Mild dissipative diffeomorphisms of the disc with zero entropy (part 1)

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Joint work with Enrique Pujals and Charles Tresser.

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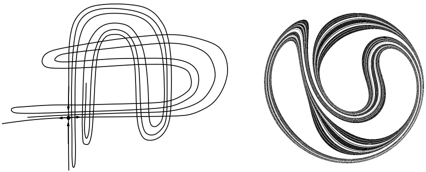
Dynamics of surface diffeomorphisms

f: a C^2-diffeomorphism of the 2-disc $\mathbb{D}$.

$h_{top}(f)$: its topological entropy.

(exponential growth rate of the number of orbits)

When $h_{top}(f) > 0$,
the dynamics is rich.



Questions:

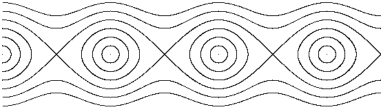
How is the dynamics when $h_{top}(f) = 0$?

What happens just before the entropy becomes positive?

Examples with zero entropy

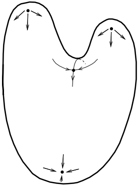
Conservative examples:

- translations on $\mathbb{T}^2$,
- hamiltonian systems.

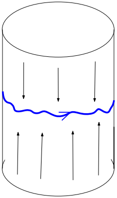


Dissipative examples:

Morse-Smale systems



irrational attractors



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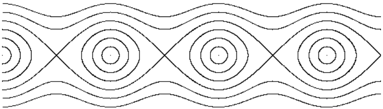

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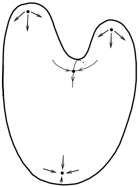

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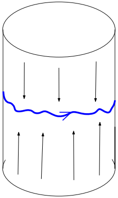
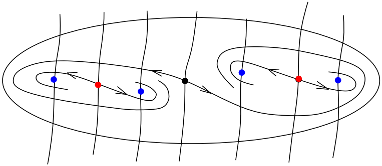


Period doubling cascade and odometer.

(Gambaudo - Tresser - Van Strien)



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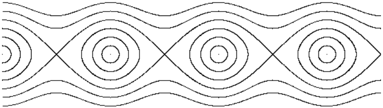

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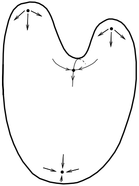

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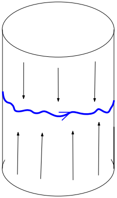

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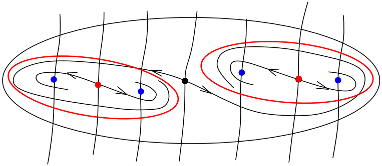

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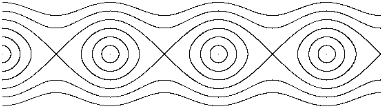

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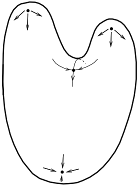

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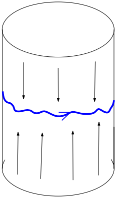

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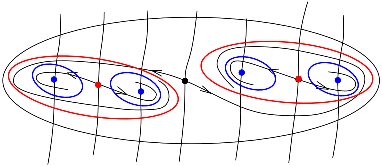

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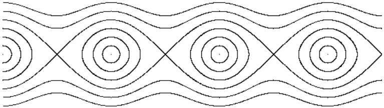

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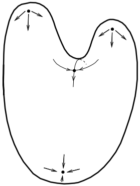

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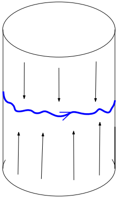

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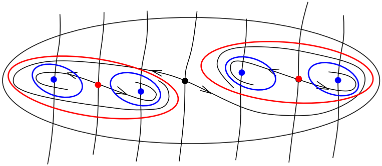

Some converse results: Franks-Handel (conservative), Le Calvez-Tal (homeo)

Set of periods when the entropy vanishes

n is a *period* of f if f^n has a fixed point x, not fixed by f^m, $m < n$.

