Quenched limit theorems for random dynamical systems via spectral method

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- *A spectral approach for quenched limit theorems for random expanding dynamical systems*, Communications in Mathematical Physics **360** (2018), 1121–1187.
- *Almost sure invariance principle for random piecewise expanding maps*, Nonlinearity **31** (2018), 2252–2280.
- *A spectral approach for quenched limit theorems for random hyperbolic dynamical systems*, preprint.

Preliminaries

Let $(X, \mathcal{B})$ be a measurable space and $T\colon X \to X$ a measurable map. For $k \in \mathbb{N} \cup \{0\}$, set

$$T^k = \begin{cases} \underbrace{T \circ \ldots \circ T}_{k} & \text{if } k \in \mathbb{N}; \\ \mathrm{Id} & \text{if } k = 0. \end{cases}$$

Let μ be a probability measure on $(X, \mathcal{B})$. We say that μ is T-**invariant** if

$$\mu(T^{-1}(A)) = \mu(A) \quad \text{for every } A \in \mathcal{B}.$$

An T-invariant probability measure μ is said to be **ergodic** if $\mu(A) \in \{0, 1\}$ for each $A \in \mathcal{B}$ such that $T^{-1}(A) = A$.

If X is a compact topological space, $\mathcal{B}$ a Borel σ-algebra and $T: X \to X$ is continuous. Then, there **exists at least one** ergodic T-invariant measure μ.

Theorem (Birkhoff)

Let μ be an T-invariant ergodic probability measure and $\phi \in L^1(X, \mu)$. Then,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi(T^k(x)) \to \int_X \phi \, d\mu, \quad \textit{for } \mu\textit{-a.e. } x \in X.$$

Hence, under the ergodicity assumption, the **deterministic process** $\phi, \phi \circ T, \phi \circ T^2, \dots$ satisfies the *law or large numbers*.

Question: how about finer limit laws?

Piecewise expanding maps

Let $X = [0,1]$ and let $\mathcal{B}$ be the associated Borel σ-algebra. We say that $T\colon X \to X$ is *piecewise expanding* if there exists a partition

$$0 = x_0 < x_1 < \ldots < x_{k-1} < x_k = 1$$

and $\alpha > 1$ such that:

- each restriction $T|_{(x_{i-1},x_i)}$ is a C^1 function which can be extended to a C^1-function on $[x_{i-1}, x_i]$;
- $|T'(x)| \geq \alpha$ for $x \in (x_{i-1}, x_i)$;
- $g(x) = \frac{1}{T'(x)}$ is a function of bounded variation.

Denote by m the Lebesque measure on X. Let $\mathcal{L}\colon L^1(m) \to L^1(m)$ be the **transfer operator** associated to T:

$$\int_X (\mathcal{L}f)g\,dm = \int_X f(g \circ T)\,dm \quad \text{for } f \in L^1(m) \text{ and } g \in L^\infty(m).$$

We have

$$(\mathcal{L}f)(x) = \sum_{y \in T^{-1}(x)} \frac{f(y)}{|T'(y)|}, \qquad \text{for } f \in L^1(m).$$

Hence, $\mathcal{L}$ is a **positive** operator.

For example, for $T(x) = 2x(mod 1)$ we have

$$(\mathcal{L}f)(x) = f(x/2) + f((x+1)/2), \qquad \text{for } f \in L^1(m).$$

Set

$$BV = \left\{ f \in L^1(m) : \inf_{f=g \ a.e.} Var(g) < \infty \right\}.$$

Then, BV is a Banach space w.r.t.

$\|f\|_{BV} := \|f\|_{L^1(m)} + \inf_{f=g \ a.e.} Var(g)$. Then, $\mathcal{L}(BV) \subset BV$ and $\mathcal{L}$ is a bounded operator on BV.

Theorem

We have that $\mathcal{L}$ is a **quasicompact** *operator of diagonal type on BV, i.e.*

$$\mathcal{L} = \sum_{k=1}^{n} \lambda_k \Pi_k + N,$$

where for each $k \in \{1, \ldots, n\}$, λ_k is an eigenvalue for $\mathcal{L}$, $|\lambda_k| = 1$, Π_k is a projection onto 1-dimensional subspace of BV. Moreover, $r(N) < 1$, $N\Pi_k = \Pi_k N = 0$ and $\Pi_i \Pi_j = \Pi_j \Pi_i = 0$ for $i \neq j$.

