore we need a third singularity p_2 since

$$\begin{aligned}\mathrm{CS}\left(\tilde{\mathcal{F}},S,p_0\right) &= \mathrm{CS}\left(\tilde{\mathcal{F}},S,p_1\right)\\ &= 0\,.\end{aligned}$$

Besides, the holonomy of $\mathcal{F}_{\psi^*\omega_0}$ along the loop $\gamma\ :\ t\in[0,2\pi]\mapsto(0,2\exp(it))$ is conjugate to $\Delta_0^{\circ 2}$, where Δ_0 is the strong holonomy of ω_0, and this holonomy must be the inverse of that of p_2 because γ winds -1 times around $\{y=0\}$. We may choose p_2 to be a resonant-saddle tangent to $x\mathrm{d}y+y\mathrm{d}x$ with holonomy $\Delta_0^{\circ -2}$ (see the realization part of [MR83]).

We invoke now an upgraded form of a realization theorem by A. Lins–Neto. This version has been written by F. Loray in [Lor10]. Instead of giving a general statement we provide one which is adapted to our framework.

Theorem. [Lor10, p159] *Let $G:=\langle\Delta_0,\cdots,\Delta_n\rangle<\mathrm{Diff}(\mathbb{C},0)$ be given with $n\in\mathbb{N}_{>0}$ and such that $\bigcirc_{\ell=0}^{n}\Delta_\ell=\mathrm{Id}$. Let a collection of reduced singular foliations $\mathcal{F}_\ell$, each one with a distinguished separatrix S_ℓ, so that in a convenient local coordinates the holonomy of $\mathcal{F}_\ell$ along S_ℓ be precisely Δ_ℓ. Suppose finally that the identity*

$$\sum_{\ell=0}^{n}\mathrm{CS}\left(\mathcal{F}_\ell,S_\ell,p_\ell\right)=-1$$

holds. Then there exists a germ of a holomorphic non-dicritic foliation $\mathcal{F}$ such that:

1. *$\mathcal{F}$ is reduced after one blow-up and admits $n+1$ singular points $(p_\ell)_{\ell\leq n}$ on the exceptional divisor $\mathcal{D}\simeq\mathbb{P}_1(\mathbb{C})$;*
2. *there exists a germ of a transverse disk Σ attached to $\mathcal{D}$ at a regular point p such that the projective holonomy representation*

   $$\pi_1\left(\mathcal{D}\backslash\left\{p_\ell\ :\ 0\leq\ell\leq n\right\},p\right)\to\mathrm{Diff}\left(\Sigma,p\right)$$

   coincides with G;
3. *the local analytical type of $E^*\mathcal{F}$ near p_ℓ is $\mathcal{F}_\ell$;*
4. *each separatrix S_ℓ is included in $\mathcal{D}$.*

Remark. The above sum is nothing else but Camacho–Sad index formula. It is in general rather easy to prove that it holds. Yet Y. Il'Yashenko [Il'97] described a subgroup $\langle\Delta_1,\Delta_2,\Delta_3\rangle$ spanned by non-linearizable quasi-resonant mappings, such that $\Delta_1\circ\Delta_2\circ\Delta_3=\mathrm{Id}$, but the sum of Camacho–Sad indices of any local realization as the holonomy of a foliation $\mathcal{F}_\ell$ is always less than -2.

