Mild dissipative diffeomorphisms of the disc with zero entropy (part 2)

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Joint work with Enrique Pujals and Charles Tresser.

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Strongly dissipative diffeomorphisms

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Definition. f is strongly dissipative if it is dissipative and for any ergodic μ ($\neq$ sink) and μ-ae x, both branches of $W^s(x)$ meet $\partial\mathbb{D}$.

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There exists a semi-conjugacy $\pi\colon (\mathbb{D}, f) \to (X, h)$
to an endomorphism on a compact real tree.
And $\pi_*(\mu) \neq \pi_*(\nu)$ *if* μ, ν *are distinct non-atomic measures.*

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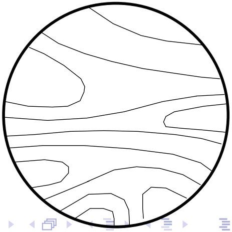

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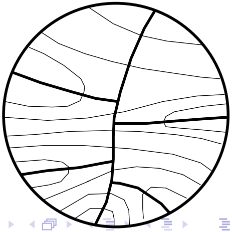

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Property 2 (closing lemma). *The support of any ergodic probability is contained in the closure of the set of periodic points.*

Renormalization

Theorem A. (C-Pujals-Tresser) *For any strongly dissipative diffeomorphism f of the disc whose topological entropy vanishes,*

- a- *either any forward orbit of f converges to a fixed point,*
- b- *or f is renormalizable: there exists a topological disc U and $m \geq 2$ such that $f^m(U) \subset U$ and $f^i(U) \cap U = \emptyset$ when $0 < i < m$.*

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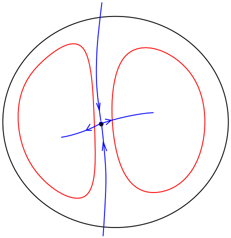
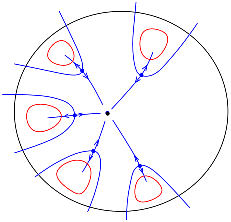
More precisely: there are finitely any renormalization domains $U_1, \dots, U_\ell$ with periods $m_1, \dots, m_\ell$ such that in $\mathbb{D} \setminus (\cup_{i,j} f^i(U_j))$, the dynamics is (generalized) Morse-Smale:

- *any forward orbit of f in $\mathbb{D} \setminus W$ converges to a periodic point,*
- *the periodic points in $\mathbb{D} \setminus W$ have uniformly bounded periods.*

Periods of renormalization



Period 2.



Period 5.
(decoration)

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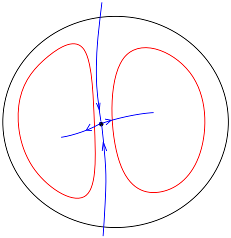

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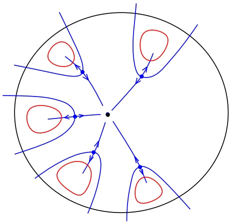

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Theorem B. *For strongly dissipative diffeomorphisms f of the disc whose topological entropy vanishes, after a large number of renormalizations, the only possible renormalization period is 2.*

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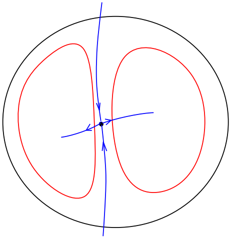

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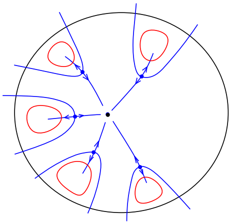

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Theorem B. *For strongly dissipative diffeomorphisms f of the disc whose topological entropy vanishes, after a large number of renormalizations, the only possible renormalization period is* 2.

$\Rightarrow$ Gambaudo-Tresser conjecture on the periods of the periodic points holds.

Zero entropy $\Rightarrow$ no cycle

Assume (to simplify) that all periodic points are hyperbolic.

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Each fixed point is: – either a *sink* (index 1),
– or a *saddle with reflexion* (index 1),
– or a *saddle with no reflexion* (index -1).

