Indecomposability of polynomials and related Diophantine equations

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Abstract

We present a new criterion for indecomposability of polynomials over $\mathbb{Z}$. Using the criterion we obtain general finiteness result on polynomial Diophantine equation $f(x) = g(y)$.

1 Introduction

The polynomial equation of the form $f(x) = g(y)$ has been studied by several authors. The essential question is whether this equation has finitely or infinitely many integer solutions.

Bilu and Tichy [3] obtained a completely explicit finiteness criterium. Their result generalize a previous one due to Schinzel [6, Theorem 8], who gave a finiteness criterium under the assumption $(\deg f, \deg g) = 1$. To formulate the criterium of Bilu and Tichy, we have to define five types of standard pairs $(f(x), g(x))$.

In what follows, a and $b \in \mathbb{Q}\backslash\{0\}, m$ and n are positive integers, and $p(x)$ is a non-zero polynomial.

A standard pair of the first kind is the pair of the form $(x^m, ax^r p(x)^m)$, or switched, $(ax^r p(x)^m, x^m)$ where $0 \le r < m, (r, m) = 1$ and $r + \deg p(x) > 0$. A standard pair of the second kind is $(x^2, (ax^2 + b)p(x)^2)$ (or switched).

Denote by $D_{m,a}(x)$ the m-th Dickson's polynomial, defined by

$$D_{m,a}(z + a/z) = z^m + (a/z)^m.$$

A standard pair of the third kind is $(D_{m,a^n}(x), D_{n,a^m}(x))$, where $\gcd(m, n) = 1$. A standard pair of the fourth kind is $\left(a^{-m/2} D_{m,a}(x), -b^{-n/2} D_{n,b}(x)\right)$, where $\gcd(m, n) = 2$.

A standard pair of the fifth kind is $((ax^2 - 1)^3, 3x^4 - 4x^3)$ (or switched).

THEOREM **1** *(*BILU-TICHY *[3].) Let $f(x), g(x) \in \mathbb{Q}[x]$ be non-constant polynomials. Then the following two assertions are equivalent.*

(a) The equation $f(x) = g(y)$ has infinitely many rational solutions with a bounded denominator.

(b) We have $f = \varphi \circ f_1 \circ \lambda$ and $g = \varphi \circ g_1 \circ \mu$, where $\lambda(x), \mu(x) \in \mathbb{Q}[x]$ are linear polynomials, $\varphi(x) \in \mathbb{Q}[x]$, and $(f_1(x), g_1(x))$ is a standard pair over $\mathbb{Q}$ such that the equation $f_1(x) = g_1(y)$ has infinitely many rational solutions with a bounded denominator.

Theorem 1 has been already applied to several Diophantine equations of the form $f_n(x) = g_m(y)$, where (f_n) and (g_n) are sequences of classical polynomials (see [1, 2, 4, 5, 9]).

As we can easily see, the conditions from Theorem 1 are closely connected with decomposability properties of polynomials f and g. In the above mentioned results, the indecomposability of corresponding polynomials was usually proved using some analytical properties of these polynomials. See [8] for systematical approach, where two such properties: simple stationary points (P1) and two-interval monotonicity (P2), were discussed in details.

The purpose of the present paper is to give general criteria for indecomposability of polynomials in terms of the degree and two leading coefficients (Corollary 1, Theorem 4 and Corollary 2). As a corollary of that results, we will obtain also general finiteness results on Diophantine equation $f(x) = g(y)$ (Theorem 7). Although these results are too restrictive to be applied to most of classical orthogonal polynomials, they are very suitable for application to polynomials defined by binary recursive relations. Details will be given in our forthcoming paper. In particular, the main result from [4] (and its very extensive generalization) follows easily from the results of the present paper.

2 A criterium for indecomposability of polynomials

A polynomial $f \in \mathbb{C}[x]$ is called *indecomposable* (over $\mathbb{C}$) if $f = g \circ h$, $g, h \in \mathbb{C}[x]$ implies $\deg g = 1$ or $\deg h = 1$.