Using the previous theorem, it is not hard to show that 1 is an eigenvalue for $\mathcal{L}$ and that the corresponding eigenvector $v \in BV$ satisfies $v \geq 0$. Thus, v is a fixed point for $\mathcal{L}$ and then it is easy to show that μ given by $d\mu = v\, dm$ is invariant for T.
Under appropriate assumptions, one can ensure that 1 is the **only** eigenvalue of $\mathcal{L}$ on the unit circle and that it is simple, i.e.

$$\mathcal{L} = \Pi + N,$$

$r(N) < 1$, $\Pi N = N\Pi = 0$, where Π is a projection onto 1-dimensional subspace of BV.

Some important consequences:

- there exists a **unique a.c.i.m.** μ for T which is **ergodic**;
- there exists $C > 0$ and $r \in (0,1)$ such that for $f \in BV$ and $g \in L^\infty(m)$,

$$\left| \int_X f(g \circ T^n)\, d\mu - \int_X f\, d\mu \int_X g\, d\mu \right| \le C r^n \|f\|_{BV} \|g\|_{L^\infty};$$

  This property is called **exponential decay of correlations**.

Moreover, we have the following result. Let $\phi \colon X \to \mathbb{R}$ be bounded and in BV, $\int_X \phi \, d\mu = 0$. Set

$$S_n := \sum_{k=0}^{n-1} \phi \circ T^k.$$

Theorem

We have that $\lim_{n\to\infty} \int_X \frac{S_n^2}{n} \, d\mu = \sigma^2$, *where*

$$\sigma^2 := \int_X \phi^2 \, d\mu + 2 \sum_{n=1}^{\infty} \int_X \phi(\phi \circ T^n) \, d\mu < \infty.$$

If $\sigma^2 > 0$, *then*

$$\frac{S_n}{\sqrt{n}} \to \mathcal{N}(0, \sigma^2) \quad \textit{in distribution.}$$

Theorem

If $\sigma^2 > 0$, then there exists $\delta > 0$ and a strictly convex, continuous and nonnegative function $c\colon (-\delta, \delta) \to \mathbb{R}$ which vanishes only at 0 such that

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(S_n > n\varepsilon) = -c(\varepsilon) \quad \text{for } \varepsilon \in (0, \delta).$$

For $\theta \in \mathbb{C}$, we define $\mathcal{L}^\theta \colon BV \to BV$ by

$$\mathcal{L}^\theta(f) = \mathcal{L}(e^{\theta\phi} f), \quad f \in BV.$$

Then, $\mathcal{L}^\theta$ is well-defined and bounded. For θ close to 0,

$$\mathcal{L}^\theta = \lambda(\theta)\Pi(\theta) + N(\theta).$$

Setting $d\mu = f\, dm$, we have

$$\begin{aligned}\lim_{n\to\infty} \int_X e^{itS_n/\sqrt{n}}\, d\mu = \lim_{n\to\infty} \int_X (\mathcal{L}^{it/\sqrt{n}})^n(f)\, dm &= \lim_{n\to\infty} \lambda(it/\sqrt{n})^n \\ &= e^{-t^2\sigma^2/2},\end{aligned}$$

for $t \in \mathbb{R}$. This yields CLT.

One has that $\lambda'(0) = 0$ and $\lambda''(0) = \sigma^2$ and that

$$\lim_{n\to\infty} \frac{1}{n} \log \int_X e^{\theta S_n}\, d\mu = \Lambda(\theta),$$

for $\theta \in \mathbb{R}$ sufficiently close to 0 and where

$$\Lambda(\theta) = \log \lambda(\theta).$$

This gives LDP.

Multiplicative ergodic theorem

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and that $\sigma \colon \Omega \to \Omega$ is an **invertible** transformation such that $\mathbb{P}$ is σ-invariant and ergodic. Furthermore, assume that Ω is a Borel subset of a separable, complete metric space. Let B be a Banach space and $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega}$ a family of bounded operators on B such that $\omega \to \mathcal{L}_\omega$ is Borel-measurable. Assume that

$$\int_\Omega \log^+ \|\mathcal{L}_\omega\| \, d\mathbb{P}(\omega) < \infty.$$

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$, the following limits exist (and are independent on ω)

$$\Lambda(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{L}_\omega^n\| \text{ and } \kappa(\mathcal{L}) := \lim_{n\to\infty} \frac{1}{n} \log ic(\mathcal{L}_\omega^n),$$

where

$$\mathcal{L}_\omega^n := \mathcal{L}_{\sigma^{n-1}\omega} \circ \ldots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega,$$

and $ic(A)$ is defined as an infimum over all $r > 0$ with the property that image of the open unit ball in B under A can be covered by finitely many open balls of radius r.