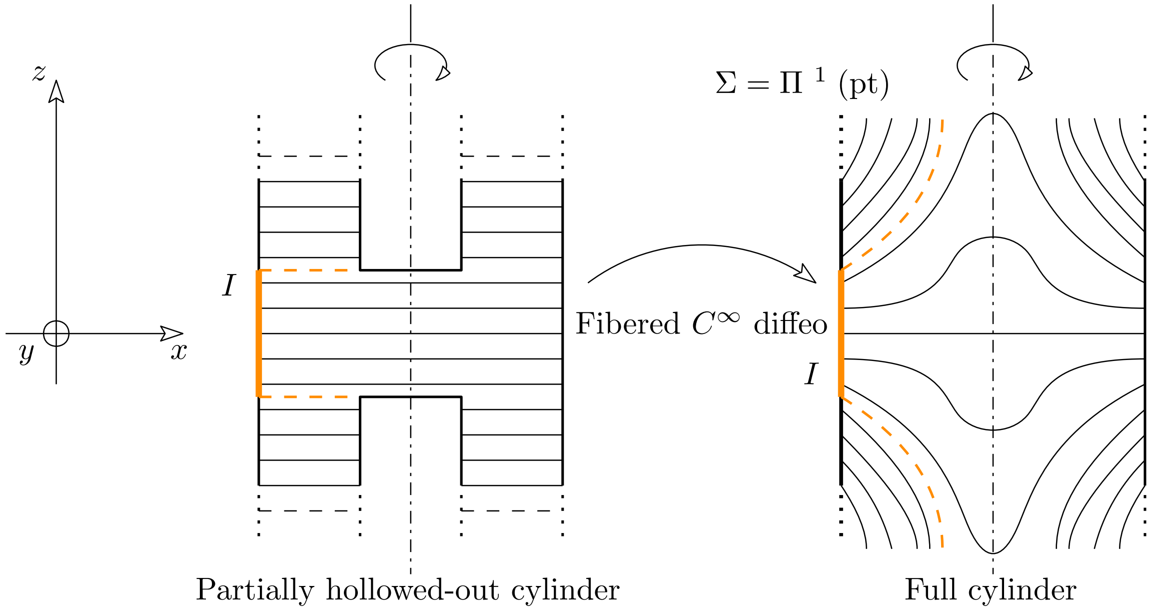

Figure 14: Basic piece of a Hector foliation

3 Movable singularities of regular foliations

3.1 Hector's example [Hec67]

It is a family of examples of a 2-dimensional, regular C^{∞} foliation of $\mathbb{R}^3$, everywhere transverse to the fibers of the projection $\Pi : (x, y, z) \mapsto (x, y)$, where we can realize any C^{∞} holonomy mapping $\mathfrak{h} : \mathbb{R} \to \mathbb{R}$ on a transverse provided it coincides with Id on an open interval.

We refer to Figure 14. Start from a H-like partially hollowed-out infinite vertical cylinder (it contains disk $\times I$ for some open interval I) and send it in a C^{∞} fashion to a full infinite vertical cylinder $\mathcal{C}$. Next, halve the cylinder to obtain two closed regions $H^{\pm}$ whose union is $\mathcal{C}$. Each one of these halves is naturally endowed with a C^{∞} foliation $\mathcal{F}^{\pm}$ coming from the horizontal foliation by planes in the H-like domain. This foliation is transverse to the fibers of Π.

If $\mathfrak{h}$ is a C^{∞}-diffeomorphism of $\mathbb{R}$ which is the identity on I then we can glue the two halves by identifying a fiber $\mathbb{R} \simeq \Sigma \subset H^{-}$ to $\mathfrak{h}(\Sigma) = \Sigma \subset H^{+}$ and gluing together the corresponding leaves of $\mathcal{F}^{\pm}$. We obtain a new C^{∞}-cylinder in $\mathbb{R}^3$, endowed with a C^{∞} foliation whose holonomy is essentially $\mathfrak{h}$.

This situation is morally wrong: one should expect non-trivial holonomy to be induced by some kind of singularity. For instance, it cannot happen with locally trivial fibrations over simply connected bases. We show below that in the (planar) holomorphic world morality is preserved from this heathen behaviors: parts of the definition set of the holonomy which corresponds to different holonomy dynamics are topologically isolated (they belong to different components).

3.2 Holonomy regions

We consider the following setting: a holomorphic foliation $\mathcal{F}$ on a product of analytic disks $\mathcal{U} \times \mathcal{V}$, which is transverse *everywhere* to the fibers of the projection

$$\begin{aligned} \Pi \, : \, \mathcal{U} \times \mathcal{V} &\longrightarrow \mathcal{U} \\ (x, y) &\longmapsto x \, . \end{aligned}$$

In particular $\mathcal{F}$ is regular. Up to uniformize the y-variable we may assume that $\mathcal{V} = \mathbb{D}$. In the following we fix $0 < r < 1$.