Two decompositions of f, say $f = g_1 \circ h_1$ and $f = g_2 \circ h_2$ are *equivalent* if there exist a linear function L such that $g_2 = g_1 \circ L$, $h_2 = L^{-1} \circ h_1$ (see [6, pp. 14–15]).

Theorem **2** *Let $f(x) \in \mathbb{Z}[x]$ be monic and decomposable over $\mathbb{C}$. Then f is decomposable over $\mathbb{Z}$ (as a composition of two monic polynomials).*

Proof.

First note that if f is decomposable over $\mathbb{C}$, then f is also decomposable over $\mathbb{Q}$ (see [7, Theorem 6]). Also, if f is decomposable over $\mathbb{Q}$, then obviously there exists a decomposition of f over $\mathbb{Q}$ with two monic polynomials as composite factors.

Let now $f = g \circ h$, where $g, h \in \mathbb{Q}[x]$ are monic and $\deg f, \deg g \geq 2$. Moreover, replacing $g(x)$ by $g(x + h(0))$ and $h(x)$ by $h(x) - h(0)$, we may assume that $h(0) = 0$.

Then $f(x) = (x - \alpha_1)\cdots(x - \alpha_n)$, $g(x) = (x - \beta_1)\cdots(x - \beta_m)$, for algebraic integers $\alpha_1, \ldots, \alpha_n$ and algebraic numbers $\beta_1, \ldots, \beta_m$. Thus,

$$f(x) = (h(x) - \beta_1)\cdots(h(x) - \beta_m).$$

The polynomials $f(x)$ and $h(x) - \beta_j$, $j = 1, 2, ..., m$ have a factorization into linear polynomials over a suitable algebraic number field K containing $\alpha_1, ..., \alpha_n; \beta_1, ..., \beta_m$. Since the factorization is unique,

$$h(x) - \beta_j = const \cdot \prod_{i \in I}(x - \alpha_i),$$

for a suitable set of indices $I \subset \{1, 2, ..., n\}$. But $h(x) - \beta_j$ is monic, which implies that $const = 1$. Since α_i's are algebraic integers and $h(0) = 0$, we conclude that β_j, $j = 1, ..., m$ (the roots of $g(x)$) are algebraic integers and also the coefficients of $h(x)$ are algebraic integers. Therefore, $g, h \in \mathbb{Z}[x]$. ■

Theorem **3** *Let $f(x) = x^n + ax^{n-1} + \cdots \in \mathbb{Z}[x]$. If $\gcd(a, n) = 1$, then f is indecomposable.*

Proof. Assume that f is decomposable. Theorem 2 implies that there exist monic polynomials $G, H \in \mathbb{Z}[x]$ such that $f = G \circ H$, $\deg G, \deg H \geq 2$. Hence,

$$f(x) = (x^k + c_{k-1}x^{k-1} + \cdots)^m + \cdots$$

where $c_{k-1} \in \mathbb{Z}$. Therefore $n = mk$ and $a = mc_{k-1}$, which implies that $\gcd(a, n) \geq m \geq 2$. ■

Corollary **1** *Let $f(x) = dx^n + ax^{n-1} + \cdots \in \mathbb{Z}[x]$. If $\gcd(a, n) = 1$, then f is indecomposable.*

Proof. We have

$$d^{n-1}f(x) = (dx)^n + a(dx)^{n-1} + \cdots = F(dx),$$

where $F(x) \in \mathbb{Z}[x]$ is monic of degree n and the coefficient with x^{n-1} equal to a. By Theorem 3, the polynomial F is indecomposable, and this clearly implies that f is indecomposable, too. ■

3 Even and odd polynomials

The results of the previous section cannot be applied to polynomials having the coefficient with x^{n-1} equal to 0. In particular, this excludes the important classes of even and odd polynomials. In this section, we show that for such polynomials the analogous results are valid.

THEOREM **4** *Let $f(x) = dx^{2n} + ax^{2n-2} + \cdots \in \mathbb{Z}[x]$ be an even polynomial and define $g(x) = f(\sqrt{x})$. Assume that $\gcd(a, n) = 1$. Then every decomposition of f is equivalent to one of the following decompositions:*

(i) $f = g(x^2)$,

(ii) $f = \big(xp(x^2)\big)^2$.