Dimension 1. Endomorphisms of $[0, 1]$.

Sharkovskii's thm. *The set of periods (if infinite) is* $\{2^n, n \in \mathbb{N}\}$.

Coullet-Tresser / Feigenbaum. *The dynamics is renormalizable.*

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Dimension 2. Diffeomorphisms of the disc.

No direct generalization of Sharkovskii!

Gambaudo-Tresser conjecture. *In the dissipative case,*
– *there exists* $k_0 \geq 1$ *such that any period has the form* $k.2^n$, $k \leq k_0$.
– *the set of periods (if infinite) contains* $\{k.2^n, n \in \mathbb{N}\}$ *for some* k.

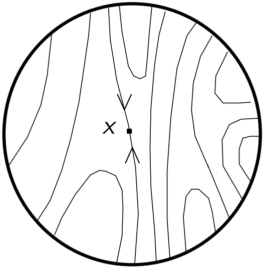
De Carvalho-Martens-Lyubich. *Renormalization for Hénon maps with jacobian* $\ll 1$.

Strongly dissipative diffeomorphisms

A diffeomorphism f is *strongly dissipative* if:

– it is dissipative: $|det(D_x f)| < 1$ for all $x \in \mathbb{D}$,

– $\forall$ ergodic measure μ ($\neq$ sink), and μ-a.e. x, $W^s(x)$ separates $\mathbb{D}$.



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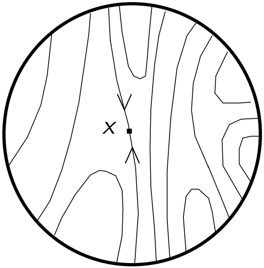
Examples.

– Hénon maps for $|b| < 1/4$:
$(x, y) \mapsto (1 - ax^2 + y, -bx)$
(uses Wiman's theorem).

– Maps close to 1D endomorphisms.

– C^2-open property.



Renormalization

Theorem A. (C-Pujals-Tresser) *For any strongly dissipative diffeomorphism f of the disc whose topological entropy vanishes,*

a- *either any forward orbit of f converges to a fixed point,*

b- *or f is renormalizable: there exists a topological disc U and $m \geq 2$ such that $f^m(U) \subset U$ and $f^i(U) \cap U = \emptyset$ when $0 < i < m$.*

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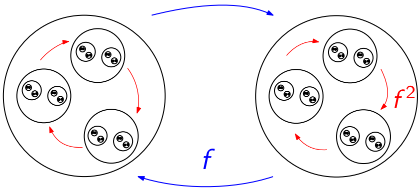
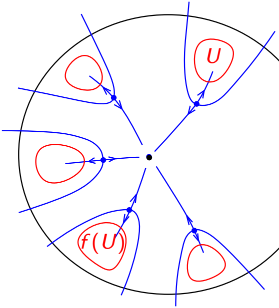
In case (b), the result also applies to the renormalization $f^m|_U$.

f is *infinitely renormalizable* when there exists an infinite sequence of successive renormalizations.



Decomposition of the dynamics: periodic structure

The renormalization domains are "attached" to particular saddle orbits that are "decorated" around a fixed point.



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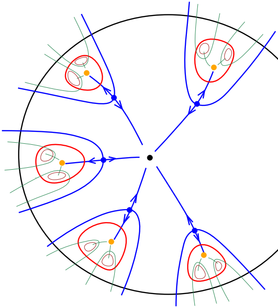


⇒ The set of periodic orbits has a hierarchical structure.