Definition. Being given a loop γ with range in $\mathcal{U}$ and base-point x_*, we consider the **holonomy region** $A_{\gamma} \subset \Pi^{-1}(x_*)$ defined as the set of initial values $y_* \in r\mathbb{D}$ such that the path γ lifts in the leaf

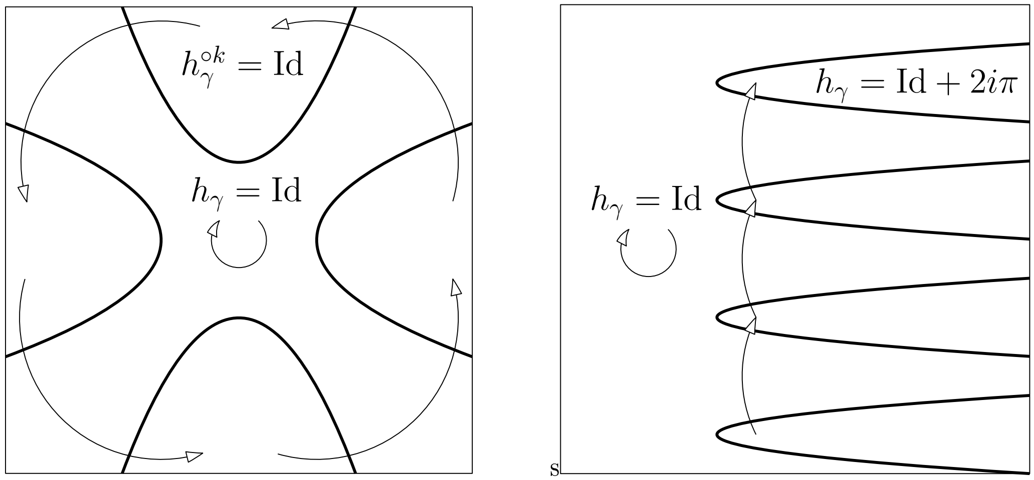

Figure 15: Left: example 1. Right: example 2. The separating bold lines are the set of initial values in $\Pi^{-1}(1)$ for which a movable singularity is encountered when lifting γ in the foliation.

of $\mathcal{F}|_{\mathcal{U}\times r\mathbb{D}}$ passing through (x_*, y_*) as a path $\gamma(y_*)$. It means that it is the maximal set on which the holonomy map

$$\begin{aligned} \mathfrak{h}_\gamma \ : \ A_\gamma &\longrightarrow r\mathbb{D} \\ y_* &\longmapsto \text{other endpoint of } \gamma(y_*) \end{aligned}$$

is well-defined.

Remark. For transversality reasons, this is equivalent to requiring that the lifted path does not reach $\mathcal{U} \times r\mathbb{S}^1$.

Proposition. *A_γ is open and simply connected.*

Proof. □

Example. For simplicity take $r := +\infty$ (*i.e.* $\mathcal{V} := \mathbb{C}$). We refer to Figure 15.

1. $x\mathrm{d}y = y^{k+1}\mathrm{d}x$ yields the identity and rotations of period k.
2. $\mathrm{d}y = \exp(y)\,\mathrm{d}x$ yields the identity and translations by $2i\pi$.

3.3 Maximal flow-box theorem

For any loop $\gamma \subset \mathcal{U}$ we define the analytic closed disc $\mathcal{U}_\gamma$ as the simply connected compact region bounded by γ. What we observed in the previous examples is that the different connected components of A_γ are separated by the trace in the transverse $\Pi^{-1}(x_*)$ of the movable singularities crossing γ. Let us make this statement more general.