The case (ii) appears if and only if $g(x) = xp(x)^2$ for some polynomial $p(x) \in \mathbb{Z}[x]$.

Proof. We have $f(x) = g(x^2)$, where, according to Corollary 1, the polynomial g is indecomposable. Now, the statement of the Theorem follows from the second Ritt's theorem [7, Theorem 8]. ■

In order to prove a similar statement for odd polynomials, we need three lemmas.

LEMMA **1** *Let f, S, T be polynomials over a field of characteristic 0, and let $\deg f \geq 1$ and $\deg S \neq \deg T$.*

(i) Assume that

$$f \circ (S + T) = f \circ (S - T).$$

Then $T = 0$, or S is a constant polynomial such that $f'(S) = f'''(S) = \cdots = 0$.

(ii) Assume that

$$f \circ (S + T) = -f \circ (S - T).$$

Then $T = 0$ and S is a constant such that $f(S) = 0$, or S is a constant such that $f(S) = f''(S) = \cdots = 0$.

Proof. Consider the Taylor's expansions

$$f \circ (S+T) = f \circ S + (f' \circ S)T + (f'' \circ S)\frac{T^2}{2!} + \cdots,$$

$$f \circ (S-T) = f \circ S - (f' \circ S)T + (f'' \circ S)\frac{T^2}{2!} - \cdots.$$

(i) From the assumption and the above formula, it follows

$$(f' \circ S)T + (f''' \circ S)\frac{T^3}{3!} + \cdots = 0. \tag{3.1}$$

With the notation $\deg f = m$, $\deg S = n$, $\deg T = k$, we have

$$\deg(f^{(2r+1)} \circ S)T^{2r+1} = (m-2r-1)n + (2r+1)k.$$

Assume that $(m-2r-1)n + (2r+1)k = (m-2l-1)n + (2l+1)k$ for positive integers r, l. Then $l = r$ or $k = n$. Since $k \neq n$ by the assumption of the lemma, we must have $l = r$. This implies that all terms in (3.1) have different degrees (if they are not zero-polynomials). Hence, we obtain a contradiction, unless $T = 0$ or $f'(S) = f'''(S) = \cdots = 0$. In the last case, S has to be a constant polynomial.

(ii) As in the case (i), we obtain

$$(f \circ S)T + (f'' \circ S)\frac{T^2}{2!} + \cdots = 0,$$

and we conclude that $T = 0$ and $f \circ S = 0$, or $T \neq 0$ and $f(S) = f''(S) = \cdots = 0$. In both cases, S has to be a constant polynomial. ■

Lemma **2** *Let $f = g \circ h$ be a decomposition of an odd polynomial f. This decomposition is equivalent to a decomposition $G \circ H$, where G and H are odd polynomials, i.e. $g = G \circ L$ and $h = L^{-1} \circ H$ for a linear polynomial L.*

Proof. We have $f(-x) = g(h(-x)) = -g(h(x))$. Let S and T be the even and odd part of h, respectively. Then $h = S + T$, and we obtain

$$g \circ (S-T) = -g \circ (S+T).$$

By Lemma 1 (ii), we conclude that S is a constant and $g(S) = g''(S) = \cdots = 0$. Hence,

$$g(x) = g'(S)(x-S) + g'''(S)(x-S)^3 + \cdots,$$

i.e. $g(x) = G(x - S)$, where G is an odd polynomial. Also, $h(x) = H(x) + S$, where $H := T$ is an odd polynomial. Finally, we have

$$f = g \circ h = G \circ L \circ L^{-1} \circ H,$$

where $L(x) = x - S$. ■

LEMMA **3** *Let $f = g \circ h$ be a decomposition of an even polynomial f. Then h is an even polynomial, or $g = G \circ L$ and $h = L^{-1} \circ H$, where G is even, H is odd and L is a linear polynomial.*