If $\kappa(\mathcal{L}) < \Lambda(\mathcal{L})$, then either there exists a finite sequence $(\lambda_i)_{i=1}^l$ such that

$$\Lambda(\mathcal{L}) = \lambda_1 > \lambda_2 > \ldots > \lambda_l > \kappa(\mathcal{L})$$

or there exists an infinite sequence $(\lambda_i)_{i\in\mathbb{N}}$ such that

$$\Lambda(\mathcal{L}) = \lambda_1 > \lambda_2 > \ldots > \kappa(\mathcal{L}) \quad \text{and} \quad \lim_{i\to\infty} \lambda_i = \kappa(\mathcal{L}).$$

Furthermore, for $\mathbb{P}$-a.e. $\omega \in \Omega$ there exists a unique splitting

$$B = V(\omega) \oplus \bigoplus_{j=1}^{l} Y_j(\omega),$$

with $l \in \mathbb{N} \cup \{\infty\}$ that depends measurably on ω such that:

- each $Y_j(\omega)$ is finite-dimensional, $\mathcal{L}_\omega(Y_j(\omega)) = Y_j(\sigma\omega)$ and for $v \in Y_j(\omega)$, $v \neq 0$ we have

$$\lim_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}_\omega^n v\| = \lambda_j;$$

- $\mathcal{L}_\omega(V(\omega)) \subset V(\sigma(\omega))$ and for every $v \in V(\omega)$,

$$\limsup_{n\to\infty} \frac{1}{n} \log\|\mathcal{L}_\omega^n v\| \leq \kappa(\mathcal{L}).$$

The numbers λ_j are called **Lyapunov exponents** of $\mathcal{L}$ w.r.t. to $\mathbb{P}$, while subspaces $Y_j(\omega)$ are called **Oseledets subspaces**.

Random piecewise expanding maps

Let T and T' be two piecewise expanding maps on $X = [0,1]$. Then, we can introduce the so-called **Rychlik distance** $d_R(T, T')$ as the infimum over all $r > 0$ with the property that there exists $A \subset [0,1]$, $m(A) > 1 - r$ and a diffeomorphism $f \colon [0,1] \to [0,1]$ such that $T'|_A = T \circ f|_A$ and such that

$$|f(x) - x| < r \quad \text{and} \quad |1/f'(x) - 1| < r,$$

for $x \in [0,1]$. Then,

$$12 d_R(T, T') = \sup_{\|g\|_{BV} \le 1} \|\mathcal{L}_T g - \mathcal{L}_{T'} g\|_1.$$

Taking a fixed piecewise expanding map T on $X = [0, 1]$ and choosing (sufficiently small) $\delta > 0$, we can consider a family $(T_\omega)_{\omega \in \Omega}$ such that $d_R(T_\omega, T) < \delta$ for each $\omega \in \Omega$ and with the property that $\omega \to T_\omega$ has countable range. Let $\mathcal{L}_\omega$ be the transfer operator associated to T_ω.

Set

$$T_\omega^n = T_{\sigma^{n-1}\omega} \circ \ldots \circ T_\omega.$$

The corresponding transfer operator of T_ω^n is

$$\mathcal{L}_\omega^n = \mathcal{L}_{\sigma^{n-1}\omega} \circ \ldots \circ \mathcal{L}_{\sigma\omega} \circ \mathcal{L}_\omega.$$

Let $\mathcal{L} = (\mathcal{L}_\omega)_{\omega \in \Omega}$. Then, we have:

- $\Lambda(\mathcal{L}) = 0$ and $\kappa(\mathcal{L}) < 0$;
- $\dim Y_1(\omega) = 1$.

Then, there exist $v_\omega \in BV$ such that $v_\omega \geq 0$, $Y_1(\omega) = \langle \{v_\omega\} \rangle$ and $\int_X v_\omega \, dm = 1$. Let μ_ω be a measure on X given by $d\mu_\omega = v_\omega dm$. We can build a measure μ on $\Omega \times X$ such that

$$\mu(A \times B) = \int_A \mu_\omega(B) \, d\mathbb{P}(\omega) \quad \text{for } A \subset \Omega \text{ and } B \subset X \text{ meas.}$$

Then, $\mu << \mathbb{P} \times m$ and $\frac{d\mu}{d(\mathbb{P} \times m)}(\omega, \cdot) = v_\omega$, $\omega \in \Omega$.