Theorem. *Take a leaf $\mathcal{L}_0$ of $\mathcal{F}$ and a loop $\Gamma \subset \mathcal{L}_0$ which is homotopically trivial in $\mathcal{L}_0$ (therefore so is $\gamma := \Pi \circ \Gamma$ in $\mathcal{U}_\gamma$). Let (x_*, y_*) be the base-point of Γ and consider the connected component A^* of A_γ containing y_*. Let $\Omega_\gamma^* := \mathrm{Sat}_{\mathcal{F}|_{\mathcal{U}_\gamma \times r\mathbb{D}}}(A^*) \subset \mathcal{U} \times r\mathbb{D}$.*

1. *The projection*

$$\Pi \ : \ \Omega_\gamma^* \longrightarrow \mathcal{U}_\gamma$$

   is a trivial fibration.

2. *There exists a holomorphic mapping $\Psi \in \mathrm{Diff}\left(\Omega_\gamma^* \to \mathbb{C}^2\right)$, fibered in the x-variable, such that $\Psi^*\mathcal{F} = \mathcal{F}_{\frac{\partial}{\partial x}}$.*

Proof. It uses the maximum modulus principle to discard Hector-like situations. □

3.4 Topological definition of a movable singularity

Definition. Fix a loop $\gamma \subset \mathcal{U}$.

1. We say that a leaf $\mathcal{L}$ of $\mathcal{F}$ **contains a movable singularity** within γ if:
   - $\mathcal{L} \cap A_\gamma \neq \emptyset$;
   - no homotopy of γ to a point in $\mathcal{U}_\gamma$ lifts to $\mathcal{L}$ through Π.
2. If moreover $\mathfrak{h}_\gamma|_{\mathcal{L}\cap A_\gamma} = \mathrm{Id}$ we say that $\mathcal{L}$ **contains an unramified movable pole** within γ.
3. We say that $\mathcal{F}$ has the **local Painlevé property** within γ if any leaf containing a movable singularity withing γ actually contains an unramified movable pole within γ.

Theorem. *Assume that $\mathcal{F}$ has the local Painlevé property within γ. Then there exists a holomorphic x-fibered mapping $\Psi \in \mathrm{Diff}\left(\mathcal{U}_\gamma \times r\mathbb{D} \to \mathbb{C}^2\right)$ such that $\Psi^*\mathcal{F}$ is a Riccati foliation.*

In particular there is no "movable pole" of order greater than 1.

3.5 Hopes and wishes

Conjecture (Ordered from almost proved to far-fetched)**.** *Let γ be a loop in $\mathcal{U}$.*

1. *If $\mathcal{L}$ and $\mathcal{L}'$ are two leaves of $\mathcal{F}|_{\Pi^{-1}(\mathcal{U}_\gamma)}$ intersecting A_γ in the* same *connected component, then they have same topology.*
2. *If $\mathfrak{h}_\gamma$ has a fixed-point $y_* \in A_\gamma$ then $\mathfrak{h}_\gamma$ is the identity on the component of A_γ containing it.*
3. *If $\mathfrak{h}_\gamma$ has a recurring point,* i.e. *$\mathfrak{h}_\gamma^{\circ m}(y)$ belongs to the same component of A_γ as y, then $\mathfrak{h}_\gamma$ is m-periodic.*
4. *Any regular holomorphic foliation on $\mathcal{U} \times \mathcal{V}$ is conjugate on that domain to a y-polynomial foliation.*

Remark.

1. (1–3) of this conjecture would bolster the definition of a **ramified movable pole** within γ as a movable singularity with holonomy $\mathfrak{h}_\gamma$ of period $m \in \mathbb{N}$, which would then be the ramification order of the singularity.
2. Claim (2) would imply that if a foliation has a non-identical holonomy $\mathfrak{h}_\gamma$ along a tangent loop Γ then there is a fixed singularity of the foliation in $\Pi^{-1}\left(\mathcal{U}_{\Pi\circ\Gamma}\right)$.

Question. *What would be a decent (and local) definition of a movable essential singularity, so that we can prove that none may exist in (local) planar foliations?*

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