Proof. If we write $h = S + T$, where S is even and T is odd part of h, then $g \circ (S + T) = g \circ (S - T)$. By Lemma 1 (i), we have two possibilities. If $T = 0$, then h is an even polynomial. If S is a constant such that $g'(S) = g'''(S) = \cdots = 0$, then

$$g(x) = g(S) + g''(S)\frac{(x - S)^2}{2!} + \cdots.$$

Therefore, $g(x) = G(x - S)$ for an even polynomial G. Also, $h(x) = H(x) + S$, where $H := T$ is an odd polynomial. ■

THEOREM **5** *Let $f(x) = x^n + ax^{n-2} + \cdots \in \mathbb{Z}[x]$ be an odd polynomial. If* $\gcd(a, n) = 1$*, then f is indecomposable.*

Proof. Assume that f is decomposable. By Lemma 2, we may assume that $f = g \circ h$, where f, g are odd polynomials. From the proofs of Lemma 2 and Theorem 2, it follows that we may also assume that $g, h \in \mathbb{Q}[x]$ are monic polynomials. As in the proof of Theorem 2, we obtain

$$f(x) = h(x)(h(x) - \beta_2) \cdots (h(x) - \beta_m).$$

We see that $h \in \mathbb{Z}[x]$. Now we have

$$f(x) = (x^k + c_{k-2}x^{k-2} + \cdots)^m + b(x^k + c_{k-2}x^{k-2} + \cdots)^{m-2} + \cdots,$$

which implies $mc_{k-2} = a$ and $\gcd(a, n) \neq 1$. ■

COROLLARY **2** *Let $f(x) = dx^n + ax^{n-2} + \cdots \in \mathbb{Z}[x]$ be an odd polynomial. If* $\gcd(da, n) = 1$*, then f is indecomposable.*

Proof. The proof is analogous to the proof of Corollary 1. ■

4 Polynomials associated with Dickson's polynomials

We say that polynomials f, g are *associated* if there exist linear polynomials L, M such that $g = L \circ f \circ M$.

Dickson's polynomials $D_{n,A}$ satisfy the recurrence $D_{0,A} = 2$, $D_{1,A} = x$, $D_{n+1,A}(x) = xD_{n,A}(x) - AD_{n-1,A}(x)$.

The following properties of Dickson's polynomials are well-known:

$$\begin{aligned} D_{n,A}(x) &= \sum_{k \le \frac{n}{2}} \frac{n}{n-k}\binom{n-k}{k}(-A)^k x^{n-2k} \qquad (4.1)\\ &= x^n - nAx^{n-2} + \frac{n(n-3)}{2!}A^2x^{n-4} - \frac{n(n-4)(n-5)}{3!}A^3x^{n-6} + \cdots \end{aligned}$$

$$D_{n,b^2A}(bx) = b^n D_{n,A}(x). \qquad (4.2)$$

From (4.2) it follows that every Dickson's polynomial $D_{n,A}$ with $A \ne 0$ is associated with some Dickson's polynomial of the form $D_{n,1}$.

THEOREM **6** *Suppose that the polynomial $f(x) = dx^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + ex^{n-4} - \cdots \in \mathbb{Z}[x]$ is associated (over $\mathbb{C}$) with $D_n := D_{n,1}$, for $n \ge 4$. Then $4a^3 \equiv 0 \pmod{n}$. If $a = 0$, then $c = 0$ and $(b, n) \ne 1$.*

Proof. Assume that $f(x) = \alpha D_n(\gamma x + \delta) + \beta$. Then

$$\begin{aligned} d &= \alpha\gamma^n,\\ a &= \alpha\gamma^{n-1}\cdot\delta n,\\ b &= \binom{n}{2}\alpha\gamma^{n-2}\delta^2 - n\alpha\gamma^{n-2},\\ c &= \binom{n}{3}\alpha\gamma^{n-3}\delta^3 - n(n-2)\alpha\gamma^{n-3}\delta. \end{aligned}$$

This leads to

$$a\gamma = dn\delta, \qquad (4.3)$$

$$b\gamma^2 = d\Big(\binom{n}{2}\delta^2 - n\Big), \qquad (4.4)$$

$$c\gamma^3 = d\Big(\binom{n}{3}\delta^3 - n(n-2)\delta\Big). \qquad (4.5)$$

From (4.3) and (4.4) we obtain

$$\delta^2((n-1)a^2 - 2bdn) = 2a^2. \tag{4.6}$$

If $a \neq 0$, then (4.5) and (4.6) imply

$$(n-1)(n-2)a^3 - 6cd^2n^2 = 3(n-2)a((n-1)a^2 - 2bdn),$$

and $4a^3 \equiv 0 \pmod{n}$.