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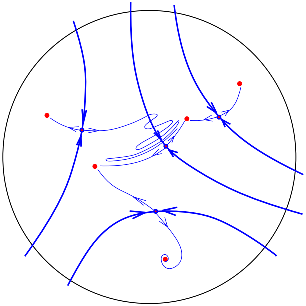
Property. *If $h(f) = 0$, there is no cycle of periodic points:*
There is no sequence of saddles $p_0, p_1, \dots, p_{n-1}, p_n = p_0$ such that
for each i, $W^u(p_i)$ accumulates on p_{i+1}.

The set of fixed points is connected

Property. *Consider* $\mathbb{D} \setminus \bigcup_{p \text{ fixed of index} -1} W^s(p)$.
Each component V *contains exactly one fixed point* q *of index* 1.
For any $p \in \partial V$ *fixed,* $W^u(p)$ *accumulates on* q.



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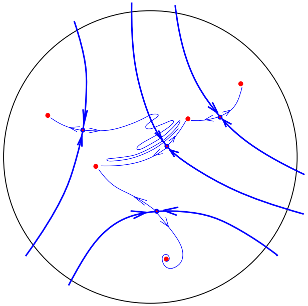
Indeed:

(1) For p fixed, the closure of each unstable branches contains a fixed point of index 1.

(2) Lefschetz formula:

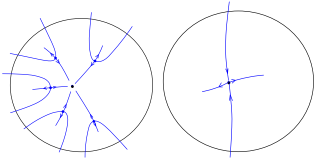
$$\sum_{x \text{ fixed}} \text{Ind}(x) = 1.$$



Structure of the periodic set

A periodic saddle q is *stabilized* if:

- either $per(q) \geq 2$ and $\overline{W^u(q)}$ contains a fixed point s of index 1,
- or q is a fixed saddle with reflexion.



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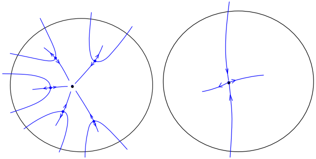
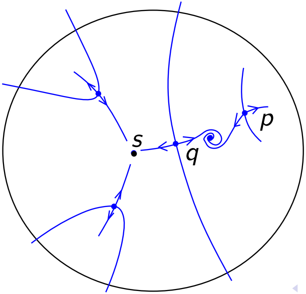


Property. *Any periodic point p is either* **fixed**, *or* **stabilized**, *or* **attached to a stabilized point** *q (by a chain of periodic points in the decorated region of q).*



Trapping discs

Property. *If Γ is an unstable branch of p, there exists a disc U which*
– *is trapped:* $f(\overline{U}) \subset U$,
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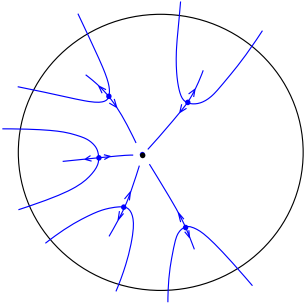
Theorem A follows:



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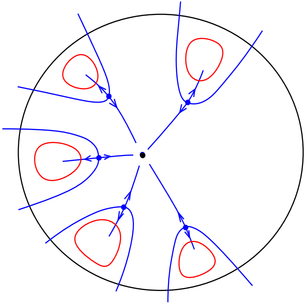

Pixton discs

p: fixed saddle of index -1.

Definition. A disc U is *Pixton* if ∂U is Jordan curve $\gamma^s \cup \gamma^u$ st:

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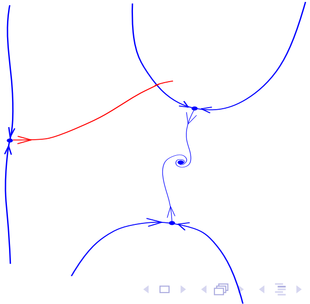

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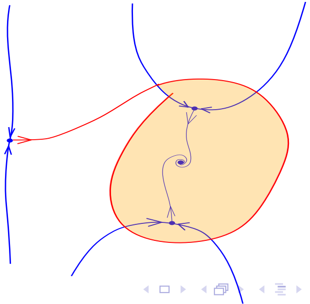

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▷ A maximal Pixton disc is trapped.