μ is invariant for $\tau\colon \Omega \times X \to \Omega \times X$ given by

$$\tau(\omega, x) = (\sigma(\omega), T_\omega(x)) \quad (\omega, x) \in \Omega \times X.$$

Furthermore, $\pi_*\mu = \mathbb{P}$, where $\pi\colon \Omega \times X \to \Omega$ is a projection.

Furthermore, μ is a **unique** probability measure on $\Omega \times X$ with the above properties.

Let $\phi\colon \Omega \times X \to \mathbb{R}$ be such that

$$\textit{esssup}_{\Omega \times X} |\phi| < \infty \quad \text{and} \quad \textit{esssup}_{\Omega} \|\phi(\omega, \cdot)\|_{BV} < \infty.$$

We assume that

$$\int_X \phi(\omega, \cdot)\, d\mu_\omega = 0 \quad \text{for } \omega \in \Omega.$$

Set

$$S_n(\omega, x) = \sum_{i=0}^{n-1} \phi(\sigma^i(\omega), T_\omega^i(x)) = \sum_{i=0}^{n-1} \phi(\tau^i(\omega, x)).$$

We are interested in **quenched** limit laws of $S_n(\omega, \cdot)$ w.r.t μ_ω for typical ω. For $\theta \in \mathbb{C}$, let $\mathcal{L}_\omega^\theta \colon BV \to BV$ be given by

$$\mathcal{L}_\omega^\theta(f) = \mathcal{L}_\omega(e^{\theta\phi(\omega,\cdot)} f), \quad f \in BV.$$

For θ close to 0, one can apply MET for the cocycle $\mathcal{L}^\theta = (\mathcal{L}_\omega^\theta)_{\omega \in \Omega}$. Let $\Lambda(\theta)$ denote the largest Lyapunov exponent of $\mathcal{L}^\theta$. Then,

$$\theta \to \Lambda(\theta) \quad \text{is a } C^2 \text{ function on a neighb. of 0.}$$

Moreover, $\Lambda'(0) = 0$ and $\Lambda''(0) = \Sigma^2$, where

$$\Sigma^2 = \int_{\Omega\times X} \phi^2 \, d\mu + 2\sum_{n=1}^{\infty} \int_{\Omega\times X} \phi(\phi\circ\tau^n) \, d\mu.$$

Theorem

For $\theta \in \mathbb{R}$ close to 0, we have that

$$\lim_{n\to\infty} \frac{1}{n} \log \int_X e^{\theta S_n(\omega,\cdot)} \, d\mu_\omega = \Lambda(\theta), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

As a direct consequence, we obtain the following LDP.

Theorem

Assume that $\Sigma^2 > 0$. Then, there exists $\delta > 0$ and a nonnegative, continuous, strictly convex function $c\colon (-\delta,\delta) \to \mathbb{R}$ vanishing only in 0 and such that

$$\lim_{n\to\infty} \frac{1}{n} \mu_\omega(S_n(\omega,\cdot) > n\varepsilon) = -c(\varepsilon) \quad \text{for } 0 < \varepsilon < \delta.$$

Theorem

Assume that $\Sigma^2 > 0$. *Then,*

$$\frac{S_n(\omega,\cdot)}{\sqrt{n}} \to \mathcal{N}(0,\Sigma^2) \quad \textit{in distribution, for } \mathbb{P}\textit{-a.e. } \omega \in \Omega.$$

Idea: For θ close to 0, $\dim Y_1^\theta(\omega) = 1$, $Y_1^\theta = \langle v_\omega^\theta \rangle$, $\int_X v_\omega^\theta \, dm = 1$.

Then,

$$\mathcal{L}_\omega^\theta v_\omega^\theta = \lambda_\omega^\theta v_{\sigma\omega}^\theta \quad \text{for some scalar } \lambda_\omega^\theta.$$

Then,

$$\lim_{n\to\infty} \int_X e^{it\frac{S_n(\omega,\cdot)}{\sqrt{n}}} \, d\mu_\omega = \lim_{n\to\infty} \int_X \mathcal{L}_\omega^{\frac{it}{\sqrt{n}},n} v_\omega \, dm = \lim_{n\to\infty} \prod_{j=0}^{n-1} \lambda_{\sigma^j\omega}^{\frac{it}{\sqrt{n}}} = \ldots.$$

Further developments:

1. local limit theorem, almost sure invariance principle;
2. multidimensional piecewise expanding maps;
3. hyperbolic case, i.e. T_ω is Anosov diffeom. In this setting, BV is replaced by anisotropic Banach spaces.