If $a = 0$, then $\delta = 0$ (by (4.3)) and $c = 0$ (by 4.5)). The relation

$$\alpha\gamma^{n-4} \cdot \frac{n(n-3)}{2} = e$$

implies $dn(n-3) = 2e\gamma^4$, and from (4.4) we obtain $-dn = b\gamma^2$. Combining these two equalities we obtain

$$b^2(n-3) = 2den. \tag{4.7}$$

Hence, if $n \not\equiv 0 \pmod{3}$ then $n|b^2$, and if $n \equiv 0 \pmod{3}$ then $\frac{n}{3}|b^2$. Since $n \geq 4$, we conclude that $\gcd(b, n) \neq 1$. ∎

COROLLARY **3** *(i) Let* $f(x) = dx^n + ax^{n-1} + \cdots \in \mathbb{Z}[x]$ *and* $\gcd(a, n) = 1$. *If* $n \geq 5$, *then* f *is not associated with any* $D_{n,A}$.

(ii) Let $f(x) = dx^n + ax^{n-2} + \cdots \in \mathbb{Z}[x]$ *be an odd or even polynomial satisfying* $\gcd(a, n) = 1$. *If* $n \geq 4$, *then* f *is not associated with any* $D_{n,A}$.

5 The equation $f(x) = g(y)$

Let us introduce the following notation:

$$\begin{aligned}
\mathcal{A} &= \{f \in \mathbb{Z}[x] : f(x) = dx^n + ax^{n-1} + \cdots,\ n \geq 3,\ \gcd(a, n) = 1\},\\
\mathcal{B} &= \{f \in \mathbb{Z}[x] : f \text{ is odd},\ f(x) = dx^n + ax^{n-2} + \cdots,\ n \geq 3,\ \gcd(da, n) = 1\},\\
\mathcal{C} &= \{f \in \mathbb{Z}[x] : f \text{ is even},\ f(x) = dx^{2n} + ax^{2n-2} + \cdots,\ n \geq 2,\ \gcd(a, n) = 1\}.
\end{aligned}$$

THEOREM **7** *The equation* $f(x) = g(y)$ *has only finitely many integer solutions if one of the following conditions are satisfied:*

a) $f, g \in \mathcal{A}$, $\deg f \geq 5$, *unless* $g(x) = f(\gamma x + \delta)$, $\gamma, \delta \in \mathbb{Q}$.

b) $f \in \mathcal{A}$*,* $g \in \mathcal{B}$ *and* $\deg f \geq 5$*, unless* $g(x) = f(\gamma x + \delta)$*,* $\gamma, \delta \in \mathbb{Q}$*.*

c) $f, g \in \mathcal{B}$*, unless* $g(x) = f(\gamma x)$*,* $\gamma \in \mathbb{Q}$*.*

d) $f \in \mathcal{A}$*,* $g \in \mathcal{C}$*, unless* $G(x) = f(\gamma x+\delta)$*,* $\gamma, \delta \in \mathbb{Q}$*, where* $G(x) := g(\sqrt{x})$*.*

e) $f \in \mathcal{B}$*,* $g \in \mathcal{C}$*,* $\deg g \neq 2$*, unless* $G(x) = f(\gamma x + \delta)$*,* $\gamma, \delta \in \mathbb{Q}$*, where* $G(x) := g(\sqrt{x})$*.*

f) $f, g \in \mathcal{C}$*, unless* $G(x) = F(\gamma x + \delta)$*, where* $G(x) := g(\sqrt{x})$*,* $F(x) := f(\sqrt{x})$*.*