Decomposition of the dynamics: aperiodic structure

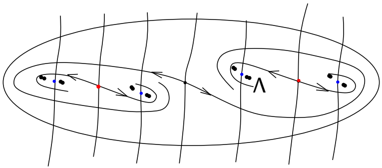
Corollary. *For f infinitely renormalizable, there is Λ compact s.t.*

– *Λ is the disjoint union of invariant compact sets K_i,*

– *any forward orbit accumulates on a periodic orbit or on some K_i,*

– *the periodic orbits of large periods are close to Λ,*

– *each K_i is semi-conjugated to an odometer: $\pi_i \colon K_i \to \mathcal{O}_i$,*
for a.e. $x \in \mathcal{O}_i$, $\pi_i^{-1}(x)$ is a singleton. (So K_i is uniq. ergodic.)



Gambaudo-Tresser conjecture

f: a strongly dissipative diffeomorphism of the disc with zero entropy.

Theorem B. *There exist W open and $m \geq 1$ such that $f(\overline{W}) \subset W$ and:*
– the periods of the periodic points of f^m in W are $\{2^n, n \in \mathbb{N}\}$,
– the periods of the periodic points in $\mathbb{D} \setminus W$ is bounded.

Corollary. There exist integers $m_1, \dots, m_\ell$ such that the set of periods of the periodic orbits of f coincides with

$$Per(f) = \{m_i.2^n, \ \ n \geq 0, 1 \leq i \leq \ell\} \cup \textit{finite set.}$$

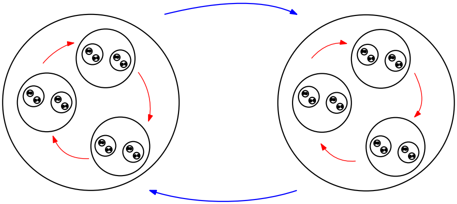
Gambaudo-Tresser conjecture: thm A $\Rightarrow$ thm B

Consider a nested sequence of renormalizations domains

$$U_k \cup f(U_k) \cup \cdots \cup f^{m_k-1}(U_k) \text{ with period } m_k \to +\infty,$$

and decreasing towards an odometer K.

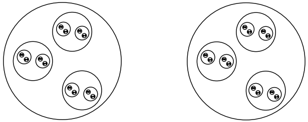
Goal. *When m_k is large, $m_{k+1} = 2m_k$.*



$\Rightarrow$ All periodic points in U_k have periods in $\{m_k.2^n \colon n \geq 0\}$.

Gambaudo-Tresser conjecture: γ-dissipation

K: a limit odometer with trapping neighborhood U.



Observation. *The dynamics on U is $9/10$-dissipative:*

$$\text{for } x, y \in U,\ u \in T^1_y M, \quad |det(D_x f)| \le \|D_y f.u\|^{9/10}.$$

Indeed: the (aperiodic) measure on K can not be hyperbolic since $h_{top} = 0$. Hence its maximal Lyapunov exponent vanishes.

Quantitative Pesin theory. *If f is $9/10$-dissipative, there is a set X s.t.*

– *$W^s(x)$ varies continuously with $x \in X$ for the C^1-topology,*

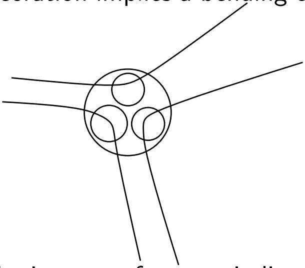
– *$\mu(X) > 2/3$ for any measure μ supported on U.*

Gambaudo-Tresser conjecture: no decoration

K: a limit odometer with trapping neighborhoods

$U_k \cup f(U_k) \cup \cdots \cup f^{m_k - 1}(U_k)$ with period $m_k \to +\infty$,

When $\frac{m_{k+1}}{m_k} \geq 3$, the decoration implies a bending of the stable manifolds..



... for at least $1/3$ of the iterates of any periodic orbit close to K.

This contradicts the uniform Pesin theory!

Conclusion: $\frac{m_{k+1}}{m_k} = 2$ for k large.

This gives thm B (Gambaudo-Tresser conjecture) from thm A (renormalization).