Proof. We apply the criterium of Bilu and Tichy (Theorem 1). By Corollaries 1 and 2, the polynomials from $\mathcal{A} \cup \mathcal{B}$ are indecomposable, and, by Theorem 5, polynomials from $\mathcal{C}$ have only trivial decompositions. The results of the previous section show that the polynomials from $\mathcal{A}\cup\mathcal{B}\cup\mathcal{C}$ with degree greater that 4 cannot be associated with Dickson's polynomials. Also they are not associated with mth power for $m \geq 3$. Indeed, the statement is trivially valid for even and odd polynomials. Assume that $\alpha(\gamma x + \delta)^m + \beta = dx^m + ax^{m-1} + bx^{m-2} + \cdots$. Then from $\alpha\delta^m = d$, $m\alpha\gamma^{m-1}\delta = a$, $\frac{m(m-1)}{2}\alpha\gamma^{m-2}\delta^2 = b$, we obtain $(m-1)a^2 = 2mbd$. Hence, $\gcd(a, m) > 1$.

Note that this argument is not valid for $m = 2$. The examples $x(x + 1)^m + 1 = 2y^2 + 1$ and $x^{2n} + x^{2n-2} + 1 = 2y^2 + 1$ show that the polynomials of degree 2 have to be excluded from the sets $\mathcal{A}$ and $\mathcal{C}$.

To exclude the standard pairs of fifth kind we will show that a polynomial $f \in \mathcal{A}$ cannot be associated to $(ux^2-1)^3$. Assume that $\alpha(u(\gamma x+\delta)^2-1)^3 + \beta = dx^6 + ax^5 + bx^4 + cx^3 + \cdots$. We have $d = \alpha u^3\gamma^6$, $a = 6\alpha u^3\gamma^5\delta$, $b = 15\alpha u^3\gamma^4\delta^2 - 3\alpha u^2\gamma^4$, $c = 20\alpha u^3\gamma^3\delta^3 - 12\alpha u^2\gamma^3\delta$. Combining these relations, we obtain $5a^3 = 9d(2ab - 3cd)$. This clearly implies that $3|a$, which contradicts the assumption that $\gcd(a, n) = 1$.

Therefore, we have excluded all standard pairs, assuming that the degrees of f and g are not very small. It remains to prove the case c) if $\deg f = \deg g = 3$. We have to check when the equation

$$dx^3 - ax = Dy^3 - by,$$

where $(ad, 3) = (bD, 3) = 1$, will have infinitely many solutions. It is easily seen that no standard pair is possible in this situation, except the trivial pair $(x, p(x))$ with p linear. Therefore

$$Dx^3 - bx = d(\gamma x + \delta)^3 - a(\gamma x + \delta).$$

This implies $\delta = 0$, $d\gamma^3 = D$ and $a\gamma = b$. ∎

As we have mentioned in the introduction, our general finiteness result from Theorem 7 is very suitable for application to polynomials defined by binary recursive relations. In our forthcoming paper, we will apply it to Diophantine equations with polynomials satisfying very general binary recurrence. Here, we give results on Fibonacci polynomials, which are related to the main result from [4].

COROLLARY **4** *Let* (F_n) *be the sequence of Fibonacci polynomials defined by* $F_0(x) = 0$, $F_1(x) = 1$, $F_{n+1} = xF_n(x) + F_{n-1}$ *for* $n \geq 1$. *Let* $Q \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, *and if* $Q \in \mathcal{A}$ *then assume that* $\deg Q \geq 5$. *If* $n \geq 4$ *and* $\deg Q \neq n-1$, *then the equation* $F_n(x) = G(y)$ *has only finitely many integer solutions.*

In particular, the equation $F_m(x) = F_n(y)$ *for* $m, n \geq 4$, $m \neq n$, *has only finitely many integer solutions.*

Proof. It is well known that

$$F_n(x) = \sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-j-1}{j} x^{n-2j-1} = x^{n-1} + (n-2)x^{n-3} + \cdots$$

Therefore, if n is even and $n \geq 4$ then $F_n \in \mathcal{B}$, and if n is odd and $n \geq 5$ then $F_n \in \mathcal{C}$. Hence, the statement of the corollary follows from Theorem 7. We only have to check that in the case e), the possibility $G(x) = f(\gamma x + \delta)$ cannot appear. This can be done by following the same approach as in the proof of Theorem 6. ∎

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